Algebraic Geometry

Based on lectures by Marc Hoyois

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Contents

Preface

These notes record and complete the *Algebraic Geometry* course taught by Professor Marc Hoyois during the Winter 2025/2026 semester at the University of Regensburg. While the official script has been publicly released, it omits proofs for most theorems. The aim of these notes is to supplement that script: we work out complete proofs based on Marc's lectures, the script itself, and supporting reading.

What is Algebraic Geometry?

Algebraic geometry, from the perspective developed in these notes, is the study of functors

$$X: \mathsf{CAlg} \to \mathsf{Set}$$

from commutative rings to sets — or, in the derived variant, to the ∞-category An of anima. Each such *algebraic functor* X assigns to every ring R a set $X(R)$ of "R-points of X". Geometry then becomes the study of these functors: their subobjects, morphisms, gluing behaviour, and invariants.

To make this functorial picture concrete, Grothendieck's *unstraightening* (equivalently, the category of elements construction) turns any X into a Grothendieck fibration

$$p: \int X \longrightarrow \mathsf{CAlg}.$$

Explicitly, $\int X$ has as objects the pairs (R, x) with $x \in X(R)$, and the fibre over each ring R is exactly the set of R-points:

$$p^{-1}(R) = X(R).$$

When X is the solution functor to a polynomial system $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$,

$$X(R) = \{a \in R^n \mid f_1(a) = \cdots = f_m(a) = 0\},$$

the fibration $\int X \to \mathsf{CAlg}$ assembles *all* solution sets over *all* rings into a single coherent geometric object, and ring homomorphisms $R \to S$ induce morphisms of R-points to S-points. Doing geometry on X is doing geometry on the whole family at once.

***Remark* 0.0.1** (Toward Derived Algebraic Geometry)**.** In derived algebraic geometry one replaces the source category CAlg by a category of "derived rings" (animated rings, or $\mathbb{E}_n$-algebras), and the target Set by the ∞-category An. The unstraightening becomes a coCartesian fibration of ∞-categories, and the examples below continue to make sense — producing richer invariants (e.g. homotopical tangent complexes, derived loop spaces) that are invisible in the classical setting.

Motivating Examples

The remainder of this preface — adapted from Marc Hoyois's script — illustrates what algebraic geometry looks like **before** any formal machinery: solution functors of concrete polynomial systems, examined across rings. These examples guide our intuition for the entire development that follows.

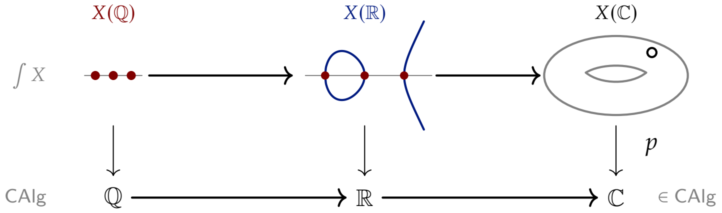
Punctured Elliptic Curves

Consider the equation $y^2 = x^3 - x$ and its solution functor

$$X(R) \;=\; \{(a,b) \in R^2 \mid b^2 = a^3 - a\}.$$

Different rings reveal different aspects of X.

Over $\mathbb{R}$. $X(\mathbb{R}) \subset \mathbb{R}^2$ is a 1-dimensional real submanifold with two connected components, diffeomorphic to $S^1 \sqcup \mathbb{R}$:



Over $\mathbb{Q}$ and $\mathbb{Z}$. The rational points are in fact integral:

$$X(\mathbb{Q}) \;=\; X(\mathbb{Z}) \;=\; \{(0,0),\ (\pm 1,0)\}.$$

Over $\mathbb{C}$. $X(\mathbb{C}) \subset \mathbb{C}^2$ is a 1-dimensional complex submanifold, diffeomorphic to a **punctured torus**. Holomorphically, it is the noncompact Riemann surface $(\mathbb{C}/\Lambda) \setminus \{0\}$, where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}i$.

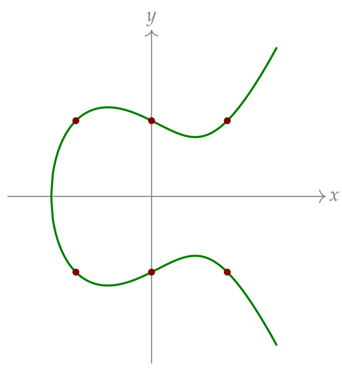
Over the dual numbers $R[\varepsilon]$, $\varepsilon^2 = 0$. A pair $(a + u\varepsilon, b + v\varepsilon)$ is a solution if and only if $(a,b) \in X(R)$ and (u,v) is a tangent vector at (a,b). Hence $X(R[\varepsilon])$ is the **tangent bundle** of $X(R)$. Over a field k, the tangent space at every point of $X(k)$ is 1-dimensional, except at $(1,0)$ in characteristic 2, where it becomes 2-dimensional. Accordingly, X has **bad reduction** at the prime 2 and **good reduction** at every other prime.

***Remark* 0.0.1** (Dual Numbers and Derivatives). For $f \in R[x_1, \ldots, x_n]$ and $a + u\varepsilon \in R[\varepsilon]^n$,

$$f(a + u\varepsilon) \;=\; f(a) \;+\; \left(\frac{\partial f}{\partial x_1}(a) \cdot u_1 + \cdots + \frac{\partial f}{\partial x_n}(a) \cdot u_n\right)\varepsilon.$$

First-order differentiation emerges purely from the algebraic relation $\varepsilon^2 = 0$, which automatically annihilates every higher-order term. The tangent space at $a \in V(f)(R)$ is therefore the kernel of the gradient $\nabla f(a) \colon R^n \to R$.

Compare this with the closely related equation $y^2 = x^3 - x + 1$, whose solution functor Y looks similar at first glance but is qualitatively different:



$Y(\mathbb{R})$ has a single connected component, diffeomorphic to $\mathbb{R}$; $Y(\mathbb{Z})$ already comprises 12 points,

$$Y(\mathbb{Z}) = \{(-1,\pm 1),\ (0,\pm 1),\ (1,\pm 1),\ (3,\pm 5),\ (5,\pm 11),\ (56,\pm 419)\},$$

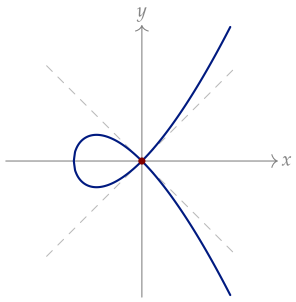
and $Y(\mathbb{Q})$ is infinite. Although $X(\mathbb{C})$ and $Y(\mathbb{C})$ are diffeomorphic — both are punctured tori — they are **not** biholomorphic: they correspond to non-isomorphic lattices in $\mathbb{C}$, a distinction invisible to real differential geometry.

Singular Cubic Curves

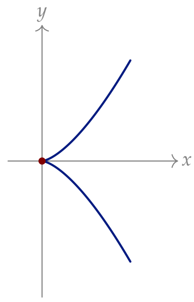
Consider the functors

$$N(R) = \{(a, b) \mid b^2 = a^3 + a^2\}, \qquad C(R) = \{(a, b) \mid b^2 = a^3\}.$$

N is the **nodal cubic** and C the **cuspidal cubic**:



Nodal: $y^2 = x^3 + x^2$



Cuspidal: $y^2 = x^3$

Both have a singularity at the origin, visible in the Jacobian: the gradient vanishes at $(0, 0)$, so the tangent space there is 2-dimensional rather than 1-dimensional.

There are two algebraic ways to address the nodal singularity:

(i) **Deformation.** View N as the $\lambda = 0$ specialisation of the family $N_\lambda = V(y^2 - x(x + \lambda)(x + 1))$. For $\lambda \neq 0$, $N_\lambda(\mathbb{C})$ is a nonsingular punctured torus; the singular fibre appears only at $\lambda = 0$.

(ii) **Blowup.** Introduce the slope coordinate $s = y/x$, in terms of which the equation rewrites as $s^2 = x + 1$ — now nonsingular. The functor $\widetilde{N}(R) = \{(a, c) \mid c^2 = a + 1\}$, equipped with the map $\widetilde{N} \to N$, $(a, c) \mapsto (a, ac)$, is the **blowup of N at the origin**.

In characteristic zero, any variety can be resolved by iterated blowups (Hironaka, 1964); whether the analogous statement holds in positive characteristic is a major open problem.

Compactifying Affine Curves

The affine curves of the previous subsections can be compactified by passing from affine n-space $\mathbb{A}^n$ to projective n-space $\mathbb{P}^n$. We recall the projective space functor in a preliminary form,

$$\mathbb{P}^n(R) \;=\; \{(a_0, \ldots, a_n) \in R^{n+1} \mid (a_0, \ldots, a_n) = R\} \;/\; R^\times,$$

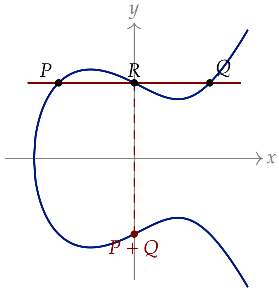
writing equivalence classes as $[a_0 : \cdots : a_n]$. A homogeneous polynomial f of degree d obeys $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$, so its vanishing locus is well-defined on $\mathbb{P}^n$.

To compactify $y^2 = x^3 - x$, we **homogenise** with a new variable w: multiply each term by an appropriate power of w to equalise the total degree, yielding $wy^2 = x^3 - w^2 x$. The resulting projective solution functor is

$$\bar{X}(R) \;=\; \{[w : x : y] \in \mathbb{P}^2(R) \mid wy^2 = x^3 - w^2 x\}.$$

Setting $w = 1$ recovers the affine X, while $w = 0$ produces the points at infinity — here a single new point $[0 : 0 : 1]$, which fills the puncture. Over $\mathbb{C}$, $\bar{X}(\mathbb{C})$ thus becomes a genuine torus, biholomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$.

Away from primes of bad reduction, $\bar{X}$ is an **elliptic curve**: once a point is fixed as the identity, there is a unique abelian group structure on $\bar{X}$ for which any three collinear points sum to zero.



Concretely, the line through two points P and Q meets the curve at a unique third point R, and $P+Q$ is the reflection of R across the x-axis. This is the **group law** on an elliptic curve. In the complex picture $\bar{X}(\mathbb{C}) \simeq \mathbb{C}/\Lambda$, the group law is inherited directly from addition on $\mathbb{C}$.

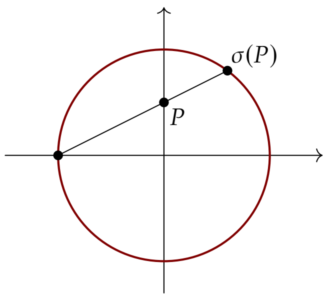
Fermat Curves and Pythagorean Triples

The Fermat functors $X_n(R) = \{[a:b:c] \in \mathbb{P}^2(R) \mid a^n = b^n + c^n\}$ interpolate between projective lines and profound number theory.

- $n = 0$: $X_0(R) = \emptyset$ for every nonzero ring R, so X_0 is the **empty scheme**.
- $n = 1$: $X_1 \simeq \mathbb{P}^1$ is a projective line.
- $n = 2$: The integral solutions $X_2(\mathbb{Z})$ are the (primitive) **Pythagorean triples**. Over any $\mathbb{Z}[\frac{1}{2}]$-algebra R there is an explicit parameterisation

  $$\sigma\colon \mathbb{P}^1(R) \xrightarrow{\sim} X_2(R), \qquad [s:t] \longmapsto [s^2+t^2 : s^2-t^2 : 2st],$$

  specialising to Euclid's formula over $\mathbb{Z}$. Geometrically, σ is the **inverse stereographic projection** from the point $(-1,0)$ on the unit circle $b^2+c^2=1$:



- $n \geq 3$: Fermat's Last Theorem (Wiles, 1995) asserts

  $$X_n(\mathbb{Q}) = X_n(\mathbb{Z}) = \begin{cases} \{[0:1:-1],\ [1:0:1],\ [1:1:0]\} & n \text{ odd}, \\ \{[1:0:\pm 1],\ [1:\pm 1:0]\} & n \text{ even}. \end{cases}$$

  Away from 3, $X_3\colon \mathsf{CAlg}_{\mathbb{Z}[1/3]} \to \mathsf{Set}$ is another elliptic curve, with a unique abelian group law having $[0:1:-1]$ as identity.

Affine, Projective, and General Schemes

The examples above touch on several overlapping classes of geometric objects, whose strict inclusions

$$\begin{array}{ccccccc} \{\text{affine schemes}\} & \subset & \{\text{quasi-affine schemes}\} & & & & \\ \cap & & \cap & & & & \\ \{\text{projective schemes}\} & \subset & \{\text{quasi-projective schemes}\} & \subset & \{\text{schemes}\} & \subset & \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \end{array}$$

form a central organising principle.

- Affine schemes are solution functors of polynomial systems in $\mathbb{A}^n$ (Chapter 1).
- Projective schemes are solution functors of **homogeneous** systems in $\mathbb{P}^n$ (Chapter 2).
- Quasi-affine and quasi-projective schemes are open subfunctors of the above.
- Schemes form the most general category — closed under gluing — and are defined in Chapter 6 as Zariski sheaves admitting an affine open cover.

A few boundary examples clarify these inclusions.

(i) **Punctured affine n-space** $\mathbb{A}^n - 0$ for $n \geq 2$ — the functor $R \mapsto \{a \in R^n \mid (a) = R\}$ — is quasi-affine but not affine.

(ii) **Punctured projective n-space** $\mathbb{P}^n - 0$ for $n \geq 2$ is quasi-projective but neither projective nor quasi-affine.

(iii) Non-quasi-projective schemes are harder to construct; the prototype is the **affine line with two origins**, obtained by gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$. This scheme is quasi-compact but not separated (Example 6.2.4).

All of these are schemes. In what follows we develop the full theory of affine schemes (Chapter 1), projective schemes (Chapter 2), and general schemes (Chapter 6), with detours through quasi-coherent modules (Chapter 3), topological realisations (Chapter 4), and the Grothendieck-topology language of sheaves (Chapter 5); their basic geometric properties are then taken up in Chapter 7.

Course Structure

The defining feature of this course is its systematic development of algebraic geometry through the *functor of points* perspective. These notes are organised into two parts.

Part I: Algebraic Geometry I (Foundations of Scheme Theory)

The first part establishes the language of schemes using the functorial approach. It covers the construction of schemes from algebraic functors and the necessary sheaf theory.

- **Chapter 1 (Affine Geometry):** Define *affine schemes* as representable functors via the Yoneda embedding $\mathrm{Spec}\colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$; prove the functorial Nullstellensatz, classify open and closed subfunctors, establish Zariski descent.
- **Chapter 2 (Projective Geometry):** $\mathbb{N}$-graded rings, projective space $\mathbb{P}^n$ and the Proj functor; functorial projective Nullstellensatz, open and closed subfunctors of $\mathrm{Proj}(A)$, projective closure, Grassmannians, Veronese and Segre embeddings.

- **Chapter 3 (Quasi-coherent Modules):** Linear algebra over algebraic functors via Mod_X; classification of open/closed subfunctors by quasi-coherent ideals, relative Spec and Proj, Zariski descent for modules, Serre twisting sheaves $\mathcal{O}(d)$.
- **Chapter 4 (Locale):** *Locales* as point-free topology; $\mathsf{Open}(X)$ as a spatial locale for any algebraic functor X; geometric realisation $|X|$, characterisation of $\mathrm{Prim}(R)$ as a spectral space.
- **Chapter 5 (Sheaves):** Grothendieck topologies, sieves, descent; sheafification, Comparison Lemma, Zariski / fppf / fpqc topologies on Aff.
- **Chapter 6 (Schemes):** Formal definition of *schemes* as Zariski sheaves with affine open covers; immersions, gluing via locally trivial equivalence relations, separated / qcqs morphisms, identification with classical locally ringed spaces.

Part II: Algebraic Geometry II (Geometry of Schemes)

The second part begins the systematic study of the geometric content of schemes and their morphisms.

- **Chapter 7 (Properties of Schemes):** Local and global properties; reduced, noetherian, irreducible, integral schemes; dimension theory, regular / Cohen–Macaulay / normal schemes, the Zariski (co)tangent space.

Subsequent chapters, corresponding to the second semester of the course and currently in preparation, will turn to the geometry of *morphisms*: **smoothness**, **étale** and **flat** morphisms, **properness**, **divisors** (Weil and Cartier), and **blowups**.

Notation and Conventions

General Philosophy. We adopt the **functor-of-points** perspective: geometric objects are functors CAlg $\to$ Set. The locally ringed space perspective is reserved for Section 6.3. All rings are commutative with unit.

Typographical Conventions

Sans serif (A) Categories and categorical constructions.
Ex: $\mathsf{Set}, \mathsf{CAlg}, \mathsf{Aff}, \mathsf{Sch}, \mathsf{QCoh}(X), \mathsf{Open}(X)$.

Roman (A) Functors, operators, and standard constructions.
Ex: $\mathrm{Spec}(R), \mathrm{Proj}(A), \mathrm{Hom}(X, Y), \mathrm{colim}, \mathrm{Prim}(R)$.

Calligraphic ($\mathcal{A}$) Sheaves of rings, sieves, and specific categories.
Ex: $\mathcal{O}_X, \mathcal{R}, \mathcal{C}$.

Blackboard Bold ($\mathbb{A}$) Standard schemes and number systems.
Ex: $\mathbb{A}^n, \mathbb{P}^n, \mathbb{G}_m, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$.

Fraktur ($\mathfrak{a}$) Ideals.
Ex: $\mathfrak{p}, \mathfrak{m}, \mathfrak{a}$.

Critical Distinctions

- **Open Subobjects:** $\mathsf{Open}(X)$ is the locale of open subfunctors of an algebraic functor X (categorical), while $\mathrm{Open}(|X|)$ is the locale of open sets of its underlying space. They agree: $\mathsf{Open}(X) \simeq \mathrm{Open}(|X|)$.
- **Structure Sheaf $\mathcal{O}_X$:** Depending on context, this is either the unit object in Mod_X or the sheaf of rings on $|X|$; these are identified in Section 6.3.
- **Equality:** We write $\simeq$ for isomorphism and $=$ for strict equality.

Abbreviations and Standard Notation

Abbr.	Meaning	Symbol	Meaning
qcqs	quasi-compact quasi-separated	$\mathrm{Spec}(R)$	Affine scheme of R
ft	finite type	$\mathrm{Proj}(A)$	Projective scheme of A
fp	finite presentation	$\lvert X\rvert$	Topological space of X
fppf	faithfully flat finite presentation	y_C	Yoneda embedding
Zar	Zariski topology	$X \times_S Y$	Fiber product
		$\mathrm{El}(F)$	Category of elements

- **Loci:** $D(f)$ is the non-vanishing locus (open), $V(I)$ the vanishing locus (closed).
- **Localization:** R_f is the localisation at f; $A_{(f)}$ is the degree-zero part $(A_f)_0$.

Diagrammatic Conventions

- **Squares:** Pullbacks are marked with $\lrcorner$, pushouts with $\ulcorner$.
- **Adjunctions:** $F \dashv G$ indicates F is left adjoint to G.
- **Maps:** $f : X \longrightarrow Y$ for morphisms, $x \longmapsto f(x)$ for elements.

Main References

These notes are primarily based on the following standard textbooks.

- **U. Görtz, T. Wedhorn**, *Algebraic Geometry I: Schemes* [GW20]. The primary guideline for the structure of the course.
- **R. Vakil**, *The Rising Sea: Foundations of Algebraic Geometry* [Vak24]. A comprehensive resource with a geometric perspective.
- **D. Eisenbud, J. Harris**, *The Geometry of Schemes* [EH00]. Highly recommended for geometric intuition through examples.
- **R. Hartshorne**, *Algebraic Geometry* [Har77]. The classic reference.
- **Yu. I. Manin**, *Introduction to the Theory of Schemes* [Man18]. A profound conceptual introduction.
- **The Stacks Project Authors**, *The Stacks Project* [The25]. The ultimate reference for technical details.

Lecture Notes and Personal References

- **P. Scholze**, *Algebraic Geometry I* (typed by Jack Davies) [Sch17].
- **D. Clausen, L. Hesselholt**, *Scheme Theory* [CH22].
- **Y. Zhao**, *Scheme Theory I* [Zha24].
- **J. E. Rodríguez Camargo**, *Notes on D-Modules via Derived Algebraic Stacks* [Cam25].

Acknowledgments

The exposition of the theory of functorial algebraic geometry presented in these notes is entirely based on the lectures and insights of **Professor Marc Hoyois**. All credit for the mathematical content belongs to him, while any errors, misconceptions, or inaccuracies that remain are solely my own responsibility.

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Part I.

Algebraic Geometry I (Foundations of Scheme Theory)

Chapter 1.

Affine Geometry

Affine geometry is where algebraic geometry begins. Classically one studies the zero sets of polynomials in k^n for k a field; modernly, the functor-of-points approach encourages us to study all solution sets at once, packaging $R \mapsto \{\text{solutions in } R^n\}$ into a single functor $\mathsf{CAlg} \to \mathsf{Set}$.

This chapter develops the framework. We start by defining *algebraic functors* and identifying the representable ones as *affine schemes* (Yoneda embedding). We then classify their subfunctors: closed ones correspond to *ideals*, open ones to *radical ideals* — the functorial Nullstellensatz. Finally we prove *Zariski descent*, the sheaf condition that lets us glue local algebraic data, and establish finiteness properties (finite type, finite presentation) that govern the size of our objects.

The organising principle throughout is that everything is a functor: Spec turns rings into functors, V and D turn ideals into subfunctors, and all naturality conditions are automatic from the functor-category framework.

1.1. Algebraic Functors and Affine Schemes

Our basic objects are functors $\mathsf{CAlg} \to \mathsf{Set}$, which we think of as "solution sets of polynomial equations parameterised by a ring R". The most concrete examples are the affine spaces $\mathbb{A}^I$; the representable ones form the category of *affine schemes*.

Definition 1.1.1 (Algebraic Functor). Let k be a ring.

- An *algebraic k-functor* is a functor $\mathsf{CAlg}_k \to \mathsf{Set}$. When $k = \mathbb{Z}$, we simply call it an *algebraic functor*.
- Given an algebraic k-functor X and a k-algebra R, we call elements of $X(R)$ the *R-points* of X.

The most fundamental algebraic functors are *affine spaces*.

Definition 1.1.2 (Affine Space). Let I be a set and k a ring. The *affine I-space over k* is the algebraic k-functor

$$\mathbb{A}^I_k \colon \mathsf{CAlg}_k \to \mathsf{Set}, \quad R \mapsto R^I.$$

When $k = \mathbb{Z}$, we simply write $\mathbb{A}^I$. For $n \geq 0$, we also write $\mathbb{A}^n_k = \mathbb{A}^{\{1,\dots,n\}}_k$ for the affine n-space over k. When $n = 1$, we call it the *affine line*; when $n = 2$, the *affine plane*.

***Remark* 1.1.3.** Unwinding the definition:

- $\mathbb{A}^0_k$ is the terminal object of $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$.
- $\mathbb{A}^1_k$ is the forgetful functor $\mathsf{CAlg}_k \to \mathsf{Set}$.
- $\mathbb{A}^{(-)}_k \colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is functorial.
- By the universal property of polynomial rings, $\mathbb{A}^I_k$ is represented by $k[x_i \mid i \in I]$, i.e., there is an isomorphism

$$\mathbb{A}^I_k \simeq \mathrm{Hom}_{\mathsf{CAlg}_k}(k[x_i \mid i \in I], -).$$

1.1.1. Presheaves and Yoneda Embedding

We begin by recalling the concept of presheaves.

Definition 1.1.4 (Presheaf). Let C be a category. A *presheaf* on C is a functor $C^{\mathrm{op}} \to \mathsf{Set}$. We denote

$$\mathsf{PShv}(C) := \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{Set})$$

for the category of presheaves on C.

Note that algebraic k-functors defined earlier are precisely presheaves on $\mathsf{CAlg}_k^{\mathrm{op}}$.

***Remark* 1.1.5.** In functor categories, limits and colimits are computed pointwise. Thus $\mathsf{PShv}(C)$ inherits many properties of the category of sets, such as: filtered colimits commute with finite limits, epimorphisms are automatically effective, etc.

We now study the canonical functor from C to $\mathsf{PShv}(C)$, called the *Yoneda embedding*.

Definition 1.1.6 (Yoneda Embedding). Let C be a category. The *Yoneda embedding* of C is the functor

$$y\colon C \to \mathsf{PShv}(C), \quad y(X) := \mathrm{Hom}_C(-, X).$$

A presheaf on C is called *representable* if it lies in the essential image of the Yoneda embedding.

For $C = \mathsf{CAlg}^{\mathrm{op}}$, the Yoneda embedding is

$$\mathsf{CAlg}^{\mathrm{op}} \to \mathsf{PShv}(\mathsf{CAlg}^{\mathrm{op}}) = \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}),$$

sending a ring R to $\mathrm{Hom}_{\mathsf{CAlg}^{\mathrm{op}}}(-, R) = \mathrm{Hom}_{\mathsf{CAlg}}(R, -)$. For this case, we use the special notation

$$\mathrm{Spec}\colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$$

to denote this embedding. Thus Remark 1.1.3 shows that $\mathbb{A}_k^I \simeq \mathrm{Spec}(k[x_i \mid i \in I])$.

1.1.2. Category of Elements

For a functor $F\colon C \to \mathsf{Set}$, we can encode its information as a category $\int_C F$. Let Set_* denote the category of pointed sets:

- Objects are pairs (X, x) where X is a set and $x \in X$ is a chosen point.
- Morphisms $(X, x) \to (Y, y)$ are maps $f\colon X \to Y$ satisfying $f(x) = y$.

Definition 1.1.7 (Category of Elements). Let C be a category and $F\colon C \to \mathsf{Set}$ a functor. The *category of elements* $\int_C F$ is defined as the pullback:

$$\begin{array}{ccc} \int_C F & \longrightarrow & \mathsf{Set}_* \\ \big\downarrow & \lrcorner & \big\downarrow \\ C & \longrightarrow & \mathsf{Set} \end{array}$$

Concretely, $\int_C F$ is the category where:

- Objects are pairs (C, x) with $C \in C$ and $x \in F(C)$.
- A morphism $(C, x) \to (C', x')$ is a morphism $f\colon C \to C'$ in C such that $F(f)\colon F(C) \to F(C')$ satisfies $F(f)(x) = x'$.

For presheaves $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, we define the category of elements similarly:

Definition 1.1.8 (Category of Elements for Presheaves). Let $\mathcal{C}$ be a category and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a functor. The *category of elements* $\int^{\mathcal{C}} F$ is defined as the pullback:

$$\begin{array}{ccc} \int^{\mathcal{C}} F & \longrightarrow & (\mathsf{Set}_*)^{\mathrm{op}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C} & \longrightarrow & \mathsf{Set}^{\mathrm{op}} \end{array}$$

Concretely, $\int^{\mathcal{C}} F$ is the category where:

- Objects are pairs (C, x) with $C \in \mathcal{C}$ and $x \in F(C)$.
- A morphism $(C, x) \to (C', x')$ is a morphism $f\colon C \to C'$ in $\mathcal{C}$ such that $F(f)\colon F(C') \to F(C)$ satisfies $F(f)(x') = x$.

The two constructions are related by a canonical equivalence of categories

$$\left(\int_{\mathcal{C}} F\right)^{\mathrm{op}} \simeq \int^{\mathcal{C}^{\mathrm{op}}} F.$$

Marc's script uses only the contravariant version $\int^{\mathcal{C}} F$, which we abbreviate as $\mathrm{El}(F) := \int^{\mathcal{C}} F$.

For an algebraic functor $X\colon \mathsf{CAlg} \to \mathsf{Set}$, the category of elements $\mathrm{El}(X)$ is therefore the contravariant version, while the construction more natural for many of our colimit arguments is $\int_{\mathsf{CAlg}} X = \mathrm{El}(X)^{\mathrm{op}}$.

***Remark* 1.1.9** (Grothendieck Construction). This construction generalizes to category-valued functors $F\colon \mathcal{C} \to \mathsf{Cat}$, yielding the *Grothendieck construction*. In the setting of ∞-categories, this extends to $F\colon \mathcal{C} \to \mathsf{An}$ and $F\colon \mathcal{C} \to \mathsf{Cat}_\infty$, giving left fibrations and coCartesian fibrations respectively.

1.1.3. Yoneda Lemma

Theorem 1.1.10 (Yoneda Lemma). *Let $\mathcal{C}$ be a category.*

(i) For $X \in \mathcal{C}$ and $F \in \mathsf{PShv}(\mathcal{C})$, there is a bijection

$$\mathrm{Hom}_{\mathsf{PShv}(\mathcal{C})}(y(X), F) \simeq F(X), \quad f \mapsto f_X(\mathrm{id}_X).$$

The inverse is given by $x \mapsto ((f : Y \to X) \mapsto f^*(x))$.

(ii) The Yoneda embedding $y\colon \mathcal{C} \to \mathsf{PShv}(\mathcal{C})$ is fully faithful.

(iii) The Yoneda embedding preserves all limits that exist in $\mathcal{C}$.

(iv) (Density of Yoneda embedding) Every presheaf $F \in \mathsf{PShv}(\mathcal{C})$ can be written as a colimit of representable functors:

$$\operatorname{colim}\left(\mathrm{El}(F) \xrightarrow{\text{forget}} \mathcal{C} \xrightarrow{y} \mathsf{PShv}(\mathcal{C})\right) \simeq F.$$

(v) (Universal property of presheaves) For any cocomplete category $\mathcal{E}$, there is a category equivalence

$$y^*\colon \mathsf{Fun}^{\mathrm{colim}}(\mathsf{PShv}(\mathcal{C}), \mathcal{E}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{C}, \mathcal{E}),$$

where $\mathsf{Fun}^{\mathrm{colim}}$ *denotes colimit-preserving functors. The inverse is constructed via left Kan extension: for* $F\colon \mathcal{C} \to \mathcal{E}$*, consider*

$$\begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle y}\downarrow & \searrow^{F} & \\ \mathsf{PShv}(\mathcal{C}) & \xrightarrow[\mathrm{Lan}_y F]{} & \mathcal{E} \end{array}$$

Since $\mathcal{E}$ *has all colimits, this extension exists automatically.*

(vi) For a category $\mathcal{E}$ *and a colimit-preserving functor* $K\colon \mathsf{PShv}(\mathcal{C}) \to \mathcal{E}$*, it has a right adjoint* $N\colon \mathcal{E} \to \mathsf{PShv}(\mathcal{C})$ *given by* $e \mapsto \mathrm{Hom}_{\mathcal{E}}(K(y(-)), e)$*. This is the* realization-nerve adjunction.

Parts (1)–(4) of this theorem are the workhorses of the entire text: (1) turns natural transformations $y(X) \to F$ *into points* $F(X)$ *and vice versa; (2) says that probing* $\mathcal{C}$ *by its Hom-functors loses no information; (4) is ubiquitous in extending a statement from representables to all presheaves; (5)–(6) provide the adjoint machinery underlying sheafification and colimit-preserving constructions.*

Proof. (1) Yoneda bijection. The fundamental natural-transformation/element correspondence. Define $\Phi\colon \mathrm{Hom}_{\mathsf{PShv}(\mathcal{C})}(y(X), F) \to F(X)$ by $\Phi(\alpha) = \alpha_X(\mathrm{id}_X)$.

For the inverse, given $x \in F(X)$, define $\Psi(x)\colon y(X) \to F$ as follows: for each $Y \in \mathcal{C}$,

$$\Psi(x)_Y\colon y(X)(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X) \to F(Y), \quad (f : Y \to X) \mapsto F(f)(x).$$

To verify $\Psi(x)$ is natural: for $g\colon Z \to Y$, we need

$$\begin{array}{ccc} \mathrm{Hom}(Y,X) & \xrightarrow{\Psi(x)_Y} & F(Y) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathrm{Hom}(Z,X) & \xrightarrow[\Psi(x)_Z]{} & F(Z) \end{array}$$

to commute. For $f \in \mathrm{Hom}(Y, X)$: $(F(g) \circ \Psi(x)_Y)(f) = F(g)(F(f)(x)) = F(f \circ g)(x) = \Psi(x)_Z(f \circ g) = (\Psi(x)_Z \circ g^*)(f)$.

Check $\Phi \circ \Psi = \mathrm{id}$: $\Phi(\Psi(x)) = \Psi(x)_X(\mathrm{id}_X) = F(\mathrm{id}_X)(x) = x$.

Check $\Psi \circ \Phi = \mathrm{id}$: For $\alpha\colon y(X) \to F$, we have $\Psi(\Phi(\alpha)) = \Psi(\alpha_X(\mathrm{id}_X))$. For any Y and $f : Y \to X$:

$$\Psi(\alpha_X(\mathrm{id}_X))_Y(f) = F(f)(\alpha_X(\mathrm{id}_X)) = \alpha_Y(y(X)(f)(\mathrm{id}_X)) = \alpha_Y(f),$$

where the second equality uses naturality of α.

(2) Full faithfulness. Yoneda is an embedding. Apply (1) with $F = y(X')$ to get $\mathrm{Hom}(y(X), y(X')) \simeq y(X')(X) = \mathrm{Hom}(X, X')$.

(3) Preservation of limits. Yoneda commutes with limits in $\mathcal{C}$*.* Limits in $\mathsf{PShv}(\mathcal{C})$ are computed pointwise. If $\lim_i X_i$ exists in $\mathcal{C}$, then for any Y:

$$y(\lim_i X_i)(Y) = \mathrm{Hom}(Y, \lim_i X_i) \simeq \lim_i \mathrm{Hom}(Y, X_i) = \lim_i y(X_i)(Y).$$

Thus $y(\lim_i X_i) \simeq \lim_i y(X_i)$ in $\mathsf{PShv}(\mathcal{C})$.

(4) Density. Every presheaf is a canonical colimit of representables — the slogan "F is determined by its values on $\mathcal{C}$*" made formal.* Recall that $\mathrm{El}(F) = \int^{\mathcal{C}} F$ is contravariant: a morphism $(C, x) \to (C', x')$ is $g\colon C \to C'$ with $F(g)(x') = x$. The colimit in question is

$$\mathrm{colim}_{(C,x)\in\mathrm{El}(F)}\, y(C),$$

which we evaluate pointwise. For any $Y \in C$,

$$\left(\operatorname{colim}_{(C,x)} y(C)\right)(Y) = \operatorname{colim}_{(C,x)} \operatorname{Hom}(Y, C).$$

A point in this colimit is represented by a triple (C, x, f) with $x \in F(C)$ and $f : Y \to C$. Two triples (C, x, f) and (C', x', f') are identified when there exists $g : C \to C'$ in $\mathrm{El}(F)$ — that is, $g : C \to C'$ with $F(g)(x') = x$ — such that $g \circ f = f'$.

We construct a bijection with $F(Y)$. Send $(C, x, f) \mapsto F(f)(x) \in F(Y)$. This is well-defined on equivalence classes: if $(C, x, f) \sim (C', x', g \circ f)$ via g as above, then

$$F(g \circ f)(x') = F(f)(F(g)(x')) = F(f)(x).$$

Surjectivity is immediate, since $y \in F(Y)$ is the image of (Y, y, id_Y). For injectivity, observe that any triple (C, x, f) is equivalent to $(Y, F(f)(x), \mathrm{id}_Y)$: the map $f : Y \to C$ provides a morphism $(Y, F(f)(x)) \to (C, x)$ in $\mathrm{El}(F)$ (since $F(f)(x) = F(f)(x)$), and $f \circ \mathrm{id}_Y = f$.

(5) and (6) Universal property and adjoints. Colimit-preserving extensions of $F : C \to \mathcal{E}$ to $\mathsf{PShv}(C)$ are determined by the values on C, computed via left Kan extension; the right adjoint is then given by Hom-sets. These follow from general Kan-extension and adjoint-functor theorem arguments. □

1.2. Affine Schemes

The Yoneda embedding $y : \mathsf{CAlg} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ has a special name on its opposite side: the Spec *functor*. Its image consists of the *affine schemes*, which play the role in algebraic geometry that rings play in commutative algebra.

1.2.1. Polynomial Equations

Definition 1.2.1 (System of Polynomial Equations)**.** Let k be a ring and let I and J be sets. A *system of J polynomial equations in I variables over k* is a J-tuple $\Sigma = (f_j)_{j \in J}$ in the polynomial ring $k[x_i \mid i \in I]$. We denote by (Σ) the ideal in $k[x_i \mid i \in I]$ generated by $(f_j)_{j \in J}$ and by $k[\Sigma]$ the k-algebra $k[x_i \mid i \in I]/(\Sigma)$.

***Remark* 1.2.2** (Systems of Equations and Presentations)**.** Every k-algebra R admits an isomorphism $R \simeq k[\Sigma]$ for some system of polynomial equations Σ: choose a generating set $(x_i)_{i \in I}$ of R as a k-algebra, and let Σ be any generating set for the kernel $k[x_i] \twoheadrightarrow R$. Thus a choice of isomorphism $R \simeq k[\Sigma]$ is precisely a *presentation* of R by generators and relations. When I is finite, R is of finite type over k; when Σ is also finite, R is of finite presentation (Definition 1.4.1).

Given a system of polynomial equations over k, we want to consider its solutions in an arbitrary k-algebra R. To that end, recall the *evaluation map*

$$k[x_i \mid i \in I] \times R^I \to R, \quad (f, a) \mapsto f(a),$$

defined as follows: for each tuple $a \in R^I$, the map $f \mapsto f(a)$ is the unique k-algebra homomorphism $k[x_i \mid i \in I] \to R$ sending x_i to a_i.

Definition 1.2.3 (Vanishing Locus)**.** Let $F \subset k[x_i \mid i \in I]$ be a subset. The *vanishing locus of F in $\mathbb{A}^I_k$* is the subfunctor $V(F) \subset \mathbb{A}^I_k$ given by

$$V(F)(R) = \{a \in R^I \mid f(a) = 0 \text{ for all } f \in F\} \subset R^I.$$

This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $R^I \to S^I$ sends $V(F)(R)$ to $V(F)(S)$.

***Remark* 1.2.4.** The vanishing locus of F manifestly depends only on the ideal it generates: if $(F) = (F')$ then $V(F) = V(F')$. Remarkably, the converse also holds — a functorial form of Hilbert's Nullstellensatz that we prove below (Corollary 1.2.11).

Definition 1.2.5 (Solution Functor)**.** Let $\Sigma = (f_j)_{j\in J}$ be a system of J polynomial equations in I variables over k. Its *solution functor* $\mathrm{Sol}_\Sigma \colon \mathsf{CAlg}_k \to \mathsf{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{A}^I_k$:

$$\mathrm{Sol}_\Sigma = V(\{f_j \mid j \in J\}) \subset \mathbb{A}^I_k.$$

By the universal property of polynomial rings, there is a one-to-one correspondence between systems of J polynomial equations in I variables and k-algebra maps

$$k[x_j \mid j \in J] \to k[x_i \mid i \in I].$$

By the Yoneda lemma, these are in turn equivalent to natural transformations

$$\mathbb{A}^I_k \to \mathbb{A}^J_k.$$

Unraveling these equivalences, the map $\mathbb{A}^I_k \to \mathbb{A}^J_k$ corresponding to a system $\Sigma = (f_j)_{j\in J}$ is given on a k-algebra R by

$$R^I \to R^J, \quad a \mapsto (f_j(a))_{j\in J}.$$

By definition, the solution functor Sol_Σ is the kernel of this map, i.e., there is a pullback square

$$\begin{array}{ccc} \mathrm{Sol}_\Sigma & \longrightarrow & \mathbb{A}^I_k \\ \big\downarrow & \lrcorner & \big\downarrow \\ 0 & \longrightarrow & \mathbb{A}^J_k \end{array} \tag{1.1}$$

where 0 is the subfunctor of $\mathbb{A}^J_k$ given by $0(R) = \{0\} \subset R^J$.

Definition 1.2.6 (Affine Scheme)**.** A functor $\mathsf{CAlg}_k \to \mathsf{Set}$ is called an *affine k-scheme* if it is isomorphic to Sol_Σ for some system of polynomial equations Σ over k. We denote by $\mathsf{Aff}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ the full subcategory spanned by the affine k-schemes. An *affine scheme* is an affine $\mathbb{Z}$-scheme.

Example 1.2.7. The affine I-space $\mathbb{A}^I_k$ is an affine k-scheme, as it is the solution functor of the empty system of equations in I variables.

Lemma 1.2.8. *Let Σ be a system of polynomial equations over k. Then the solution functor Sol_Σ is represented by the k-algebra $k[\Sigma]$, i.e., there is an isomorphism*

$$\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma]) \colon \mathsf{CAlg}_k \to \mathsf{Set}.$$

Proof. By definition, $k[\Sigma] = k[x_i \mid i \in I]/(\Sigma)$. For any k-algebra R, a k-algebra map $\varphi \colon k[\Sigma] \to R$ corresponds to a k-algebra map $\tilde{\varphi} \colon k[x_i \mid i \in I] \to R$ that sends each $f_j \in \Sigma$ to zero.

By the universal property of the polynomial ring, $\tilde{\varphi}$ is determined by the tuple $a = (\tilde{\varphi}(x_i))_{i\in I} \in R^I$. The condition that $\tilde{\varphi}(f_j) = 0$ for all $j \in J$ is exactly the condition that $f_j(a) = 0$ for all $j \in J$, i.e., $a \in \mathrm{Sol}_\Sigma(R)$.

Thus we have a bijection

$$\mathrm{Hom}_{\mathsf{CAlg}_k}(k[\Sigma], R) \xrightarrow{\sim} \mathrm{Sol}_\Sigma(R), \quad \varphi \mapsto (\varphi(x_i))_{i\in I},$$

which is natural in R. Therefore $\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])$. □

Theorem 1.2.9 (Characterization of Affine Schemes). *Let k be a ring. The following conditions are equivalent for an algebraic k-functor $X\colon \mathsf{CAlg}_k \to \mathsf{Set}$:*

(i) X is an affine k-scheme.

(ii) X is representable, i.e., isomorphic to $\mathrm{Spec}(A)$ for some k-algebra A.

(iii) X preserves limits and is accessible.

Proof. *(i) ⇒ (ii):* If X is an affine k-scheme, then $X \simeq \mathrm{Sol}_\Sigma$ for some system of polynomial equations Σ. By Lemma 1.2.8, $X \simeq \mathrm{Spec}(k[\Sigma])$, so X is representable.

(ii) ⇒ (iii): If $X \simeq \mathrm{Spec}(A)$ for some k-algebra A, then $X(R) = \mathrm{Hom}_{\mathsf{CAlg}_k}(A, R)$. Since $\mathrm{Hom}_{\mathsf{CAlg}_k}(A, -)$ is a representable functor, it preserves all limits (by the Yoneda lemma).

For accessibility: A functor $F\colon \mathsf{CAlg}_k \to \mathsf{Set}$ is accessible if there exists a regular cardinal κ such that F preserves κ-filtered colimits. For a representable functor $\mathrm{Hom}(A, -)$, we can choose κ to be larger than the cardinality of A. Then $\mathrm{Hom}(A, -)$ preserves κ-filtered colimits because such colimits are computed pointwise, and elements of A are finitely presented.

(iii) ⇒ (i): Suppose $X\colon \mathsf{CAlg}_k \to \mathsf{Set}$ preserves limits and is accessible. We invoke the *representability theorem* for Set-valued functors (a special case of the adjoint functor theorem): a functor from a presentable category to Set is representable if and only if it preserves small limits and is accessible. Since CAlg_k is presentable, the hypotheses give a k-algebra A together with a natural isomorphism $X \simeq \mathrm{Hom}_{\mathsf{CAlg}_k}(A, -) = \mathrm{Spec}(A)$. By Remark 1.2.2 we may write $A \simeq k[\Sigma]$ for some system of polynomial equations Σ, so $X \simeq \mathrm{Sol}_\Sigma$ by Lemma 1.2.8.

Concretely, the algebra A is constructed as follows. Accessibility lets us write $X = \mathrm{colim}_\alpha \mathrm{Spec}(A_\alpha)$ as a small colimit of representables *in* $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$; this colimit is computed pointwise and is generally not itself representable. However, X being limit-preserving forces it to admit a left adjoint $L\colon \mathsf{Set} \to \mathsf{CAlg}_k$, and we may take $A = L(\{*\})$. The Yoneda lemma then gives

$$X(R) = \mathrm{Hom}_{\mathsf{Set}}(\{*\}, X(R)) = \mathrm{Hom}_{\mathsf{CAlg}_k}(L(\{*\}), R) = \mathrm{Hom}_{\mathsf{CAlg}_k}(A, R). \qquad \square$$

Corollary 1.2.10. *The Yoneda embedding of $\mathsf{CAlg}_k^{\mathrm{op}}$ induces an equivalence of categories*

$$\mathrm{Spec}\colon \mathsf{CAlg}_k^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set}).$$

Under this equivalence, the affine k-scheme Sol_Σ corresponds to the k-algebra $k[\Sigma]$.

Proof. This follows immediately from Theorem 1.2.9. The Yoneda embedding is fully faithful and essentially surjective onto the representable functors, which by (i) ⇔ (ii) are exactly the affine k-schemes. □

Under the equivalence of Corollary 1.2.10, the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ of affine k-schemes corresponds to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[\Sigma]$. This implies the following result:

Corollary 1.2.11 (Functorial Nullstellensatz). *Sending a subset $F \subset k[x_i \mid i \in I]$ to its vanishing locus $V(F) \subset \mathbb{A}^I_k$ induces an order-reversing bijection*

$$V\colon \{\textit{ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{A}^I_k\}.$$

Proof. By Remark 1.2.4, $V(F)$ depends only on the ideal (F), so we can view V as a map from ideals.

Injectivity: Suppose $V(F) = V(F')$ for ideals (F) and (F'). Under the equivalence $\mathrm{Spec}\colon \mathsf{CAlg}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}$, the closed embedding $V(F) \hookrightarrow \mathbb{A}^I_k$ corresponds to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[x_i \mid i \in I]/(F)$, and similarly for $V(F')$. Since $V(F) = V(F')$ as subfunctors, the two quotient maps are isomorphic over $k[x_i \mid i \in I]$, which forces $(F) = (F')$.

Surjectivity: Let $Z \subset \mathbb{A}^I_k$ be a vanishing locus. By definition, there exists a system of polynomial equations Σ such that $Z = \mathrm{Sol}_\Sigma = V(\Sigma)$. Thus $Z = V((\Sigma))$ for the ideal (Σ).

The order-reversing property is clear: if $(F) \subset (F')$, then $V(F') \subset V(F)$. □

Example 1.2.12. Consider the following systems of polynomial equations over $\mathbb{R}$ in one variable:

$$\Sigma_1 = (x^2+1), \quad \Sigma_2 = ((x^2+1)^2), \quad \Sigma_3 = (x^2+x+1), \quad \Sigma_4 = (x^4+1).$$

Then:

$$\begin{aligned} &\mathrm{Sol}_{\Sigma_1}(\mathbb{R}) = \varnothing, && \mathrm{Sol}_{\Sigma_2}(\mathbb{R}) = \varnothing, && \mathrm{Sol}_{\Sigma_3}(\mathbb{R}) = \varnothing, && \mathrm{Sol}_{\Sigma_4}(\mathbb{R}) = \varnothing, \\ &\mathrm{Sol}_{\Sigma_1}(\mathbb{C}) = \{\pm i\}, && \mathrm{Sol}_{\Sigma_2}(\mathbb{C}) = \{\pm i\}, && \mathrm{Sol}_{\Sigma_3}(\mathbb{C}) = \{\zeta_3, \bar{\zeta}_3\}, && \mathrm{Sol}_{\Sigma_4}(\mathbb{C}) = \{\pm\zeta_8, \pm\bar{\zeta}_8\}, \end{aligned}$$

where $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$. All four equations have the same solutions in $\mathbb{R}$. However, as the four ideals (Σ_i) in $\mathbb{R}[x]$ are pairwise distinct, they define four different subfunctors of $\mathbb{A}^1_\mathbb{R}$ by Corollary 1.2.11. The solutions in $\mathbb{C}$ distinguish them, except for Sol_{Σ_1} and Sol_{Σ_2}. To see that $\mathrm{Sol}_{\Sigma_1} \neq \mathrm{Sol}_{\Sigma_2}$ as subfunctors of $\mathbb{A}^1_\mathbb{R}$, we can compute the solutions in the $\mathbb{R}$-algebra $\mathbb{C}[\varepsilon]$ of dual complex numbers (where $\varepsilon^2 = 0$):

$$\mathrm{Sol}_{\Sigma_1}(\mathbb{C}[\varepsilon]) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_2}(\mathbb{C}[\varepsilon]) = \{\pm i + a\varepsilon \mid a \in \mathbb{C}\}.$$

On the other hand, the associated $\mathbb{R}$-algebras are

$$\mathbb{R}[\Sigma_1] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_2] \simeq \mathbb{C}[\varepsilon], \quad \mathbb{R}[\Sigma_3] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_4] \simeq \mathbb{C} \times \mathbb{C}.$$

By Lemma 1.2.8, Sol_{Σ_1} and Sol_{Σ_3} are both isomorphic to $\mathrm{Spec}(\mathbb{C})$. The different ideals (Σ_1) and (Σ_3) correspond to two different embeddings of the affine $\mathbb{R}$-scheme $\mathrm{Spec}(\mathbb{C})$ into $\mathbb{A}^1_\mathbb{R}$, and the systems Σ_1 and Σ_3 themselves are two different presentations of the $\mathbb{R}$-algebra $\mathbb{C}$.

***Remark* 1.2.13.** In summary, given a system of polynomial equations Σ over k, we have the following relations between Σ and Sol_Σ:

(i) The data of the pullback square (1.1) is equivalent to the data of Σ itself.

(ii) The data of the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ is equivalent to the data of the ideal (Σ) in the polynomial ring $k[x_i \mid i \in I]$.

(iii) The data of the affine k-scheme Sol_Σ alone is equivalent to the data of the k-algebra $k[\Sigma]$.

This can be compared with the following types of data in differential geometry:

(i) A smooth manifold M given as the vanishing locus of a smooth function $\mathbb{R}^n \to \mathbb{R}^m$.

(ii) A smooth manifold M given as a closed submanifold of $\mathbb{R}^n$.

(iii) A smooth manifold M.

Smooth manifolds are the basic objects of interest in differential geometry. Embedding a manifold M into a Euclidean space or realizing it as the vanishing locus of a function are often useful ways to understand M, but we do not consider this additional data to be part of the manifold M itself. The situation in algebraic geometry is entirely similar: the basic objects of interest are affine schemes. Any affine scheme X can be embedded into an affine space ($X \hookrightarrow \mathbb{A}^I$) or realized as the solution functor of a system of polynomial equations ($X \simeq \mathrm{Sol}_\Sigma$), but this data is not part of the affine scheme X itself.

1.2.2. Examples

Example 1.2.14 (Terminal Scheme $*$). The algebraic functor $\mathsf{CAlg} \to \mathsf{Set}, R \mapsto \{*\}$ is isomorphic to $\mathrm{Spec}(\mathbb{Z})$, hence is an affine scheme. In fact, it is the terminal object of $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$.

Example 1.2.15 (Initial Scheme $\varnothing$). This example shows that the Yoneda embedding $\mathrm{Spec}\colon \mathsf{Aff} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ does not preserve colimits.

- In Aff, the initial object is the functor

$$\mathsf{CAlg} \to \mathsf{Set}, \quad R \mapsto \begin{cases} \varnothing, & R \neq 0 \\ \{*\}, & R = 0, \end{cases}$$

  which is isomorphic to $\mathrm{Spec}(0)$.

- However, in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$, the initial object is the constant functor

$$\mathsf{CAlg} \to \mathsf{Set}, \quad R \mapsto \varnothing,$$

  which is not representable, hence not an affine scheme.

Example 1.2.16 (Idempotent Classifier Idem). Let $\mathrm{Idem}\colon \mathsf{CAlg} \to \mathsf{Set}$ be the functor sending R to its set of idempotent elements $\mathrm{Idem}(R) = \{e \in R \mid e^2 = e\}$. There is a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathsf{CAlg}}(\mathbb{Z} \times \mathbb{Z}, R) &\xrightarrow{\sim} \mathrm{Idem}(R) \\ \varphi &\mapsto \varphi(1, 0). \end{aligned}$$

Since this is natural in R, we have $\mathrm{Idem} \simeq \mathrm{Spec}(\mathbb{Z} \times \mathbb{Z})$, hence affine.

Example 1.2.17 (Multiplicative Group $\mathbb{G}_m$). Let $\mathbb{G}_m\colon \mathsf{CAlg} \to \mathsf{Ab}$ be the functor sending a ring R to its group of units $R^\times$. We claim this functor is represented by the *monoid ring* $\mathbb{Z}[\mathbb{Z}]$.

First, recall that the monoid ring functor $\mathbb{Z}[-]\colon \mathsf{CMon} \to \mathsf{CAlg}$ is left adjoint to the underlying multiplicative monoid functor $\mathsf{CAlg} \to \mathsf{CMon}$, $R \mapsto (R, \cdot)$. For any ring R and commutative monoid M:

$$\mathrm{Hom}_{\mathsf{CAlg}}(\mathbb{Z}[M], R) \simeq \mathrm{Hom}_{\mathsf{CMon}}(M, (R, \cdot)),$$

where $(R, \cdot)$ denotes the multiplicative monoid structure on R.

Taking $M = \mathbb{Z}$ (the additive group, viewed as a monoid): since $\mathbb{Z}$ is a group, any monoid homomorphism $\mathbb{Z} \to (R, \cdot)$ factors through $R^\times$. Moreover, $\mathbb{Z}$ is the free group on one generator, so group homomorphisms $\phi\colon \mathbb{Z} \to R^\times$ are determined by $\phi(1) \in R^\times$.

Therefore:

$$\mathbb{G}_m(R) = R^\times \simeq \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, R^\times) \simeq \mathrm{Hom}_{\mathsf{CAlg}}(\mathbb{Z}[\mathbb{Z}], R).$$

This proves $\mathbb{G}_m \simeq \mathrm{Spec}(\mathbb{Z}[\mathbb{Z}])$. Concretely, $\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t, t^{-1}]$ (Laurent polynomials).

***Remark* 1.2.18** ($\mathbb{G}_m$ as Tate Circle). In higher category theory, the classifying space of $\mathbb{Z}$ is homotopy equivalent to the circle: $B\mathbb{Z} \simeq S^1$. For a presentable monoidal category $\mathcal{C}$, we have

$$\mathsf{Fun}(B\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{Z}}(\mathcal{C}) \simeq \mathrm{Mod}_{\mathbb{1}[\mathbb{Z}]}(\mathcal{C}),$$

where $\mathbb{1}[\mathbb{Z}]$ is the monoid ring object in $\mathcal{C}$.

Since $\mathbb{Z}[\mathbb{Z}]$ is the coordinate ring $\mathcal{O}(\mathbb{G}_m)$, this suggests a deep duality between the topological circle S^1 (corresponding to $\mathbb{Z}$) and the algebraic $\mathbb{G}_m$ in algebraic geometry. Therefore, $\mathbb{G}_m$ is often called the *Tate circle*.

In motivic homotopy theory $\mathsf{H}(S)$, we have two types of "circles":

- *Simplicial circle:* $S^{1,0} := S^1$ (constant sheaf).
- *Tate circle:* $S^{1,1} := \mathbb{G}_m$.

The motivic (i, j)-sphere is defined as:

$$S^{i,j} := (S^1)^{\wedge(i-j)} \wedge \mathbb{G}_m^{\wedge j}.$$

The projective line satisfies $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m \simeq S^{2,1}$.

The stable motivic homotopy category $\mathsf{SH}(S)$ is constructed by formally inverting $(\mathbb{P}^1, \infty) \simeq S^{2,1}$ in the pointed motivic homotopy category $\mathsf{H}(S)_*$.

Example 1.2.19 (Additive Group $\mathbb{G}_a$). The functor $\mathbb{G}_a \colon \mathsf{CAlg} \to \mathsf{Ab}$ sending R to $(R, +)$ is called the *additive group*. The composition

$$\mathsf{CAlg} \xrightarrow{\mathbb{G}_a} \mathsf{Ab} \xrightarrow{\text{forget}} \mathsf{Set}$$

is simply the forgetful functor, a.k.a. the affine line $\mathbb{A}^1$. Hence $\mathbb{G}_a$ is an affine group scheme.

Example 1.2.20 (The Matrix Ring Mat_n). Let $n \geq 0$ and let $\mathrm{Mat}_n \colon \mathsf{CAlg} \to \mathsf{Alg}$ be the functor sending a ring R to the associative ring of $n \times n$ matrices over R. This is an associative ring object in the category of affine schemes. Indeed, since an $n \times n$ matrix over R is simply a family of n^2 elements of R, the composition of Mat_n with the forgetful functor $\mathsf{Alg} \to \mathsf{Set}$ is isomorphic to the affine space $\mathbb{A}^{n^2}$. Thus, Mat_n is representable by the polynomial ring $\mathbb{Z}[x_{ij} \mid 1 \leq i, j \leq n]$.

Example 1.2.21 (The General Linear Group GL_n). Let $n \geq 0$ and let $\mathrm{GL}_n \colon \mathsf{CAlg} \to \mathsf{Grp}$ be the functor sending R to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices. Then GL_n is an affine group scheme. Let $A = \mathbb{Z}[x_{ij} \mid 1 \leq i, j \leq n]$ be the ring representing Mat_n, and let $X = (x_{ij})_{i,j}$ be the universal $n \times n$ matrix. A matrix $M \in \mathrm{Mat}_n(R)$ is invertible if and only if its determinant $\det(M) \in R$ is a unit. Hence, for any ring R, there is a natural isomorphism:

$$\mathrm{Hom}_{\mathsf{CAlg}}(A_{\det(X)}, R) \xrightarrow{\sim} \mathrm{GL}_n(R), \quad \varphi \mapsto (\varphi(x_{ij}))_{i,j}$$

where $A_{\det(X)}$ is the localization of A at the element $\det(X)$. Therefore, $\mathrm{GL}_n \simeq \mathrm{Spec}(A_{\det(X)})$.

Example 1.2.22 (The Special Linear Group SL_n). Let $n \geq 0$ and let $\mathrm{SL}_n \colon \mathsf{CAlg} \to \mathsf{Grp}$ be the functor sending R to the group $\mathrm{SL}_n(R)$ of $n \times n$ matrices with determinant 1. Using the universal matrix X as defined in Example 1.2.21, we have:

$$\mathrm{Hom}_{\mathsf{CAlg}}(A/(\det(X) - 1), R) \xrightarrow{\sim} \mathrm{SL}_n(R), \quad \varphi \mapsto (\varphi(x_{ij}))_{i,j}$$

Consequently, $\mathrm{SL}_n \simeq \mathrm{Spec}(A/(\det(X) - 1))$. The sequence of subfunctor inclusions $\mathrm{SL}_n \subset \mathrm{GL}_n \subset \mathrm{Mat}_n$ corresponds to the following sequence of ring maps:

$$A \longrightarrow A_{\det(X)} \longrightarrow A/(\det(X) - 1).$$

***Remark* 1.2.23** (Categorical Perspective of GL_n and SL_n). The group schemes GL_n and SL_n can be elegantly characterized as fiber products (pullbacks) in the category of algebraic functors. Specifically, GL_n is defined by the pullback square:

$$\begin{array}{ccc} \mathrm{GL}_n & \hookrightarrow & \mathrm{Mat}_n \\ \downarrow & \lrcorner & \downarrow \det \\ \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1 \end{array}$$

Since the Spec functor preserves limits (and hence fiber products), and all other objects in the diagram are affine, it follows that GL_n is affine.

Similarly, SL_n is characterized by the following pullback:

$$\begin{array}{ccc} \mathrm{SL}_n & \longrightarrow & \mathrm{Mat}_n \\ \downarrow & \lrcorner & \downarrow \det \\ * & \xrightarrow{1} & \mathbb{A}^1 \end{array}$$

where $* \simeq \mathrm{Spec}(\mathbb{Z})$ is the final object, representing the condition that the determinant is fixed to 1.

Example 1.2.24 (The Affine Space of a Module $\mathbb{A}(M)$). Let k be a ring and M be a k-module. Consider the functor $\mathbb{A}(M)\colon \mathsf{CAlg}_k \to \mathrm{Mod}_k$ defined by

$$\mathbb{A}(M)(R) = \{k\text{-linear maps } M \to R\} = (M \otimes_k R)^\vee.$$

Then $\mathbb{A}(M)$ is a k-module object in affine k-schemes. Indeed, if $\mathrm{Sym}_k(M)$ is the free k-algebra on M, there is a bijection

$$\mathrm{Hom}_{\mathsf{CAlg}_k}(\mathrm{Sym}_k(M), R) \xrightarrow{\sim} \mathbb{A}(M)(R), \quad \varphi \mapsto \varphi\,|_M,$$

so that $\mathbb{A}(M) \simeq \mathrm{Spec}(\mathrm{Sym}_k(M))$. The affine I-space is a special case of this construction $\mathbb{A}^I_k \simeq \mathbb{A}(k^{(I)})$.

***Remark* 1.2.25.** Let k be a ring and let M be a k-module. We can consider the "predual" of $\mathbb{A}(M)$: define $\mathbb{A}^\vee(M)\colon \mathsf{CAlg}_k \to \mathrm{Mod}_k$ by

$$\mathbb{A}^\vee(M)(R) = M \otimes_k R.$$

There is a canonical map $\mathbb{A}^\vee(M^\vee) \to \mathbb{A}(M)$, which is an isomorphism if and only if M is a vector space (Definition 2.1.4). Otherwise, $\mathbb{A}^\vee(M)$ does not preserve limits and hence is not an affine k-scheme (in fact, it is not even a scheme). For that reason, the functor $\mathbb{A}^\vee(M)$ is rarely used.

1.2.3. Base Change

Given a ring map $\varphi\colon k \to k'$, we can transform any system of polynomial equations Σ over k into a system $\varphi^*(\Sigma)$ over k' by applying φ to all the coefficients. More generally, many types of data over k can be transformed into data over k' using φ, a process known as *base change*, *change of coefficients*, or *extension of scalars*.

Other examples include:

- The functor $\varphi^*\colon \mathrm{Mod}_k \to \mathrm{Mod}_{k'}$ sending a k-module M to the k'-module $M \otimes_k k'$
- The functor $\varphi^*\colon \mathsf{CAlg}_k \to \mathsf{CAlg}_{k'}$ sending a k-algebra A to the k'-algebra $A \otimes_k k'$

Unraveling these constructions, we see that there is a canonical isomorphism of k'-algebras:

$$k'[\varphi^*(\Sigma)] \simeq \varphi^*(k[\Sigma]).$$

In this section, we investigate the related process of transforming an algebraic k-functor into an algebraic k'-functor.

Theorem 1.2.26 (Functoriality of Presheaves). *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let*

$$u^*\colon \mathsf{PShv}(\mathcal{D}) \to \mathsf{PShv}(\mathcal{C}), \quad F \mapsto F \circ u,$$

be the "restriction along u" functor.

(i) *u^* admits a left adjoint $u_\sharp$ and a right adjoint u_* given by:*

$$u_\sharp(F)(d) = \operatorname{colim}\left((C \times_{\mathcal{D}} \mathcal{D}_{d/})^{\mathrm{op}} \to C^{\mathrm{op}} \xrightarrow{F} \mathsf{Set}\right) = \operatorname{colim}_{d \to u(c)} F(c),$$
$$u_*(F)(d) = \lim\left((C \times_{\mathcal{D}} \mathcal{D}_{/d})^{\mathrm{op}} \to C^{\mathrm{op}} \xrightarrow{F} \mathsf{Set}\right) = \lim_{u(c) \to d} F(c),$$

provided these colimits and limits exist (e.g., if C is small).

(ii) *The functor $u_\sharp$ extends u: there is a canonical isomorphism*

$$u_\sharp \circ y_C \simeq y_{\mathcal{D}} \circ u.$$

(iii) *If u is fully faithful, then $u_\sharp$ and u_* are fully faithful.*

(iv) *If u is a localization, then u^* is fully faithful.*

(v) *If the functor u has a left adjoint u_L (resp. a right adjoint u_R), then there is a canonical isomorphism $u_\sharp \simeq u_L^*$ (resp. $u_* \simeq u_R^*$).*

Proof. (i) The adjunctions $u_\sharp \dashv u^*$ and $u^* \dashv u_*$ follow from the general theory of Kan extensions. For any $F \in \mathsf{PShv}(C)$ and $G \in \mathsf{PShv}(\mathcal{D})$, we have:

$$\operatorname{Hom}_{\mathsf{PShv}(\mathcal{D})}(u_\sharp(F), G) \simeq \operatorname{Hom}_{\mathsf{PShv}(C)}(F, u^*(G))$$
$$\operatorname{Hom}_{\mathsf{PShv}(C)}(u^*(G), F) \simeq \operatorname{Hom}_{\mathsf{PShv}(\mathcal{D})}(G, u_*(F))$$

The explicit formulas follow from the pointwise computation of left Kan extensions. For $d \in \mathcal{D}$:

$$u_\sharp(F)(d) = \operatorname{colim}_{(c,\alpha\colon d \to u(c))} F(c)$$

where the colimit is taken over the comma category $C \times_{\mathcal{D}} \mathcal{D}_{d/}$. Similarly for u_*.

(ii) By the universal property of the Yoneda embedding, for any $c \in C$:

$$u_\sharp(y_C(c))(d) = \operatorname{colim}_{d \to u(c')} \operatorname{Hom}_C(c', c) \simeq \operatorname{Hom}_{\mathcal{D}}(d, u(c)) = y_{\mathcal{D}}(u(c))(d).$$

(iii) If u is fully faithful, then the comma categories $C \times_{\mathcal{D}} \mathcal{D}_{d/}$ and $C \times_{\mathcal{D}} \mathcal{D}_{/d}$ have initial and terminal objects respectively when non-empty. This implies that $u_\sharp$ and u_* are fully faithful.

(iv) If u is a localization, then u^* has a fully faithful left adjoint by definition.

(v) If $u_L \dashv u$, then by uniqueness of adjoints, we have $u_\sharp \simeq u_L^*$. Similarly for right adjoints.

□

***Remark* 1.2.27.** (i) Given $F\colon C^{\mathrm{op}} \to \mathsf{Set}$, the presheaves $u_\sharp(F)$ and $u_*(F)$ are instances of *Kan extensions*: $u_\sharp(F)$ is the left Kan extension of F along u^{op}, and $u_*(F)$ is the right Kan extension of F along u^{op}.

(ii) By the universal property of y_C, the functor $u_\sharp$ is the unique colimit-preserving extension of u (up to unique isomorphism). There is no analogous characterization of u_*.

Corollary 1.2.28 (Base Change for Slice Categories). *Let C be a category, let $Y \to X$ be a morphism in C, and let $u\colon C_{/Y} \to C_{/X}$ be the forgetful functor.*

(i) *The restriction functor $u^*\colon \mathsf{PShv}(C_{/X}) \to \mathsf{PShv}(C_{/Y})$ has a left adjoint $u_\sharp$ given by*

$$u_\sharp(F)(U \to X) = \coprod_{\substack{\text{maps } U \to Y \\ \text{over } X}} F(U \to Y).$$

(ii) If pullbacks along $Y \to X$ exist in C, then u^* also has a right adjoint u_*, given by

$$u_*(F)(U \to X) = F(U \times_X Y \to Y).$$

Proof. This is Theorem 1.2.26 with the comma categories computed explicitly for slice categories. □

For a ring k we have $\mathsf{CAlg}_k \simeq \mathsf{CAlg}_{k/}$, hence $\mathsf{CAlg}_k^{\mathrm{op}} \simeq (\mathsf{CAlg}^{\mathrm{op}})_{/k}$. Specializing Corollary 1.2.28 to $C = \mathsf{CAlg}^{\mathrm{op}}$ yields:

Corollary 1.2.29 (Base Change for Algebras)**.** *Let $\varphi\colon k \to k'$ be a ring map. Then there is a triple of adjoint functors*

$$\mathsf{Fun}(\mathsf{CAlg}_{k'},\mathsf{Set}) \;\begin{matrix}\xrightarrow{\;\varphi_\sharp\;}\\ \xleftarrow{\;\varphi^*\;}\\ \xrightarrow{\;\varphi_*\;}\end{matrix}\; \mathsf{Fun}(\mathsf{CAlg}_k,\mathsf{Set})$$

described as follows:

- *φ^* is precomposition with the forgetful functor $\mathsf{CAlg}_{k'} \to \mathsf{CAlg}_k$; it is the unique colimit-preserving extension of $\varphi^*\colon \mathsf{CAlg}_k^{\mathrm{op}} \to \mathsf{CAlg}_{k'}^{\mathrm{op}}$.*
- *φ_* is precomposition with $\varphi^*\colon \mathsf{CAlg}_k \to \mathsf{CAlg}_{k'}$.*
- *$\varphi_\sharp$ is given by*

$$\varphi_\sharp(X)(A) = \coprod_{\substack{k'\text{-algebra}\\ \text{structures on } A}} X(A);$$

it is the unique colimit-preserving extension of the forgetful functor $\mathsf{CAlg}_{k'}^{\mathrm{op}} \to \mathsf{CAlg}_k^{\mathrm{op}}$.

Proof. Apply Corollary 1.2.28 to the ring map $\varphi\colon k \to k'$ viewed as an object $k' \in \mathsf{CAlg}_{k/}$. For the formula for $\varphi_\sharp$: a k'-algebra structure on A is the same data as a map $U \to k'$ in CAlg_k, where U is the underlying k-algebra of A. The coproduct ranges over all such structures. □

Definition 1.2.30 (Base Change and Weil Restriction)**.** Let $\varphi\colon k \to k'$ be a ring map, giving rise to the adjoint triple of Corollary 1.2.29.

(i) The functor φ^* is called *base change* or *extension of scalars* along φ and is also denoted by $X \mapsto X_{k'}$.

(ii) The functor φ_* is called *Weil restriction* or *restriction of scalars* along φ and is also denoted by R_φ or $R_{k'/k}$.

***Remark* 1.2.31** (Preservation of Affine Schemes)**.** Corollary 1.2.29 implies in particular that φ^* and $\varphi_\sharp$ both preserve affine schemes:

(i) For a k-algebra A, $\varphi^*(\mathrm{Spec}(A)) \simeq \mathrm{Spec}(A \otimes_k k')$ in $\mathsf{Fun}(\mathsf{CAlg}_{k'},\mathsf{Set})$. Consequently, for any system of polynomial equations Σ over k, we have $\varphi^*(\mathrm{Sol}_\Sigma) \simeq \mathrm{Sol}_{\varphi^*(\Sigma)}$.

(ii) For a k'-algebra B, $\varphi_\sharp(\mathrm{Spec}(B)) \simeq \mathrm{Spec}(B)$ in $\mathsf{Fun}(\mathsf{CAlg}_k,\mathsf{Set})$.

The third functor φ_*, however, does not always preserve affine schemes; the next example illustrates a case where it does.

Example 1.2.32 (Weil Restriction of $\mathbb{G}_m$). Consider $\mathbb{G}_m\colon \mathsf{CAlg} \to \mathsf{Set}$, $R \mapsto R^\times$. Its base change $\mathbb{G}_{m,\mathbb{C}}\colon \mathsf{CAlg}_\mathbb{C} \to \mathsf{Set}$ is the restriction of $\mathbb{G}_m$ along the forgetful functor $\mathsf{CAlg}_\mathbb{C} \to \mathsf{CAlg}$. The Weil restriction $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})\colon \mathsf{CAlg}_\mathbb{R} \to \mathsf{Set}$ is then

$$R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})(A) = \mathbb{G}_{m,\mathbb{C}}(A \otimes_\mathbb{R} \mathbb{C}) = (A \otimes_\mathbb{R} \mathbb{C})^\times.$$

This functor is an affine $\mathbb{R}$-scheme: for any $\mathbb{R}$-algebra A, there is a natural bijection

$$\mathrm{Hom}_{\mathsf{CAlg}_\mathbb{R}}\big(\mathbb{R}[x,y,z,w]/(xz-yw-1,\ yz+xw),\ A\big) \xrightarrow{\sim} (A \otimes_\mathbb{R} \mathbb{C})^\times, \quad \varphi \mapsto \varphi(x) + i\varphi(y).$$

The two relations encode the requirement that $\varphi(x)+i\varphi(y)$ have an inverse $\varphi(z)+i\varphi(w)$. More generally, for any finite field extension $k \subset k'$, the Weil restriction $R_{k'/k}$ sends affine k'-schemes to affine k-schemes.

Verification of the bijection. Given $\varphi\colon \mathbb{R}[x,y,z,w]/(xz-yw-1,\ yz+xw) \to A$, set $\alpha = \varphi(x)+i\varphi(y)$ and $\beta = \varphi(z)+i\varphi(w)$ in $A \otimes_\mathbb{R} \mathbb{C}$. Direct computation gives

$$\begin{aligned}\alpha\beta &= \varphi(x)\varphi(z) - \varphi(y)\varphi(w) + i\big(\varphi(y)\varphi(z) + \varphi(x)\varphi(w)\big)\\ &= \varphi(xz-yw) + i\,\varphi(yz+xw) = 1 + i\cdot 0 = 1,\end{aligned}$$

so $\alpha \in (A\otimes_\mathbb{R}\mathbb{C})^\times$ with inverse β.

Conversely, given $\alpha = a+ib \in (A\otimes_\mathbb{R}\mathbb{C})^\times$ with inverse $\beta = c+id$, the relation $\alpha\beta = 1$ amounts to

$$ac - bd = 1, \quad ad + bc = 0.$$

Setting $\varphi(x) = a$, $\varphi(y) = b$, $\varphi(z) = c$, $\varphi(w) = d$ produces the inverse map. □

A basic fact about sets is that maps and families are interchangeable: there is an equivalence of categories

$$\mathsf{Ar}(\mathsf{Set}) \simeq \mathsf{Fam}(\mathsf{Set}),$$

where $\mathsf{Ar}(\mathsf{Set}) = \mathsf{Fun}(\{0 \to 1\}, \mathsf{Set})$ is the arrow category, and $\mathsf{Fam}(\mathsf{Set})$ has as objects the set-indexed families $(X_i)_{i\in I}$ with morphisms $(X_i)_{i\in I} \to (Y_j)_{j\in J}$ consisting of a map $u\colon I \to J$ together with maps $X_i \to Y_{u(i)}$. A map $f\colon X \to I$ corresponds to the family of its fibers $(f^{-1}\{i\})_{i\in I}$, and a family $(X_i)_{i\in I}$ corresponds to the projection $\coprod_{i\in I} X_i \to I$. Fixing the target I specializes this to

$$\mathsf{Set}_{/I} \simeq \mathsf{Set}^I.$$

The following proposition generalizes this to presheaves of sets; we recover the equivalence above by taking $C = *$.

Proposition 1.2.33 (Slices of Presheaf Categories). *Let C be a category and $F \in \mathsf{PShv}(C)$. Then there is an equivalence of categories*

$$\mathsf{PShv}(C)_{/F} \underset{\mathrm{fib}_F}{\overset{\coprod_F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathsf{PShv}(\mathrm{El}(F))$$

described as follows. Let $u\colon \mathrm{El}(F) \to C$ be the forgetful functor; the tautological map $* \to u^*(F)$ has adjoint $u_\sharp(*) \to F$, which is an isomorphism.

- *For $H \in \mathsf{PShv}(\mathrm{El}(F))$, the presheaf $\coprod_F H$ over F is $u_\sharp(H) \to u_\sharp(*) \simeq F$. Explicitly,*

$$\Big(\coprod_F H\Big)(X) = \coprod_{x\in F(X)} \big(H(X,x) \to *\big).$$

- *For $G \in \mathsf{PShv}(C)_{/F}$, the presheaf* $\mathrm{fib}_F(G)$ *on* $\mathrm{El}(F)$ *is the pullback* $u^*(G) \times_{u^*(F)} *$. *Explicitly,*

$$\mathrm{fib}_F(G)(X,x) = G(X) \times_{F(X)} \{x\} \simeq \left\{ \begin{array}{ccc} & & G \\ & \nearrow & \downarrow \\ y(X) & \xrightarrow{x} & F \end{array} \right\}.$$

Proof. We verify that $\coprod_F$ and fib_F are mutually inverse.

Step 1: $\mathrm{fib}_F \circ \coprod_F \simeq \mathrm{id}$. For $H \in \mathsf{PShv}(\mathrm{El}(F))$,

$$\begin{aligned} \mathrm{fib}_F\Big(\coprod_F H\Big)(X,x) &= \Big(\coprod_F H\Big)(X) \times_{F(X)} \{x\} \\ &= \Big(\coprod_{y \in F(X)} H(X,y)\Big) \times_{F(X)} \{x\} = H(X,x). \end{aligned}$$

Step 2: $\coprod_F \circ \mathrm{fib}_F \simeq \mathrm{id}$. For $G \in \mathsf{PShv}(C)_{/F}$,

$$\begin{aligned} \Big(\coprod_F \mathrm{fib}_F(G)\Big)(X) &= \coprod_{x \in F(X)} \mathrm{fib}_F(G)(X,x) \\ &= \coprod_{x \in F(X)} \big(G(X) \times_{F(X)} \{x\}\big) = G(X), \end{aligned}$$

with the map to $F(X)$ given by the natural projection. □

Specializing to the case of a representable presheaf, we get:

Corollary 1.2.34 (Slice Categories of Presheaves). *Let C be a category, $X \in C$, and $u : C_{/X} \to C$ the forgetful functor. Then $u_\sharp$ induces an equivalence*

$$\mathsf{PShv}(C_{/X}) \xrightarrow{\sim} \mathsf{PShv}(C)_{/y(X)}.$$

Proof. Apply Proposition 1.2.33 to $F = y(X)$ and use $\mathrm{El}(y(X)) \simeq C_{/X}$. □

Specializing to $C = \mathsf{CAlg}^{\mathrm{op}}$ gives the following key result:

Corollary 1.2.35 (Algebraic Geometry over a Base Ring). *Let k be a ring and $\varphi : \mathbb{Z} \to k$ the unique ring map. Then $\varphi_\sharp$ induces an equivalence*

$$\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set}) \xrightarrow{\sim} \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{Spec}(k)}.$$

Proof. The special case of Corollary 1.2.34 with $C = \mathsf{CAlg}^{\mathrm{op}}$ and $X = \mathrm{Spec}(k)$. □

***Remark* 1.2.36** (Base-Independent Algebraic Geometry). By Corollary 1.2.35, algebraic geometry over a base ring k is subsumed by algebraic geometry over $\mathbb{Z}$: working in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ rather than $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ costs no generality, and we will often do so from now on. Furthermore, the Spec functor is independent of the base, in the sense that the following square commutes:

$$\begin{array}{ccc} \mathsf{CAlg}_k^{\mathrm{op}} & \xrightarrow{\sim} & (\mathsf{CAlg}^{\mathrm{op}})_{/k} \\ {\scriptstyle \mathrm{Spec}}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}} \\ \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set}) & \xrightarrow{\sim} & \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{Spec}(k)} \end{array}$$

1.2.4. Functions on Algebraic Functors

Recall that $\mathbb{A}^1 : \mathsf{CAlg} \to \mathsf{Set}$ is the forgetful functor $R \mapsto R$. Thus $\mathbb{A}^1$ is a ring object in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$:

$$\begin{array}{ccc} & & \mathsf{CAlg} \\ & \overset{\mathrm{id}}{\nearrow} & \big\downarrow \scriptstyle\text{forget} \\ \mathsf{CAlg} & \xrightarrow[\mathbb{A}^1]{} & \mathsf{Set} \end{array}$$

For any algebraic functor X, we have $\mathrm{Hom}(X, \mathbb{A}^1) \in \mathsf{CAlg}$. Pointwise, for a ring R:

$$\mathrm{Hom}(X(R), \mathbb{A}^1(R)) = \mathrm{Hom}_{\mathsf{Set}}(X(R), R).$$

Each $f \in \mathrm{Hom}(X, \mathbb{A}^1)$ functorially maps an R-point x to an element $f(x) \in R$. Thus we can view $\mathrm{Hom}(X, \mathbb{A}^1)$ as "functions" on X.

Similarly, $\mathbb{G}_m : \mathsf{CAlg} \to \mathsf{Set}$ factors as

$$\mathsf{CAlg} \xrightarrow{R \mapsto R^\times} \mathsf{Ab} \xrightarrow{\text{forget}} \mathsf{Set}.$$

For $f \in \mathrm{Hom}(X, \mathbb{G}_m)$, we have $f(x) \in R^\times$ for any $x : \mathrm{Spec}(R) \to X$ (with $R \neq 0$), so $f(x) \neq 0$. Thus f represents a "nowhere-vanishing function" on X.

Definition 1.2.37 (Functions). Let X be an algebraic functor.

(i) A *function* on X is a morphism $X \to \mathbb{A}^1$. We denote

$$\mathcal{O}(X) \coloneqq \mathrm{Hom}(X, \mathbb{A}^1)$$

for the ring of all functions on X, called the *coordinate ring* of X.

(ii) A *nowhere-vanishing function* on X is a morphism $X \to \mathbb{G}_m$. We denote

$$\mathcal{O}^\times(X) \coloneqq \mathrm{Hom}(X, \mathbb{G}_m)$$

for the group of all nowhere-vanishing functions on X.

***Remark* 1.2.38.** We note the following:

- Since $\mathbb{G}_m \subset \mathbb{A}^1$ is a monomorphism, $\mathcal{O}^\times(X) \subset \mathcal{O}(X)$.
- $\mathcal{O}(\mathrm{Spec}(R)) = \mathrm{Hom}(\mathrm{Spec}(R), \mathbb{A}^1) \simeq \mathbb{A}^1(R) \simeq R$, by the Yoneda lemma.
- Writing any presheaf as a colimit of representables,

$$\mathcal{O}(X) = \mathrm{Hom}\left(\mathrm{colim}_{x:\ \mathrm{Spec}(R)\to X} \mathrm{Spec}(R),\ \mathbb{A}^1\right) \simeq \lim_{x:\ \mathrm{Spec}(R)\to X} R,$$

  so a function $f \in \mathcal{O}(X)$ is encoded by the family $(f \circ x)_x$. Similarly,

$$\mathcal{O}^\times(X) \simeq \lim_{x:\ \mathrm{Spec}(R)\to X} R^\times \simeq \mathcal{O}(X)^\times,$$

  where the last isomorphism uses that taking units commutes with limits (since $R^\times \simeq \mathrm{Hom}(\mathbb{Z}[u^{\pm 1}], R)$ is itself a limit-preserving functor in R).
- The functor

$$\mathcal{O} : \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}} \to \mathsf{CAlg}, \quad X \mapsto \mathcal{O}(X)$$

  preserves limits, and is the unique such extension of $\mathrm{id}_{\mathsf{CAlg}}$ along Spec: under the universal property

$$\mathsf{Fun}^{\lim}(\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}}, \mathsf{CAlg}) \simeq \mathsf{Fun}(\mathsf{CAlg}, \mathsf{CAlg}),$$

  $\mathcal{O}$ corresponds to $\mathrm{id}_{\mathsf{CAlg}}$.

***Warning* 1.2.39.** In Remark 1.2.38, since $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is a large category, the indexing category for $\mathcal{O}(X)$ may become large, so $\mathcal{O}(X)$ might not be an object of CAlg but rather a large ring (an object of $\widehat{\mathsf{CAlg}}$).

There are two approaches to resolve this issue:

(i) Introduce $\widehat{\mathsf{Set}}$ and $\widehat{\mathsf{CAlg}}$, allowing us to describe $\mathcal{O}$ as a functor

$$\mathcal{O}\colon \mathsf{Fun}(\mathsf{CAlg}, \widehat{\mathsf{Set}})^{\mathrm{op}} \to \widehat{\mathsf{CAlg}}.$$

This would be the unique large-limit-preserving extension of $\mathsf{CAlg} \hookrightarrow \widehat{\mathsf{CAlg}}$.

(ii) Consider the full subcategory $\mathsf{Fun}^{\mathrm{acc}}(\mathsf{CAlg}, \mathsf{Set})$ of accessible functors. Recall that for a presentable category $\mathcal{C}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *accessible* if there exists κ such that F preserves κ-filtered colimits. For $\mathcal{D} = \mathsf{Set}$ (which is a topos), κ-filtered colimits commute with κ-small limits, so F can be written as a κ-small colimit of representables. For $X \in \mathsf{Fun}^{\mathrm{acc}}(\mathsf{CAlg}, \mathsf{Set})$, it can be written as a small colimit of $\mathrm{Spec}(R_i)$, giving a functor

$$\mathcal{O}\colon \mathsf{Fun}^{\mathrm{acc}}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}} \to \mathsf{CAlg}.$$

This approach is more common in modern derived algebraic geometry.

For a ring A, we have:

$$\begin{aligned}
\mathrm{Hom}(X, \mathrm{Spec}(A)) &\simeq \lim_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Hom}(\mathrm{Spec}(R), \mathrm{Spec}(A)) \\
&\simeq \lim_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Hom}(A, R) \\
&\simeq \mathrm{Hom}(A, \lim_x R) \\
&\simeq \mathrm{Hom}(A, \mathcal{O}(X)).
\end{aligned}$$

Thus we have an adjoint pair:

$$\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{CAlg}$$

1.2.5. Loci

The two distinguished subfunctors of $\mathbb{A}^1$,

$$0 \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{G}_m,$$

let us cut out two natural subfunctors from any algebraic functor X via a function $f\colon X \to \mathbb{A}^1$: the preimage $f^{-1}(0)$ — where f vanishes — and $f^{-1}(\mathbb{G}_m)$ — where f is invertible (*nowhere-vanishing* in our terminology). We package this for arbitrary subsets of functions.

Definition 1.2.40 (Loci). Let X be an algebraic functor and $F \subset \mathcal{O}(X)$ a subset of functions on X.

(i) The *vanishing locus* of F is the subfunctor $V(F) \subset X$ defined by

$$V(F)(R) := \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in F\}.$$

(ii) The *non-vanishing locus* of F is the subfunctor $D(F) \subset X$ defined by

$$D(F)(R) := \{x \in X(R) \mid (f(x))_{f\in F} \text{ generates the unit ideal in } R\}.$$

***Remark* 1.2.41.** Note that $V(F)$ depends only on the ideal $(F) \subset O(X)$. If $\sqrt{(G)} = \sqrt{(F)}$, then $D(F) = D(G)$, so $D(F)$ depends only on $\sqrt{(F)}$.

Example 1.2.42 (Punctured Affine Space)**.** We have $\mathbb{A}^I = \operatorname{Spec}(\mathbb{Z}[x_i \mid i \in I])$. Define $\mathbb{A}^I - 0$ as the algebraic functor

$$(\mathbb{A}^I - 0)(R) = \{a \in R^I \mid (a_i)_{i\in I} \text{ generates } R\}.$$

Here (a) denotes the ideal generated by all components of $a \in R^I$.

The coordinate functions $x_i \colon \mathbb{A}^I \to \mathbb{A}^1$ send $a \in \mathbb{A}^I(R)$ to its i-th component a_i. Therefore:

$$\mathbb{A}^I - 0 = D(\{x_i \mid i \in I\}) = \{a \in R^I \mid (x_i(a))_{i\in I} \text{ generates } R\}.$$

Note that when $I = \{1\}$, we have $\mathbb{A}^1 - 0 \simeq \mathbb{G}_m$.

A subset $F \subset O(X)$ assembles into a single morphism

$$\mathrm{ev}_F \colon X \to \mathbb{A}^F, \quad (x\colon \operatorname{Spec}(R) \to X) \mapsto (g(x))_{g\in F},$$

in terms of which

$$V(F) = \mathrm{ev}_F^{-1}(0) = \{x \in X(R) \mid g(x) = 0 \text{ for all } g \in F\},$$
$$D(F) = \mathrm{ev}_F^{-1}(\mathbb{A}^F - 0) = \{x \in X(R) \mid (g(x))_{g\in F} \text{ generates } R\}.$$

***Warning* 1.2.43.** It is tempting to think that $D(F)$ is somehow the "complement" of $V(F)$ in X, but this is false. While this holds over fields, it fails over general rings. For example, consider $R = 0$: then $D(F)(0) = V(F)(0) = X(0)$, so they are not even disjoint.

However, in certain subcategories of Fun(CAlg, Set), such as Sch (schemes), this does hold.

Similarly, $V(F)$ is not the complement of $D(F)$, because $D(F) = D(F')$ can hold while $V(F) \neq V(F')$ (since V depends on ideals while D depends on radical ideals).

Proposition 1.2.44 (Properties of Loci)**.** *Let X be an algebraic functor.*

(i) For any family $(F_i)_{i\in I}$ of subsets of $O(X)$:

$$\bigcap_{i\in I} V(F_i) = V\left(\bigcup_{i\in I} F_i\right)$$

and

$$\bigcup_{i\in I} D(F_i) \subset D\left(\bigcup_{i\in I} F_i\right).$$

The inclusion becomes equality when evaluated on local rings.

(ii) For finitely many subsets $F_1, \ldots, F_n \subset O(X)$:

$$D(F_1) \cap \cdots \cap D(F_n) = D(F_1 \cdots F_n),$$

and

$$V(F_1) \cup \cdots \cup V(F_n) \subset V(F_1 \cdots F_n).$$

The inclusion becomes equality when evaluated on integral domains.

Proof. We verify these pointwise on all rings R.

(1) For the vanishing loci:

$$\begin{aligned}\bigcap_{i\in I} V(F_i)(R) &= \bigcap_{i\in I}\{x \in X(R) \mid f_i(x) = 0 \text{ for all } f_i \in F_i\} \\ &= \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in \bigcup_{i\in I} F_i\} \\ &= V\left(\bigcup_{i\in I} F_i\right)(R).\end{aligned}$$

For the non-vanishing loci:

$$\begin{aligned}\bigcup_{i\in I} D(F_i)(R) &= \bigcup_{i\in I}\{x \in X(R) \mid (f_i(x))_{f_i\in F_i} = R\} \\ &\subset \{x \in X(R) \mid (f(x))_{f\in\bigcup_{i\in I} F_i} = R\} \\ &= D\left(\bigcup_{i\in I} F_i\right)(R).\end{aligned}$$

The inclusion holds because if $(f_i(x))_{f_i\in F_i} = R$ for some i, then certainly $\bigcup_i F_i$ generates R.

When R is a local ring with unique maximal ideal $\mathfrak{m}$: if $(f(x))_{f\in\bigcup F_i} = R$, then at least one $f(x)$ is a unit. This f belongs to some F_i, so $(f_i(x))_{f_i\in F_i} = R$, hence $x \in D(F_i)(R)$. Thus equality holds.

(2) The equation $D(F_1) \cap \cdots \cap D(F_n) = D(F_1 \cdots F_n)$ is straightforward: x satisfies $(f_i(x))_{f_i\in F_i} = R$ for all i if and only if $(f_1(x)\cdots f_n(x))_{f_i\in F_i} = R$.

For the vanishing loci, the inclusion $V(F_1) \cup \cdots \cup V(F_n) \subset V(F_1 \cdots F_n)$ is clear.

When R is an integral domain: if $f(x) = 0$ for all $f \in F_1 \cdots F_n$, write $f = f_1 \cdots f_n$ with $f_i \in F_i$. Since R is a domain and $f_1(x)\cdots f_n(x) = 0$, at least one $f_i(x) = 0$. Thus $x \in V(F_i)(R)$ for some i. □

Example 1.2.45. We have $\mathbb{Z} = (2, 3)$, but $V(2) \cap V(3) = V(1)$, while $D(2) \cup D(3) \neq D(1)$. This confirms that $D(F)$ is not the complement of $V(F)$.

Now we characterize open and closed subfunctors of affine schemes.

Proposition 1.2.46 (Subfunctors of $\operatorname{Spec}(A)$). *Let A be a ring.*

(i) For any subset $F \subset A$, the quotient map $A \longrightarrow A/(F)$ induces an isomorphism

$$\operatorname{Spec}(A/(F)) \xrightarrow{\sim} V(F) \subset \operatorname{Spec}(A).$$

Therefore $V(F)$ is an affine scheme.

(ii) For any $f \in A$, the localization map $A \longrightarrow A_f$ induces an isomorphism

$$\operatorname{Spec}(A_f) \xrightarrow{\sim} D(f) \subset \operatorname{Spec}(A).$$

Therefore $D(f)$ is an affine scheme.

Proof. *(1)* For $F \subset A$:

$$V(F)(R) = \{\varphi \in \operatorname{Hom}(A, R) \mid \varphi(f) = 0 \text{ for all } f \in F\}.$$

This condition is equivalent to φ factoring through $A/(F)$, i.e., $\varphi \in \operatorname{Hom}(A/(F), R)$.

(2) For a ring R, a map $\psi\colon A_f \longrightarrow R$ corresponds to a map $\tilde{\psi}\colon A \longrightarrow R$ sending f to a unit in R. Thus $\operatorname{Hom}(A_f, R)$ is in bijection with $D(f)(R)$. □

***Remark* 1.2.47.** For a general subset $F \subset A$, $D(F)$ is typically not affine. For instance, Example 1.2.42 shows $\mathbb{A}^I - 0 \simeq D(\{x_i \mid i \in I\})$ is not affine when $|I| > 1$.

1.2.6. Open and Closed Subfunctors

Definition 1.2.48 (Open and Closed Subfunctors)**.** Let X be an algebraic functor and $Z \subset X$ a subfunctor.

(i) Z is a *closed subfunctor* of X if for every R-point $x\colon \operatorname{Spec}(R) \to X$, there exists $F \subset R$ such that $x^{-1}(Z) \simeq V(F)$.

(ii) Z is an *open subfunctor* of X if for every R-point $x\colon \operatorname{Spec}(R) \to X$, there exists $F \subset R$ such that $x^{-1}(Z) \simeq D(F)$.

Proposition 1.2.49 (Classification of Subfunctors of $\operatorname{Spec}(A)$)**.** *Let A be a ring.*

(i) $F \mapsto V(F)$ induces an order-reversing bijection

$$\{\textit{ideals in } A\} \xrightarrow{\sim} \{\textit{closed subfunctors of } \operatorname{Spec}(A)\}.$$

(ii) $F \mapsto D(F)$ induces an order-preserving bijection

$$\{\textit{radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \operatorname{Spec}(A)\}.$$

Proof. (1) First, $V(F) \subset \operatorname{Spec}(A)$ is indeed a closed subfunctor: for any $x\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ (corresponding to $\varphi\colon A \to R$):

$$x^{-1}(V(F)) \simeq V(\varphi(F)) \subset \operatorname{Spec}(R).$$

Injectivity: We claim that $f \in (F)$ if and only if $V(F) \subset V(f)$, for any $f \in A$ and ideal $(F) \subset A$. The forward direction is clear: if $f = \sum r_i f_i$ with $f_i \in F$, then f vanishes whenever all f_i do. For the converse, take the canonical point $\pi\colon A \to A/(F)$, which lies in $V(F)$ tautologically; if $V(F) \subset V(f)$, then $\pi(f) = 0$, i.e., $f \in (F)$.

Now suppose $V(F) = V(F')$. For any $f \in F$, the inclusion $V(F') = V(F) \subset V(f)$ gives $f \in (F')$, so $(F) \subset (F')$. By symmetry $(F') \subset (F)$, hence $(F) = (F')$.

Surjectivity: For any closed subfunctor $Z \subset \operatorname{Spec}(A)$, consider $x = \mathrm{id}_{\operatorname{Spec}(A)}$. By definition, $\mathrm{id}^{-1}(Z) \simeq V(F)$ for some $F \subset A$. Thus $Z \simeq V(F)$.

(2) For F with $\sqrt{(F)} = F$, the well-definedness and surjectivity are clear from the definition.

Injectivity: We must show $f \in F$ if and only if $D(f) \subset D(F)$.

($\Leftarrow$) Assume $D(f) \subset D(F)$. Evaluating at $R = A_f$, we have $D(f)(A_f)$ contains the canonical map $\varphi\colon A \to A_f$. Since $D(f) \simeq \operatorname{Spec}(A_f)$, this is the unique element of $D(f)(A_f)$.

The inclusion $D(f) \subset D(F)$ means $\varphi \in D(F)(A_f)$, i.e., $(\varphi(F)) = A_f$. Therefore, there exist $a \in F$ and $n \in \mathbb{N}$ such that

$$\frac{a}{f^n} = 1 \in A_f.$$

Clearing the denominator gives $f^k a = f^{n+k}$ for some $k \geq 0$, so $f^{n+k} = f^k a \in (F)$. Since F is a radical ideal, this forces $f \in F$.

($\Rightarrow$) is immediate from the definition of D. □

Definition 1.2.50 (Immersions)**.** Let $Y \to X$ be a morphism of algebraic functors.

(i) $Y \to X$ is a *closed immersion* if it is a monomorphism with closed image.

(ii) $Y \to X$ is an *open immersion* if it is a monomorphism with open image.

Definition 1.2.51 (Locally Closed Subfunctors). Let X be an algebraic functor.

(i) A subfunctor $Y \subset X$ is *locally closed* if there exists an open subfunctor $U \subset X$ such that $Y \subset U$ is a closed subfunctor of U.

(ii) A morphism $Y \to X$ is an *immersion* if it is a monomorphism with locally closed image.

Proposition 1.2.52 (Stability of Immersions). *(i) (Base change) Consider a pullback square:*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

If f is a closed immersion, then so is f'. The same holds for open immersions and immersions.

(ii) (Composition) Consider a commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & {}_{h}\searrow \quad \swarrow_{f} & \\ & X & \end{array}$$

If f and g are closed immersions, then so is h. If h is a closed immersion and f is a monomorphism, then g is a closed immersion. The same holds for open immersions and immersions.

Proof. *(1)* We prove for closed immersions; the other cases are similar.

If f is a closed immersion, then f is a monomorphism and has closed image. Monomorphisms are stable under pullback, so f' is a monomorphism.

For the image, let $x' \colon \operatorname{Spec}(R) \to X'$ be an R-point, and let $x \colon \operatorname{Spec}(R) \to X$ be the composition. By the pasting lemma:

$$\begin{array}{ccc} x'^{-1}(f'(Y')) & \longrightarrow & \operatorname{Spec}(R) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle x'} \\ Y' & \xrightarrow{f'} & X' \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

the outer rectangle is a pullback, so $x^{-1}(f(Y)) \simeq x'^{-1}(f'(Y'))$. Since $f(Y)$ is closed in X, we have $x^{-1}(f(Y)) \simeq V(I)$ for some ideal $I \subset R$. Thus $f'(Y')$ is closed in X'.

(2 - Composition)

Closed case: If f and g are closed immersions, they are both monomorphisms, so $h = f \circ g$ is a monomorphism (monomorphisms are closed under composition).

For the image: for $x \colon \operatorname{Spec}(R) \to X$, we have

$$x^{-1}(h(Z)) = x^{-1}(f(g(Z))).$$

Since f has closed image, $x^{-1}(f(Y)) \simeq \operatorname{Spec}(R/I_1)$ for some ideal $I_1 \subset R$. This gives $y \colon \operatorname{Spec}(R/I_1) \to Y$.

Since g has closed image in Y:

$$y^{-1}(g(Z)) \simeq \operatorname{Spec}(R/(I_1 + I_2))$$

for some ideal I_2. Thus $x^{-1}(f(g(Z))) \simeq \mathrm{Spec}(R/(I_1 + I_2))$, which is closed.

Open case: Assume $X \simeq \mathrm{Spec}(A)$. We must show: for any subset $F \subset A$ and open subfunctor $U \subset D(F)$, U is an open subfunctor of $\mathrm{Spec}(A)$.

For each $f \in F$, we have $D(f) \subset \mathrm{Spec}(A)$ with $D(f) \simeq \mathrm{Spec}(A_f)$. Consider $U \cap D(f)$. Since $U \subset D(F)$ is open, $U \cap D(f)$ is an open subfunctor of $\mathrm{Spec}(A_f)$, so we can write

$$U \cap D(f) = D(\bar{H}_f) \subset \mathrm{Spec}(A_f)$$

for some $\bar{H}_f \subset A_f$. Choose $H_f \subset A$ whose image in A_f equals $\bar{H}_f$ (essentially clearing denominators). Then

$$D(fH_f) = D(H_f) \cap D(f) = D(\bar{H}_f) = U \cap D(f).$$

Let $H = \bigcup_{f \in F} fH_f \subset A$. We claim $U = D(H)$.

For $\varphi \colon A \to R$ (i.e., $\mathrm{Spec}(\varphi) \colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$):

($U \subset D(H)$): If $\mathrm{Spec}(\varphi) \in U$, we need $(\varphi(H)) = R$. Since $U \subset D(F)$, we have $(\varphi(F)) = R$. For any $f \in F$, the map $\mathrm{Spec}(R_{\varphi(f)}) \to \mathrm{Spec}(A_f)$ lies in $U \cap D(f) = D(fH_f)$. Thus $(\varphi(fH_f)) = R_{\varphi(f)}$, so $\varphi(fH_f)$ contains some power of $\varphi(f)$, i.e., $\varphi(f) \in \sqrt{(\varphi(fH_f))}$.

Taking the union over all $f \in F$:

$$R = \sqrt{(\varphi(F))} \subset \sqrt{(\varphi(H))}.$$

($D(H) \subset U$): If $(\varphi(H)) = R$, then since $H \subset (F)$, we have $D(H) \subset D(F)$. Since $U \subset D(F)$ is open, $\mathrm{Spec}(\varphi)^{-1}(U) = D(I)$ for some radical ideal $I \subset R$.

We need $I = R$. Consider the pullback with $D(f)$:

$$D(\varphi(f)I) = \mathrm{Spec}(\varphi)^{-1}(U \cap D(f)) = D(\varphi(fH_f)).$$

Thus $\sqrt{(\varphi(f)I)} = \sqrt{(\varphi(fH_f))}$. Taking the union over $f \in F$:

$$I \supset \sqrt{(\varphi(F)I)} = \sqrt{(\varphi(H))} = R.$$

Therefore $I = R$.

immersion case: Let $X = \mathrm{Spec}(A)$, $Z \subset X$ closed, and $U \subset Z$ open. We show $U \subset X$ is locally closed.

Write $Z = \mathrm{Spec}(A/I)$ and $U = D(F)$ where $F \subset A/I$. Let $\hat{F}$ be the preimage of F in A and $\hat{U} = D(\hat{F})$. Then $\hat{U} \cap Z = U$:

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \downarrow & \lrcorner & \downarrow \\ \hat{U} & \longrightarrow & X \end{array}$$

Since $Z \to X$ is a closed immersion, so is $U \to \hat{U}$. Thus $U \to X$ is locally closed.

(2 - Decomposition) Follows from Theorem 1.2.53 below. □

Many classes of morphisms in algebraic geometry (such as immersions) satisfy certain "cancellation laws". To avoid repeating similar diagram chases for each property, we state a general result below. This theorem gives a criterion for when a property satisfies a "2-out-of-3" type condition, leveraging the behavior of the diagonal morphism.

Theorem 1.2.53 (The Diagonal Argument). *Let C be a category with fiber products. Let $\mathcal{P}$ be a property of morphisms in C that is:*

(i) Stable under composition.

(ii) Stable under base change.

Let $g\colon Z \to Y$ and $f\colon Y \to X$ be morphisms. If the composite $f \circ g$ has property $\mathcal{P}$ and the diagonal $\Delta_f\colon Y \to Y \times_X Y$ has property $\mathcal{P}$, then g has property $\mathcal{P}$.

Proof. Factor g through its graph Γ_g:

$$g\colon Z \xrightarrow{\Gamma_g} Z \times_X Y \xrightarrow{\mathrm{pr}_Y} Y.$$

(i) The projection $\mathrm{pr}_Y\colon Z \times_X Y \to Y$ is the base change of $f \circ g\colon Z \to X$ along f:

$$\begin{array}{ccc} Z \times_X Y & \longrightarrow & Z \\ \downarrow{\scriptstyle \mathrm{pr}_Y} & & \downarrow{\scriptstyle f\circ g} \\ Y & \xrightarrow{f} & X \end{array}$$

Since $f \circ g$ has property $\mathcal{P}$ and $\mathcal{P}$ is stable under base change, pr_Y has property $\mathcal{P}$.

(ii) The graph morphism Γ_g is the base change of the diagonal Δ_f along $(f \circ g) \times \mathrm{id}_Y$:

$$\begin{array}{ccc} Z & \xrightarrow{\Gamma_g} & Z \times_X Y \\ \downarrow{\scriptstyle f\circ g} & & \downarrow \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y \end{array}$$

(More precisely, it is the base change of Δ_f along the map $Z \times Y \to Y \times Y$ induced by $f \circ g$ and 1_Y). Since Δ_f has property $\mathcal{P}$ and $\mathcal{P}$ is stable under base change, Γ_g has property $\mathcal{P}$.

Since $\mathcal{P}$ is stable under composition, $g = \mathrm{pr}_Y \circ \Gamma_g$ has property $\mathcal{P}$. □

1.2.7. Quasi-Affine Schemes

Definition 1.2.54 (Quasi-Affine Scheme). Let k be a ring. An algebraic k-functor X is a *quasi-affine k-scheme* if there exist a k-algebra A and a finite subset $F \subset A$ such that $X \simeq D(F) \subset \mathrm{Spec}(A)$.

Denote $\mathsf{QAff}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ for the full subcategory spanned by quasi-affine k-schemes.

Proposition 1.2.55 (Closed Subfunctors of Quasi-Affine Schemes). *Let X be a quasi-affine k-scheme and $Z \hookrightarrow X$ a closed immersion. Then Z is also a quasi-affine k-scheme.*

Proof. We prove for $k = \mathbb{Z}$; the general case is identical. Since X is quasi-affine, there exist a ring A and finite $F \subset A$ with $X \simeq D(F)$.

Let Y be the image of $Z \hookrightarrow X$, so Y is a closed subfunctor of $D(F)$. For any $f \in F$, $Y \cap D(f)$ is a closed subfunctor of $D(f)$. Let $J(f) \subset A_f$ be the corresponding ideal, and let $I(f) \subset A$ be its preimage under $A \to A_f$. Then $I(f)_f = J(f)$, so

$$V(I(f)) \cap D(f) = Y \cap D(f).$$

Define $I = \bigcap_{f \in F} I(f)$.

Claim 1.2.56. *$Y \simeq D(\bar{F}) \subset \mathrm{Spec}(A/I) = V(I)$, where $\bar{F}$ is the image of F in A/I.*

Proof of Claim. Let $\bar{Y}$ be the closure of Y in $\mathrm{Spec}(A)$ (the smallest closed subfunctor containing Y).

Step 1: $\bar{Y} = V(I)$.

First, $Y \subset V(I)$: Let $\varphi\colon A \to R$ be a point in $Y(R)$. Since $Y \subset D(F)$, we have $(\varphi(F)) = R$. We need $(\varphi(I)) = 0$.

For any $f \in F$, consider the composition $A \xrightarrow{\varphi} R \to R_{\varphi(f)}$. This corresponds to a point in $Y \cap D(f) = V(I(f)) \cap D(f)$ evaluated at $R_{\varphi(f)}$. Thus $\varphi(I(f)) = 0$ in $R_{\varphi(f)}$ for each f, which implies $\varphi(I(f)) = 0$ in R. Taking the intersection over $f \in F$ gives $\varphi(I) = 0$.

Conversely, $V(I) \subset \bar{Y}$: By Proposition 1.2.49, closed subfunctors of $\mathrm{Spec}(A)$ correspond to ideals. If $Y \subset V(K)$, we need $K \subset I$. By definition of I, this reduces to showing $K_f \subset J(f)$ in A_f.

From $V(J(f)) = Y \cap D(f) \subset V(K_f)$, we get $K_f \subset J(f)$.

Step 2: $Y = \bar{Y} \cap D(F)$.

First, we show $I_g = J(g)$ for all $g \in F$. One inclusion is clear. For $J(g) \subset I_g$, note that for any $f \in F$:

$$J(g)_f = J(f)_g \quad \text{in } A_{fg}$$

(both equal the ideal cutting out $Y \cap D(fg)$).

Since localization is exact, $I(f)_g$ is the preimage of $J(f)_g$ under $A_g \to A_{fg}$. Thus $J(g) \subset I(f)_g$ for all f, so $J(g) \subset I_g$.

A point $x\colon \mathrm{Spec}(R) \to \bar{Y} \cap D(F)$ corresponds to $\varphi\colon A \to R$ with $\varphi(I) = 0$ and $(\varphi(F)) = R$.

Since $Y \subset D(F)$ is closed in $D(F)$, we have $Y \subset \bar{Y} \cap D(F)$ is also closed. To show equality, consider the pullback $x^{-1}(Y) \simeq V(G)$ for some $G \subset R$. We need $G = 0$.

By Zariski descent (Proposition 1.3.9), it suffices to show $G_{\varphi(f)} = 0$ for all $f \in F$. But

$$V(G_{\varphi(f)}) = x^{-1}(Y) \cap D(\varphi(f)) = x^{-1}(Y \cap D(f)) = x^{-1}(V(I_f)) = V(\varphi(I_f)) = V(0).$$

Thus $G_{\varphi(f)} = 0$. □

Since F is finite, so is $\bar{F}$, and hence $Y \simeq D(\bar{F}) \subset \mathrm{Spec}(A/I)$ is quasi-affine. □

1.3. Zariski Descent

Let $(f_i)_{i\in I}$ be a family of elements in a ring R generating the unit ideal. In this case,

$$\bigcap_{i\in I} V(f_i) = V\left(\bigcup_{i\in I}\{f_i\}\right) = \emptyset.$$

However, as mentioned in Warning 1.2.43, $D(f_i)$ is not the complement of $V(f_i)$. Thus, in general, the equality $\bigcup_{i\in I} D(f_i) = \mathrm{Spec}(R)$ does not hold in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$.

In this section, we explain that if we restrict to the category of affine schemes $\mathsf{Aff} \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$, then $\mathrm{Spec}(R)$ is indeed the union (i.e., colimit) of the $D(f_i)$. More precisely, a morphism $\varphi\colon \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is uniquely determined by a family $\{\varphi_i\colon D(f_i) \to \mathrm{Spec}(S)\}_{i\in I}$ satisfying the gluing condition: for all $i, j \in I$,

$$\varphi_i|_{D(f_i)\cap D(f_j)} = \varphi_j|_{D(f_i)\cap D(f_j)}.$$

For two elements $f, g \in R$, the statement "$D(f) \cup D(g) = \mathrm{Spec}(R)$ in Aff" is equivalent to saying that

$$\begin{array}{ccc} D(f) \cap D(g) & \hookrightarrow & D(f) \\ \downarrow & \ulcorner & \downarrow \\ D(g) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

is a pushout diagram in Aff. Equivalently, for any affine scheme X,

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_f) \\ \downarrow & \lrcorner & \downarrow \\ X(R_g) & \longrightarrow & X(R_{fg}) \end{array}$$

is a pullback (i.e., limit) diagram in Set.

More generally, the condition "$\bigcup_{i\in I} D(f_i) = \mathrm{Spec}(R)$ in Aff" is equivalent to the statement that for any affine scheme X,

$$X(R) \longrightarrow \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j})$$

is an equalizer diagram. We begin by establishing this for modules.

Theorem 1.3.1 (Zariski Descent for Modules). *Let R be a ring and let $(f_i)_{i\in I}$ be a family of elements in R generating the unit ideal.*

(i) (Descent for morphisms) *For R-modules M and N, the diagram*

$$\mathrm{Hom}_R(N, M) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{R_{f_i}}(N_{f_i}, M_{f_i}) \rightrightarrows \prod_{i,j\in I} \mathrm{Hom}_{R_{f_i f_j}}(N_{f_i f_j}, M_{f_i f_j})$$

is an equalizer in Set.

(ii) (Descent for objects) *Given the following data:*

- *For each $i \in I$, an R_{f_i}-module M_i;*
- *For each pair $(i, j) \in I \times I$, an $R_{f_i f_j}$-linear isomorphism $\alpha_{ij} : (M_i)_{f_j} \xrightarrow{\sim} (M_j)_{f_i}$;*
- (Cocycle condition) *For each triple $(i, j, k) \in I \times I \times I$, we have*

$$\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \quad \text{in } \mathrm{Hom}_{R_{f_i f_j f_k}}((M_i)_{f_j f_k}, (M_k)_{f_i f_j}).$$

Then there exists an R-module M and, for each $i \in I$, an R_{f_i}-linear isomorphism $\beta_i : M_{f_i} \xrightarrow{\sim} M_i$ such that $\alpha_{ij} \circ \beta_i = \beta_j$ for all i, j. Moreover, M is unique up to unique isomorphism.

***Remark* 1.3.2.** For the higher categorical version, one uses faithfully flat descent, since the flat topology is finer than the Zariski topology. This reduces Zariski descent to faithfully flat descent via a standard argument.

Proof. (1) Descent for morphisms:

We write the equalizer explicitly as

$$E = \left\{ (\varphi_i : N_{f_i} \to M_{f_i})_{i\in I} \;\middle|\; \forall i, j \in I,\ (\varphi_i)_{f_j} = (\varphi_j)_{f_i} \text{ in } \mathrm{Hom}_{R_{f_i f_j}}(N_{f_i f_j}, M_{f_i f_j}) \right\}.$$

We must show that the natural map $\mathrm{Hom}_R(N, M) \to E$ is a bijection.

Setting $N = R$, the problem reduces to proving descent for elements of M. We establish this first.

Descent for elements (Existence): Given $(x_i)_{i\in I} \in \prod_{i\in I} M_{f_i}$ satisfying the compatibility condition $(x_i)_{f_j} = (x_j)_{f_i}$ for all $i, j \in I$.

Since $(f_i)_{i\in I}$ generates the unit ideal, we can choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ still generates the unit ideal. For each $j \in J$, write $x_j = a_j / f_j^{n_j}$ where $a_j \in M$ and $n_j \geq 0$.

Choose a sufficiently large uniform integer $N \geq \max_j n_j$ such that for all $i, j \in J$:

- Each x_i can be written as $x_i = a'_i / f_i^N$ for some $a'_i \in M$ (by multiplying numerator and denominator by $f_i^{N-n_i}$);
- The compatibility condition $(x_i)_{f_j} = (x_j)_{f_i}$ translates to: there exists $m \geq 0$ such that

  $$(f_i f_j)^m (a'_j f_i^N - a'_i f_j^N) = 0 \quad \text{in } M.$$

  After possibly increasing N, we may assume this holds with $m = 0$, i.e.,

  $$a'_j f_i^N = a'_i f_j^N \quad \text{for all } i, j \in J.$$

- There exist $b_i \in R$ such that $\sum_{i\in J} b_i f_i^N = 1$.

Define $x = \sum_{i\in J} b_i a'_i \in M$. For any $k \in J$, we verify that the image of x in M_{f_k} equals x_k:

$$\begin{aligned} x &= \frac{\sum_{i\in J} b_i a'_i}{1} = \frac{\sum_{i\in J} b_i a'_i f_k^N}{f_k^N} \\ &= \frac{\sum_{i\in J} b_i (a'_k f_i^N)}{f_k^N} \quad \text{(using } a'_i f_k^N = a'_k f_i^N\text{)} \\ &= \frac{a'_k \sum_{i\in J} b_i f_i^N}{f_k^N} = \frac{a'_k \cdot 1}{f_k^N} = \frac{a'_k}{f_k^N} = x_k. \end{aligned}$$

Thus $x|_{f_k} = x_k$ for all $k \in J$. For any $i \in I$, since $(f_j)_{j\in J}$ generates the unit ideal, by repeating the above argument on the covering $\{D(f_j) \cap D(f_i)\}_{j\in J}$ of $D(f_i)$, we conclude $x|_{f_i} = x_i$.

Descent for elements (Uniqueness): Suppose $x \in M$ satisfies $x|_{f_i} = 0$ for all $i \in I$. Equivalently, for each i, there exists $n_i \geq 0$ such that $f_i^{n_i} x = 0$ in M.

Choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ generates the unit ideal. Choose N large enough so that $f_j^N x = 0$ for all $j \in J$ and $\sum_{j\in J} c_j f_j^N = 1$ for some $c_j \in R$. Then:

$$x = 1 \cdot x = \left(\sum_{j\in J} c_j f_j^N \right) x = \sum_{j\in J} c_j (f_j^N x) = \sum_{j\in J} c_j \cdot 0 = 0.$$

General case for morphisms: Now let N be arbitrary and suppose given compatible morphisms $\varphi_i : N_{f_i} \to M_{f_i}$. Define $\varphi : N \to M$ as follows: for $n \in N$, take $\varphi(n) \in M$ to be the unique element (provided by descent for elements) whose image in each M_{f_i} is $\varphi_i(n|_{f_i})$. Uniqueness then forces φ to be R-linear.

(2) Descent for objects:

Define M as the equalizer of the following diagram:

$$M := \operatorname{eq}\left(\prod_{i\in I} M_i \overset{a}{\underset{b}{\rightrightarrows}} \prod_{(i,j)\in I\times I} (M_i)_{f_j} \right),$$

where the morphisms are defined componentwise:

- The (i, j)-component of a is the localization $M_i \to (M_i)_{f_j}$;

- The (i,j)-component of b is the composition $M_j \to (M_j)_{f_i} \xrightarrow{\alpha_{ij}^{-1}} (M_i)_{f_j}$.

By construction, there are canonical morphisms $\beta_i \colon M_{f_i} \to M_i$ induced by the projection $M \to M_i$ followed by localization.

Case 1: I is finite.

Fix $0 \in I$. Since I is finite and localization commutes with finite products, we have $(-)_{f_0}(\prod_{i\in I} M_i) \simeq \prod_{i\in I}(M_i)_{f_0}$.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
M_{f_0} & \longrightarrow & \prod_{i\in I}(M_i)_{f_0} & \overset{a_{f_0}}{\underset{b_{f_0}}{\rightrightarrows}} & \prod_{i,j\in I}(M_i)_{f_j f_0} \\
{\scriptstyle \beta_0}\downarrow & (1) & \simeq\downarrow{\scriptstyle \alpha_{i0}} & (2) & \simeq\downarrow{\scriptstyle (\alpha_{i0})_{f_j}} \\
M_0 & \longrightarrow & \prod_{i\in I}(M_0)_{f_i} & \rightrightarrows & \prod_{i,j\in I}(M_0)_{f_i f_j}
\end{array}$$

Two preliminary observations on the cocycle data, both forced by $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$:

- Setting $(i,j,k) = (i,i,i)$ gives $\alpha_{ii} \circ \alpha_{ii} = \alpha_{ii}$, so $\alpha_{ii} = \mathrm{id}$.
- Setting $(i,j,k) = (i,j,i)$ then gives $\alpha_{ji} \circ \alpha_{ij} = \mathrm{id}$, hence $\alpha_{ji} = \alpha_{ij}^{-1}$.

We verify that the diagram commutes square by square.

(1) The left square commutes by the compatibility $\alpha_{ij} \circ \beta_i = \beta_j$ together with the definition of M as an equalizer.

(2) For the right square, we compare both parallel maps componentwise. On the (i,j)-component, the a-path and the b-path of the top row are

$$a_{f_0}\big|_{(i,j)} \colon (M_i)_{f_0} \xrightarrow{\mathrm{loc}_{f_j}} (M_i)_{f_j f_0},$$
$$b_{f_0}\big|_{(i,j)} \colon (M_j)_{f_0} \xrightarrow{\mathrm{loc}_{f_i}} (M_j)_{f_i f_0} \xrightarrow{(\alpha_{ij}^{-1})_{f_0}} (M_i)_{f_j f_0},$$

followed by $(\alpha_{i0})_{f_j} \colon (M_i)_{f_j f_0} \xrightarrow{\sim} (M_0)_{f_i f_j}$. Commutativity of the a-side is just naturality of localization. For the b-side, we must check that

$$(\alpha_{i0})_{f_j} \circ (\alpha_{ij}^{-1})_{f_0} = (\alpha_{j0})_{f_i} \colon (M_j)_{f_i f_0} \xrightarrow{\sim} (M_0)_{f_i f_j}.$$

Applying the cocycle condition with the triple $(j,i,0)$ gives $\alpha_{i0} \circ \alpha_{ji} = \alpha_{j0}$ at the level of $R_{f_i f_j f_0}$-modules. Substituting $\alpha_{ji} = \alpha_{ij}^{-1}$ and rearranging yields the desired identity.

The bottom row is the trivial descent diagram for the R_{f_0}-module M_0 — every module descends to itself — hence is an equalizer. Since the vertical maps are isomorphisms, $\beta_0 \colon M_{f_0} \to M_0$ is an isomorphism.

Uniqueness: Given two solutions $(M, \{\beta_i\})$ and $(N, \{\gamma_i\})$, define for each $i \in I$:

$$\varphi_i \colon M_{f_i} \xrightarrow{\beta_i} M_i \xrightarrow{\gamma_i^{-1}} N_{f_i}.$$

The compatibility $\alpha_{ij} \circ \beta_i = \beta_j$ and $\alpha_{ij} \circ \gamma_i = \gamma_j$ ensures that $(\varphi_i)_{f_j} = (\varphi_j)_{f_i}$.

By descent for morphisms (part 1), there exists a unique morphism $\varphi \colon M \to N$ with $\varphi_{f_i} = \varphi_i$ for all i. Since $\ker(\varphi)_{f_i} = \ker(\varphi_i) = 0$ and $\operatorname{coker}(\varphi)_{f_i} = \operatorname{coker}(\varphi_i) = 0$ for all i, and both kernel and cokernel satisfy descent (as they are determined by exactness of sequences), we have $\ker(\varphi) = 0$ and $\operatorname{coker}(\varphi) = 0$. Thus φ is an isomorphism.

Case 2: I is arbitrary.

Choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ generates the unit ideal. For any finite $J \subset J' \subset I$, define $M_{J'}$ as the equalizer constructed from the data restricted to J'. The inclusion $J \subset J'$ induces a map $M_{J'} \to M_J$, and by the finite case, both are isomorphic to some M_k (where $k \in J$). Thus $M_{J'} \simeq M_J$ canonically.

Since limits commute with limits, we have

$$M = \mathrm{eq}\left(\prod_{i\in I} M_i \rightrightarrows \prod_{i,j\in I} (M_i)_{f_j}\right) = \lim_{\substack{J\subset I \\ J \text{ finite}}} \mathrm{eq}\left(\prod_{i\in J} M_i \rightrightarrows \prod_{i,j\in J} (M_i)_{f_j}\right) \simeq M_J.$$

This completes the proof. □

Corollary 1.3.3 (Descent for Module Elements). *Let R be a ring, and $(f_i)_{i\in I}$ a family of elements generating the unit ideal of R. For an R-module M, the diagram*

$$M \longrightarrow \prod_{i\in I} M_{f_i} \rightrightarrows \prod_{i,j\in I} M_{f_i f_j}$$

is an equalizer in Mod_R.

Proof. This is the special case of Theorem 1.3.1(1) with $N = R$. □

Taking $M = R$ in Corollary 1.3.3, we obtain:

Corollary 1.3.4 (Zariski Descent for Rings). *Let R be a ring, and let $(f_i)_{i\in I}$ be a family of elements generating the unit ideal of R. Then the diagram*

$$R \longrightarrow \prod_{i\in I} R_{f_i} \rightrightarrows \prod_{i,j\in I} R_{f_i f_j}$$

is an equalizer in CAlg.

Corollary 1.3.5 (Zariski Descent for Affine Schemes). *Let R be a ring, and let $(f_i)_{i\in I}$ be a family of elements generating the unit ideal of R. Then for any affine scheme X, the diagram*

$$X(R) \longrightarrow \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j})$$

is an equalizer in Set.

Proof. Write $X \simeq \mathrm{Spec}(A)$ for some ring A. Since $\mathrm{Spec}(A)(R) = \mathrm{Hom}_{\mathsf{CAlg}}(A, R)$ and the functor $\mathrm{Hom}_{\mathsf{CAlg}}(A, -)$ preserves limits (being representable), the result follows from Corollary 1.3.4. □

Corollary 1.3.6 (Descent for Non-vanishing Loci). *Let R be a ring and $F \subset R$ a subset. Then for any affine scheme X, the diagram*

$$\mathrm{Hom}(D(F), X) \longrightarrow \prod_{f\in F} X(R_f) \rightrightarrows \prod_{f,g\in F} X(R_{fg})$$

is an equalizer in Set.

Proof. Recall from Theorem 1.1.10 that $D(F)$ can be written as a colimit:

$$D(F) \simeq \mathrm{colim}_{\substack{R\to S \\ F\mapsto S^\times}} \mathrm{Spec}(S).$$

Let C_F denote the indexing category of ring maps $R \to S$ that send all elements of F to units. Since $\mathrm{Hom}(-, X)$ converts colimits to limits:

$$\mathrm{Hom}(D(F), X) \simeq \mathrm{Hom}\left(\mathrm{colim}_{S \in C_F} \mathrm{Spec}(S), X\right) \simeq \lim_{S \in C_F} X(S).$$

By Corollary 1.3.5, for each S with $R \to S$ sending F to units, the diagram

$$X(S) \longrightarrow \prod_{f \in F} X(S_f) \rightrightarrows \prod_{f,g \in F} X(S_{fg})$$

is an equalizer. Taking limits over $S \in C_F$ and using the fact that limits commute with limits:

$$\lim_{S \in C_F} X(S) \longrightarrow \prod_{f \in F} \lim_{S \in C_F} X(S_f) \rightrightarrows \prod_{f,g \in F} \lim_{S \in C_F} X(S_{fg})$$

is an equalizer.

It remains to show $\lim_{S \in C_F} X(S_g) \simeq X(R_g)$ for each $g \in F$.

For $g \in F$, consider the localization $S \to S_g$. In the category C_F, the map $R \to S_g$ sends F to units (since g is already a unit in S and other elements of F are sent to units in S, hence also in S_g). Moreover, for $R \to S \to S_g$, we have $X((S_g)_g) \simeq X(S_g)$ since g is already a unit in S_g.

This shows that the subcategory of C_F consisting of ring maps $R \to S$ where g is sent to a unit is cofinal. Therefore:

$$\lim_{\substack{R \to S \\ F \mapsto S^\times}} X(S_g) \simeq \lim_{\substack{R \to S \\ g \mapsto S^\times}} X(S_g).$$

The latter category has an initial object R_g, so the limit is simply $X(R_g)$. □

Example 1.3.7. In Corollary 1.3.6, let $X = \mathbb{A}^1$. For $|I| \geq 2$, we have an isomorphism induced by the inclusion $\mathbb{A}^I - 0 \hookrightarrow \mathbb{A}^I$:

$$\mathcal{O}(\mathbb{A}^I) \xrightarrow{\sim} \mathcal{O}(\mathbb{A}^I - 0).$$

By the equivalence of categories $\mathsf{Aff}^{\mathrm{op}} \simeq \mathsf{CAlg}$ (Corollary 1.2.10), this shows that $\mathbb{A}^I - 0$ is not an affine scheme when $|I| \geq 2$, even though it is a quasi-affine scheme. Thus we have a proper inclusion:

$$\mathsf{Aff} \subsetneq \mathsf{QAff}.$$

1.3.1. Zariski Local Properties

A key observation in the proof of Theorem 1.3.1 is that a morphism φ is an isomorphism if and only if all its localizations φ_{f_i} are isomorphisms. This property of being verifiable locally on R_{f_i} is ubiquitous in algebraic geometry. In this section, we systematically study such properties.

Definition 1.3.8 (Zariski Local Property). Let $\mathcal{P}$ be a property of modules (or linear maps, algebras, etc.). We say that $\mathcal{P}$ is *Zariski local* if for any ring R and any family $(f_i)_{i \in I}$ of elements generating the unit ideal, an R-module M (or R-linear map, R-algebra, etc.) satisfies $\mathcal{P}$ if and only if for each $i \in I$, the localized object M_{f_i} (or corresponding localized object) satisfies $\mathcal{P}$.

Proposition 1.3.9 (Examples of Zariski Local Properties). *The following properties are Zariski local:*

Properties of modules:

(a) Being zero.

(b) Finite generation.

(c) *Finite presentation.*

(d) *Projectivity.*

(e) *Flatness.*

Properties of module homomorphisms:

(f) *Being zero.*

(g) *Injectivity.*

(h) *Surjectivity.*

(i) *Being an isomorphism.*

Properties of algebras:

(j) *Finite generation.*

(k) *Finite presentation.*

Properties of sequences:

(l) *Exactness.*

***Remark* 1.3.10.** Proving that projectivity satisfies Zariski descent is more subtle and beyond our current scope. For a classical reference (in French), see Raynaud-Gruson [RG71, p.81]. Modern approaches typically deduce Zariski descent from faithfully flat descent; see Stacks Project [The25, Tag 058B, Tag 05A5].

Proof. We prove several representative cases. Throughout, let $(f_i)_{i\in I}$ generate the unit ideal of R.

(a) Being zero: A module $M = 0$ if and only if $M_{f_i} = 0$ for all $i \in I$. This follows from the injectivity of $M \hookrightarrow \prod_{i\in I} M_{f_i}$ (Corollary 1.3.3).

(g), (h), (i) Properties of maps: For an R-linear map $\varphi\colon M \to N$:

- φ is injective $\Leftrightarrow \ker(\varphi) = 0 \Leftrightarrow \ker(\varphi)_{f_i} = 0$ for all i (by exactness of localization) $\Leftrightarrow \ker(\varphi_{f_i}) = 0$ for all $i \Leftrightarrow \varphi_{f_i}$ is injective for all i.
- Similarly for surjectivity using $\operatorname{coker}(\varphi)$.
- Being an isomorphism is equivalent to being both injective and surjective.

(e) Flatness: Assume M_{f_i} is a flat R_{f_i}-module for all $i \in I$. To show M is flat over R, consider any injection $\varphi\colon N \hookrightarrow N'$ of R-modules. We must show $\varphi \otimes \mathrm{id}_M \colon N \otimes_R M \to N' \otimes_R M$ is injective.

By Zariski locality of injectivity, it suffices to verify injectivity after localizing at each f_i. Since tensor product commutes with localization:

$$(\varphi \otimes \mathrm{id}_M)_{f_i} \simeq \varphi_{f_i} \otimes \mathrm{id}_{M_{f_i}} \colon N_{f_i} \otimes_{R_{f_i}} M_{f_i} \to N'_{f_i} \otimes_{R_{f_i}} M_{f_i}.$$

Since localization preserves injectivity and M_{f_i} is flat over R_{f_i}, this map is injective. Thus M is flat.

(b) Finite generation: Assume M_{f_i} is finitely generated over R_{f_i} for all i. Choose a finite subset $J \subset I$ such that $(f_j)_{j\in J}$ generates R.

For each $j \in J$, choose finitely many elements $m_{j,1}, \ldots, m_{j,n_j} \in M$ whose images generate M_{f_j} (clearing denominators if necessary). Define

$$\phi_j \colon R^{n_j} \to M, \quad (r_1, \ldots, r_{n_j}) \mapsto \sum_k r_k m_{j,k}.$$

Consider $\Phi = \sum_{j \in J} \phi_j \colon \bigoplus_{j \in J} R^{n_j} \to M$. To show Φ is surjective, it suffices to show Φ_{f_k} is surjective for each $k \in J$ (by Zariski locality of surjectivity). But $(\phi_k)_{f_k}$ is surjective by construction, so Φ_{f_k} is surjective. Thus M is finitely generated.

(c) Finite presentation: Assume M_{f_i} is finitely presented over R_{f_i} for all i. By (b), M is finitely generated. Choose a surjection $\pi \colon R^n \twoheadrightarrow M$ with kernel K, giving an exact sequence

$$0 \to K \to R^n \xrightarrow{\pi} M \to 0.$$

Localizing (exactness is preserved):

$$0 \to K_{f_i} \to R^n_{f_i} \to M_{f_i} \to 0.$$

Since M_{f_i} is finitely presented and $R^n_{f_i}$ is finitely generated free, the kernel K_{f_i} is finitely generated over R_{f_i} (this is the definition of finite presentation). By (b), K is finitely generated over R. Therefore M is finitely presented.

(l) Exactness: A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is exact at M if and only if $\ker(g) = \operatorname{im}(f)$, which holds if and only if $\ker(g)_{f_i} = \operatorname{im}(f)_{f_i}$ for all i (by part (a)), if and only if the localized sequence is exact at M_{f_i} for all i. □

Proposition 1.3.11 (Zariski-Local Nature of Immersions). *Let R be a ring and $u \colon X \to \operatorname{Spec}(A)$ be a map of algebraic functors. Suppose that X satisfies Zariski descent. Let $(f_i)_{i \in I}$ generate the unit ideal in A, and let $u_i \colon X \times_{\operatorname{Spec}(A)} D(f_i) \to D(f_i)$ be the base change of u to $D(f_i)$. For each of the following classes of maps, if each u_i belongs to the class, so does u:*

(i) monomorphism,

(ii) closed immersions,

(iii) open immersions,

(iv) immersions

Proof. (i) *Monomorphisms.* Let $x, y \in X(R)$ be such that $u(x) = u(y) = \varphi \in \operatorname{Spec}(A)(R)$. We want to show that $x = y$. Since X is a Zariski sheaf, for any ring R and a family $(g_j)_{j \in J}$ generating the unit ideal, we have an equalizer diagram:

$$X(R) \longrightarrow \prod_{j \in J} X(R_{g_j}) \rightrightarrows \prod_{j,k \in J} X(R_{g_j g_k})$$

Thus, it suffices to show that for every $j \in J$, we have $x_{g_j} = y_{g_j}$.

Recall that we are given a family $(f_i)_{i \in I}$ generating the unit ideal in A. Since $\varphi \in \operatorname{Spec}(A)(R)$ corresponds to a ring map $\varphi^* \colon A \to R$, consider $g_i := \varphi^*(f_i)$. One can easily check that $(g_i)_{i \in I}$ generates the unit ideal of R. So, we can use the index set I for the equalizer diagram.

Fix $i \in I$ and consider the localization $R \to R_{g_i}$. The map $\varphi_{g_i} \colon \operatorname{Spec}(R_{g_i}) \to \operatorname{Spec}(A)$ factors through $\operatorname{Spec}(A_{f_i}) \simeq D(f_i)$ (by Proposition 1.2.46). Therefore, the pairs (x_{g_i}, φ_{g_i}) and (y_{g_i}, φ_{g_i}) define elements in the fiber product $\left(X \times_{\operatorname{Spec}(A)} D(f_i)\right)(R_{g_i})$. By assumption, u_i is a monomorphism, which implies $x_{g_i} = y_{g_i}$. By the sheaf property, $x = y$.

(ii) *Closed immersions.* Assume that each u_i is a closed immersion. Since $D(f_i) \simeq \operatorname{Spec}(A_{f_i})$, there exists an ideal $K_i \subset A_{f_i}$ such that $X \times_{\operatorname{Spec}(A)} D(f_i) \simeq \operatorname{Spec}(A_{f_i}/K_i)$.

Claim 1.3.12. *There is a unique ideal $K \subset A$ such that $K_{f_i} = K_i$ for all i.*

Proof of the claim. The functor $R \mapsto \{\text{Ideals in } R\}$ satisfies Zariski descent. Indeed, viewing ideals as submodules, by Theorem 1.3.1 (Zariski descent for modules), we can glue the modules K_i to a global module K. To see that K is an ideal (submodule of A), consider the inclusion maps $\iota_i \colon K_i \hookrightarrow A_{f_i}$. By full faithfulness of localization (descent for morphisms), these glue to a unique map $\iota\colon K \to A$. Since injectivity is a local property (kernel being zero is local), ι is injective. □

Let $Y = \mathrm{Spec}(A/K)$. There is a canonical map $Y \to X$ induced by the local isomorphisms $Y \times_{\mathrm{Spec}(A)} D(f_i) \simeq \mathrm{Spec}(A_{f_i}/K_i) \simeq X \times_{\mathrm{Spec}(A)} D(f_i)$.

We verify $X \simeq Y$ by checking the functor of points. For every ring R and morphism $\varphi\colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$:

- *Factorization through Y:* By definition, φ factors through Y iff $\varphi^*(K) = 0$.
- *Factorization through X:* Since $X \subset \mathrm{Spec}(A)$ is a subfunctor, φ factors through X iff it does so locally. Consider the cover $g_i = \varphi^*(f_i)$. The restriction $\varphi|_{D(g_i)}$ factors through X iff the map $A_{f_i} \to R_{g_i}$ kills K_i.

Since $K_{f_i} = K_i$, the condition $\varphi^*(K) = 0$ is equivalent to locally killing K_i. Thus φ factors through X iff it factors through Y. Hence $X \simeq Y$, so u is a closed immersion.

(iii) *Open immersions.* By (1), u is a monomorphism. We must show its image corresponds to an open subscheme $D(J) \subset \mathrm{Spec}(A)$.

Since u_i is an open immersion, its image in $D(f_i) \simeq \mathrm{Spec}(A_{f_i})$ is of the form $D(J_i)$ for some radical ideal $J_i \subset A_{f_i}$. Note that $D(J_i)$ is the complement of $V(J_i)$. The closed sets $V(J_i)$ must glue to a global closed set $V(J)$.

Claim 1.3.13. *There exists a unique radical ideal $J \subset A$ such that $\sqrt{J \cdot A_{f_i}} = J_i$ for all i.*

Proof of the claim. The functor $R \mapsto \{\text{Radical Ideals in } R\}$ satisfies Zariski descent. Specifically, the ideals J_i satisfy the compatibility condition on overlaps: $V(J_i) \cap D(f_j) = V(J_j) \cap D(f_i)$ inside $D(f_i f_j)$. Using the same logic as in the closed immersion case (descent for modules/ideals), we can glue the ideals J_i to a global ideal J'. Let $J = \sqrt{J'}$. Then J is the unique radical ideal with the required local property. □

Let $U = D(J) \subset \mathrm{Spec}(A)$. We verify $\mathrm{im}(u) \simeq U$. Since u is a monomorphism, it suffices to check the image locally. The restriction of U to $D(f_i)$ is $D(J \cdot A_{f_i}) = D(J_i)$, which is exactly the image of u_i. Thus, the image of u is the open subfunctor $D(J)$, so u is an open immersion.

(iv) *Immersions.* Since u_i is an immersion, it factors as a closed immersion followed by an open immersion:

$$X \times_{\mathrm{Spec}(A)} D(f_i) \xrightarrow{\text{closed}} W_i \xrightarrow{\text{open}} D(f_i).$$

This means the image of u_i is a locally closed subset of $D(f_i)$. Algebraically, this corresponds to an ideal $K_i \subset A_{f_i}$ (defining the closure of the image) and a radical ideal $J_i \subset A_{f_i}$ (defining the open neighborhood W_i). Specifically, $X|_{D(f_i)} \simeq V(K_i) \cap D(J_i) \simeq \mathrm{Spec}((A_{f_i}/K_i)_{J_i})$.

By the descent of ideals (from Step 2) and descent of radical ideals (from Step 3):

- The ideals K_i glue to a unique global ideal $K \subset A$.
- The radical ideals J_i glue to a unique global radical ideal $J \subset A$.

Let $Z = V(K) \cap D(J)$. This is a locally closed subscheme of $\mathrm{Spec}(A)$. Locally on $D(f_i)$, Z becomes $V(K_{f_i}) \cap D(J_{f_i}) = V(K_i) \cap D(J_i)$, which is isomorphic to the image of u_i. Since u is a monomorphism and its image coincides with Z locally, u induces an isomorphism $X \simeq Z$. Thus u is an immersion.

□

1.4. Finiteness Properties

Recall that a k-algebra is of *finite presentation* if it is isomorphic to $k[\Sigma]$ where Σ is a system of finitely many polynomial equations in finitely many variables, and it is of *finite type* if it is isomorphic to $k[\Sigma]$ where Σ has finitely many variables (but possibly infinitely many equations). We denote the corresponding full subcategories of CAlg_k by $\mathsf{CAlg}_k^{\mathrm{fp}}$ and $\mathsf{CAlg}_k^{\mathrm{ft}}$. It turns out that these finiteness conditions are encoded geometrically by properties of the functor $\mathrm{Spec}(A)\colon \mathsf{CAlg}_k \to \mathsf{Set}$.

Definition 1.4.1 (Locally of Finite Presentation/Type)**.** Let $X\colon \mathsf{CAlg}_k \to \mathsf{Set}$ be an algebraic k-functor.

(i) X is *locally of finite presentation* if it preserves filtered colimits.

(ii) X is *locally of finite type* if it preserves colimits of filtered diagrams whose transition maps are all injective.

Proposition 1.4.2. *Let k be a ring and A a k-algebra.*

(i) A is of finite presentation if and only if the functor $\mathrm{Spec}(A)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ is locally of finite presentation (i.e., preserves filtered colimits).

(ii) A is of finite type if and only if the functor $\mathrm{Spec}(A)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ is locally of finite type (i.e., preserves filtered colimits of monomorphisms).

Proof. (i) *(Finite Presentation)*

Direction $\Rightarrow$: Assume that A is of finite presentation.

Step 1: Free algebras are compact. Let Σ be a finite set. The functor defined by the polynomial ring $k[\Sigma]$ is given by

$$\mathrm{Hom}_{\mathsf{CAlg}_k}(k[\Sigma], R) \simeq R^{\Sigma} \simeq \prod_{s\in\Sigma} R.$$

Since finite products commute with filtered colimits in Set, for any filtered diagram $F_\bullet\colon \mathcal{I} \to \mathsf{CAlg}_k$, we have

$$\begin{aligned}\mathrm{Hom}(k[\Sigma], \mathrm{colim}_i F_i) &\simeq (\mathrm{colim}_i F_i)^{\Sigma}\\ &\simeq \mathrm{colim}_i (F_i^{\Sigma})\\ &\simeq \mathrm{colim}_i \mathrm{Hom}(k[\Sigma], F_i).\end{aligned}$$

Thus, $k[\Sigma]$ is a compact object.

Step 2: General case via Coequalizer. Since A is of finite presentation, we can choose a presentation

$$k[y_1, \dots, y_m] \rightrightarrows k[x_1, \dots, x_n] \to A,$$

which expresses A as a coequalizer of two maps between free algebras on finite sets. Let $P_1 = k[x_1, \dots, x_n]$ and $P_0 = k[y_1, \dots, y_m]$. We have an exact sequence of functors (equalizer diagram in Set):

$$\mathrm{Hom}(A, -) \to \mathrm{Hom}(P_1, -) \rightrightarrows \mathrm{Hom}(P_0, -).$$

Now consider the filtered colimit. Since filtered colimits in Set commute with finite limits (specifically equalizers), we have the following commutative diagram where the rows are equalizers:

$$\begin{array}{ccccc}\mathrm{colim}_i \mathrm{Hom}(A, F_i) & \longrightarrow & \mathrm{colim}_i \mathrm{Hom}(P_1, F_i) & \rightrightarrows & \mathrm{colim}_i \mathrm{Hom}(P_0, F_i)\\ \downarrow & & \downarrow\wr & & \downarrow\wr\\ \mathrm{Hom}(A, \mathrm{colim}_i F_i) & \longrightarrow & \mathrm{Hom}(P_1, \mathrm{colim}_i F_i) & \rightrightarrows & \mathrm{Hom}(P_0, \mathrm{colim}_i F_i)\end{array}$$

The vertical arrows for P_1 and P_0 are isomorphisms by Step 1. By the universal property of equalizers, the induced map for A (the dashed arrow) must also be an isomorphism.

Direction $\Leftarrow$*:* Conversely, assume that $\mathrm{Spec}(A)$ preserves filtered colimits. Recall that every k-algebra can be written as the canonical filtered colimit of its finitely presented objects over it:

$$A \simeq \mathrm{colim}_{(B,\varphi)\in(\mathsf{CAlg}_k^{\mathrm{fp}})_{/A}} B.$$

By assumption, the functor $\mathrm{Hom}(A, -)$ preserves this colimit:

$$\mathrm{Hom}_{\mathsf{CAlg}_k}(A, A) \simeq \mathrm{Hom}_{\mathsf{CAlg}_k}(A, \mathrm{colim}_B B) \simeq \mathrm{colim}_B \mathrm{Hom}_{\mathsf{CAlg}_k}(A, B).$$

Consider the identity morphism id_A on the left. It must map to some element in the colimit on the right. This implies there exists a finitely presented algebra A_0 (in the diagram) and a morphism $u\colon A \to A_0$ such that the composition

$$A \xrightarrow{u} A_0 \xrightarrow{\varphi} A$$

is equal to the identity id_A. Thus, A is a *retract* of the finitely presented algebra A_0. Since a retract of a finitely presented object is finitely presented, A is of finite presentation.

(ii) *(Finite Type)*

Direction $\Rightarrow$*:* Assume A is of finite type. Then there exists a surjective homomorphism $\pi\colon k[\Sigma] \twoheadrightarrow A$ for some finite set Σ. Finite type corresponds to the property that $\mathrm{Spec}(A)$ is *locally of finite type*, which means the functor $\mathrm{Hom}(A, -)$ preserves filtered colimits of *monomorphisms*.

Let $F_\bullet\colon \mathcal{I} \to \mathsf{CAlg}_k$ be a filtered diagram where all transition maps are injective. We want to show the map

$$\Phi\colon \mathrm{colim}_i \mathrm{Hom}(A, F_i) \to \mathrm{Hom}(A, \mathrm{colim}_i F_i)$$

is bijective.

Injectivity: Since $k[\Sigma] \to A$ is an epimorphism, the map $\mathrm{Hom}(A, R) \to \mathrm{Hom}(k[\Sigma], R)$ is injective. The injectivity of Φ follows from the injectivity of the corresponding map for $k[\Sigma]$ (which is an isomorphism) and the exactness properties of filtered colimits.

Surjectivity: Let $\varphi\colon A \to \mathrm{colim}_i F_i$ be a map. The composition $k[\Sigma] \to A \to \mathrm{colim}_i F_i$ lifts to some map $\psi_j\colon k[\Sigma] \to F_j$ (since $k[\Sigma]$ is compact). We check if ψ_j descends to A, i.e., if it vanishes on the kernel $I = \ker(k[\Sigma] \to A)$. Let $f \in I$ be a generator. Its image maps to 0 in $\mathrm{colim}\, F_i$. Since the transition maps in $F_\bullet$ are *injective*, f must map to 0 in F_j itself (there is no "transient" kernel in the colimit). Since this holds for all generators, ψ_j factors through A, providing the required lift.

□

Example 1.4.3. (i) $\mathbb{A}_k^I$ is locally of finite type if and only if I is a finite set, in which case it is also locally of finite presentation. The same holds for the punctured affine spaces $\mathbb{A}_k^I - 0$.

(ii) The affine k-space $\mathbb{A}(M)$ (Example 1.2.24) is locally of finite presentation (resp. of finite type) if and only if the k-module M is of finite presentation (resp. of finite type).

(iii) The algebraic k-functor $\mathbb{A}^\vee(M)$ (Remark 1.2.25) is locally of finite presentation for any k-module M, since the tensor product functor is left adjoint.

(iv) The affine $\mathbb{Z}$-schemes $*$, $\varnothing$, Idem, $\mathbb{G}_m$, $\mathbb{G}_a$, Mat_n, GL_n, and SL_n are all locally of finite presentation.

***Remark* 1.4.4.** Filtered colimits commute with finite limits in Set, and finite limits in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ are computed pointwise. Consequently, being locally of finite presentation, or locally of finite type, is preserved under finite limits in $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$.

***Remark* 1.4.5** (Compatibility with Base Change). Let $\varphi\colon k \to k'$ be a ring map. Both the forgetful functor $\mathsf{CAlg}_{k'} \to \mathsf{CAlg}_k$ and its left adjoint $\varphi^*\colon \mathsf{CAlg}_k \to \mathsf{CAlg}_{k'}$ preserve filtered colimits, so by Corollary 1.2.29, base change along φ and Weil restriction along φ both preserve the property of being locally of finite presentation or locally of finite type. The third functor $\varphi_\sharp$ usually does not: for instance, $\mathrm{Spec}(\mathbb{C})$ is locally of finite presentation as an affine $\mathbb{R}$-scheme but not as an affine $\mathbb{Q}$-scheme.

Recall that an algebraic k-functor can be thought of as a map of algebraic functors $X \to \mathrm{Spec}(k)$ (Corollary 1.2.35). Remark 1.4.5 implies that, for any cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k), \end{array}$$

if f is locally of finite presentation or of finite type, so is f'. Consequently, we can extend Definition 1.4.1 to arbitrary maps of algebraic functors as follows:

Definition 1.4.6 (Morphism Locally of Finite Presentation/Type). Let $f\colon X \to S$ be a map of algebraic functors.

(i) f is *locally of finite presentation* if, for every ring k and every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite presentation.

(ii) f is *locally of finite type* if, for every ring k and every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite type.

Concretely, under the equivalence of Corollary 1.2.35, the base change of $f\colon X \to S$ along $s\colon \mathrm{Spec}(k) \to S$ is the algebraic k-functor $\mathsf{CAlg}_k \to \mathsf{Set}$ given by

$$(\varphi\colon k \to R) \mapsto \{x \in X(R) \mid f(x) = s(\varphi) \text{ in } S(R)\} = \left\{ \begin{array}{ccc} \mathrm{Spec}(R) & \overset{x}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \mathrm{Spec}(\varphi)} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k) & \overset{s}{\longrightarrow} & S \end{array} \right\}.$$

Example 1.4.7. Any open immersion is locally of finite presentation, and any closed immersion is locally of finite type. A closed immersion $i\colon Z \hookrightarrow X$ is locally of finite presentation if and only if, for every ring R and every $x\colon \mathrm{Spec}(R) \to X$, there exists a *finite* subset $F \subset R$ such that $x^{-1}(i(Z)) = V(F)$.

Proposition 1.4.8 (Closure properties of Morphisms Locally of Finite Presentation/Type). *(i)* ***(Base Change)*** *Consider a cartesian square of algebraic functors*

$$\begin{array}{ccc} Y' & \overset{f'}{\longrightarrow} & X' \\ \downarrow & \lrcorner & \downarrow \\ Y & \overset{f}{\longrightarrow} & X. \end{array}$$

If f is locally of finite presentation, so is f'. The same holds for "locally of finite type".

(ii) ***(Cancellation / 2-out-of-3)*** *Consider a commutative triangle of algebraic functors*

$$\begin{array}{ccccc} Z & & \overset{g}{\longrightarrow} & & Y \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle f} & \\ & & X. & & \end{array}$$

If f is locally of finite presentation, then g is locally of finite presentation if and only if h is. The same holds for "locally of finite type".

Proof. (i) Let f be locally of finite presentation. We want to show that f' is also locally of finite presentation. For any ring k and any k-point $x\colon \operatorname{Spec}(k) \to X'$, we must show that the fiber $X'_x := x^{-1}(Y') \simeq Y' \times_{X'} \operatorname{Spec}(k)$ is locally of finite presentation (as a k-functor).

Consider the following diagram:

$$\begin{array}{ccccc} x^{-1}(Y') & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow f \\ \operatorname{Spec}(k) & \xrightarrow{x} & X' & \longrightarrow & X \end{array}$$

By the **Pasting Lemma**, the outer rectangle is a pullback square. Thus, the fiber $x^{-1}(Y')$ is isomorphic to the fiber of f over the composite point $\operatorname{Spec}(k) \xrightarrow{x} X' \to X$. Since f is locally of finite presentation, its fibers are locally of finite presentation by definition. Thus f' satisfies the condition.

(ii) We prove the equivalence.

($\Rightarrow$) **(Composition)** Assume g is locally of finite presentation. We verify $h = f \circ g$ is locally of finite presentation. Let $z\colon \operatorname{Spec}(k) \to X$ be a point. The fiber $z^{-1}(Z)$ can be computed as:

$$z^{-1}(Z) := Z \times_X \operatorname{Spec}(k) \simeq Z \times_Y (Y \times_X \operatorname{Spec}(k)).$$

Since f is LFP, the fiber $Y_z := z^{-1}(Y) = Y \times_X \operatorname{Spec}(k)$ is LFP over k. Since g is LFP, the base change map $Z \times_Y Y_z \to Y_z$ is LFP (by Base Change, proved in part 1). Since LFP morphisms are closed under composition (a standard algebraic fact: if $A \to B$ is fp and $B \to C$ is fp, then $A \to C$ is fp), the composite $Z \times_Y Y_z \to Y_z \to \operatorname{Spec}(k)$ makes the fiber an LFP k-functor. Thus h is LFP.

($\Leftarrow$) **(Cancellation)** Assume h is locally of finite presentation. We must show that for every k-point $y\colon \operatorname{Spec}(k) \to Y$, the fiber $y^{-1}(Z) = Z \times_Y \operatorname{Spec}(k)$ is locally of finite presentation over k.

Set $x = f \circ y\colon \operatorname{Spec}(k) \to X$. Consider the following diagram of fibers:

$$\begin{array}{ccc} y^{-1}(Z) & \longrightarrow & x^{-1}(Z) \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec}(k) & \xrightarrow{\bar{y}} & x^{-1}(Y) \end{array}$$

where $x^{-1}(Y)$ is the fiber of f over x, $x^{-1}(Z)$ is the fiber of h over x, and $\bar{y}$ is the k-point of $x^{-1}(Y)$ induced by y. This square is cartesian because

$$y^{-1}(Z) = Z \times_Y \operatorname{Spec}(k) \simeq (Z \times_X \operatorname{Spec}(k)) \times_{(Y \times_X \operatorname{Spec}(k))} \operatorname{Spec}(k) = x^{-1}(Z) \times_{x^{-1}(Y)} \operatorname{Spec}(k).$$

We now verify the hypotheses needed to conclude:

Step 1: $\bar{y}$ is LFP. Since f is LFP, the fiber $x^{-1}(Y)$ is an LFP k-functor. In particular, the structure map $x^{-1}(Y) \to \operatorname{Spec}(k)$ is LFP. The map $\bar{y}\colon \operatorname{Spec}(k) \to x^{-1}(Y)$ is a section of this structure map. It corresponds to a *closed immersion of finite presentation*. Concretely, writing $x^{-1}(Y) \simeq \operatorname{Spec}(B)$ locally with B a finitely presented k-algebra, the point $\bar{y}$ corresponds to a surjection $\varphi\colon B \twoheadrightarrow k$. The kernel $I = \ker \varphi$ is a finitely generated ideal (since B is Noetherian-like over k). For any B-algebra R, the fiber is $\operatorname{Spec}(R \otimes_B k) = \operatorname{Spec}(R/IR)$, which is finitely presented over R. Thus $\bar{y}$ is LFP.

Step 2: Conclude by base change and composition. By base change (part 1), since $\bar{y}$ is LFP, the map $y^{-1}(Z) \to x^{-1}(Z)$ is LFP. Since h is LFP, the fiber $x^{-1}(Z) \to \operatorname{Spec}(k)$ is LFP. By the composition result ($\Rightarrow$), the composite $y^{-1}(Z) \to x^{-1}(Z) \to \operatorname{Spec}(k)$ is LFP. Hence $y^{-1}(Z)$ is an LFP k-functor, so g is locally of finite presentation.

□

1.5. Nullstellensatz

Consider a monic polynomial $f \in k[x]$ over a field k. The functorial Nullstellensatz (Corollary 1.2.11) tells us that f is determined by its zero sets in all k-algebras. On the other hand, elementary field theory tells us that f splits into linear factors over some finite field extension of k. Combining the two: knowing the zero sets of f over finite field extensions of k determines f provided it is separable (i.e., has no multiple roots), and in general determines the radical of f — the product of the distinct prime factors of f.

Hilbert's Nullstellensatz generalizes this latter statement to systems of polynomial equations in several variables: given $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, the sets of common zeros in all finite field extensions of k determine the radical ideal $\sqrt{(f_1, \dots, f_m)}$. This was the cornerstone of classical algebraic geometry, which restricted attention to solutions in fields. We record the result here, while pointing out some of its limitations.

For a ring k, define the maps:

$$\{\text{subsets of } k[x_1, \dots, x_n]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{subsets of } k^n\}$$

as follows:

$$V(F) = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in F\},$$
$$I(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

Note that both maps are order-reversing and that $F \subset I(V(F))$ and $X \subset V(I(X))$ (in other words, this is an adjunction between posets). Note also that $I(X)$ is always an ideal in $k[x_1, \dots, x_n]$, and it is even a radical ideal if k is reduced (if a power of f vanishes on X, so does f). Call a subset $X \subset k^n$ *algebraic* if it lies in the image of V, or equivalently if $X = V(I(X))$.

Theorem 1.5.1 (Hilbert's Nullstellensatz). *Let k be an algebraically closed field and let $n \in \mathbb{N}$. For any subset $F \subset k[x_1, \dots, x_n]$, we have:*

$$I(V(F)) = \sqrt{(F)}.$$

Consequently, the maps V and I define a one-to-one correspondence:

$$\{\textit{radical ideals in } k[x_1, \dots, x_n]\} \underset{I}{\overset{V}{\underset{\sim}{\rightleftarrows}}} \{\textit{algebraic subsets of } k^n\}$$

Proof. The inclusion $\sqrt{(F)} \subset I(V(F))$ is clear: if $f^m \in (F)$ for some $m \geq 1$, then f^m vanishes on $V(F)$, so f vanishes on $V(F)$ since k is a field (hence reduced).

For the reverse inclusion, we use the following key lemma:

Lemma 1.5.2 (Key Lemma for Nullstellensatz). *Let k be an algebraically closed field and let $I \subsetneq k[x_1, \dots, x_n]$ be a proper ideal. Then $V(I) \neq \emptyset$.*

Proof of Lemma. Let $\mathfrak{m}$ be a maximal ideal containing I. We claim that $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1, \dots, a_n) \in k^n$.

The quotient $k[x_1, \dots, x_n]/\mathfrak{m}$ is a field extension of k. By Zariski's lemma, this is a finite extension. Since k is algebraically closed, we must have $k[x_1, \dots, x_n]/\mathfrak{m} = k$.

Let a_i be the image of x_i in k. Then $x_i - a_i \in \mathfrak{m}$ for all i, so $(x_1 - a_1, \dots, x_n - a_n) \subset \mathfrak{m}$. But the left ideal is maximal (its quotient is k), so equality holds.

Therefore, $(a_1, \dots, a_n) \in V(\mathfrak{m}) \subset V(I)$. □

Now suppose $f \in I(V(F))$ but $f \notin \sqrt{(F)}$. Consider the ideal $J = (F, 1 - yf) \subset k[x_1, \dots, x_n, y]$ where y is a new variable.

If J were the unit ideal, then we would have:

$$1 = \sum_i g_i f_i + h(1 - yf)$$

for some $g_i, h \in k[x_1, \dots, x_n, y]$ and $f_i \in F$. Setting $y = 1/f$ (where we work in the localization), we get:

$$1 = \sum_i g_i(x, 1/f) f_i$$

which means $f^N \in (F)$ for some N, contradicting $f \notin \sqrt{(F)}$.

Therefore, $J \neq k[x_1, \dots, x_n, y]$. By the lemma, there exists $(a_1, \dots, a_n, b) \in k^{n+1}$ with $(a_1, \dots, a_n, b) \in V(J)$. This means:

- $f_i(a_1, \dots, a_n) = 0$ for all $f_i \in F$
- $1 - bf(a_1, \dots, a_n) = 0$

The first condition says $(a_1, \dots, a_n) \in V(F)$, so $f(a_1, \dots, a_n) = 0$ by assumption. But then the second condition gives $1 = 0$, a contradiction. □

We can upgrade this result to an equivalence of categories as follows. Define the category AffSet_k of *affine algebraic sets* over k as follows:

- An object of AffSet_k is a pair (n, X) with $n \geq 0$ and $X \subset k^n$ an algebraic subset.
- A morphism $(n, X) \to (m, Y)$ is a map $f : X \to Y$ such that there exists a polynomial map $k^n \to k^m$ extending f.

Recall that a ring R is *reduced* if 0 is the only nilpotent element of R. We denote by $\mathsf{CAlg}_k^{\mathrm{red}}$ the category of reduced k-algebras.

Corollary 1.5.3 (Categorical Nullstellensatz). *Let k be an algebraically closed field. Then there is an equivalence of categories:*

$$\mathsf{AffSet}_k \xrightarrow{\sim} (\mathsf{CAlg}_k^{\mathrm{ft,red}})^{\mathrm{op}}, \quad (n, X) \mapsto k[x_1, \dots, x_n]/I(X).$$

Hence, AffSet_k is equivalent to the full subcategory of Aff_k spanned by the reduced affine k-schemes of finite type.

Proof. We must show this functor is fully faithful and essentially surjective.

Fully faithful: Let (n, X) and (m, Y) be affine algebraic sets. A morphism $f : X \to Y$ extendable by polynomials corresponds to a k-algebra map:

$$k[y_1, \dots, y_m]/I(Y) \to k[x_1, \dots, x_n]/I(X).$$

By Theorem 1.5.1, $I(X)$ and $I(Y)$ are radical ideals, so the quotients are reduced. The correspondence is given by sending f to the map induced by the polynomial extension.

Essentially surjective: Any reduced finitely generated k-algebra has the form $k[x_1, \dots, x_n]/I$ where I is a radical ideal. By the Nullstellensatz, $I = I(V(I))$, so this algebra corresponds to the affine algebraic set $(n, V(I))$. □

***Remark* 1.5.4** (Finite Type vs Finite Presentation). By Hilbert's Basissatz, "finite type" and "finite presentation" are equivalent for algebras over a field (and more generally over a Noetherian ring).

We can also formulate a Nullstellensatz for an arbitrary field k as follows. Denote by $\mathsf{Field}_k^{\mathrm{fin}} \subset \mathsf{CAlg}_k$ the full subcategory of finite field extensions of k.

Corollary 1.5.5 (Nullstellensatz for Arbitrary Fields). *Let k be a field and let $n \in \mathbb{N}$. Then there is an order-reversing bijection:*

$$V: \{\textit{radical ideals in } k[x_1, \dots, x_n]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{A}^n : \mathsf{Field}_k^{\mathrm{fin}} \to \mathsf{Set}\}.$$

Proof. A radical ideal $I \subset k[x_1, \dots, x_n]$ determines a vanishing locus:

$$V(I): \mathsf{Field}_k^{\mathrm{fin}} \to \mathsf{Set}, \quad K \mapsto \{x \in K^n \mid f(x) = 0 \text{ for all } f \in I\}.$$

Conversely, given a vanishing locus $X: \mathsf{Field}_k^{\mathrm{fin}} \to \mathsf{Set}$, we can recover the ideal:

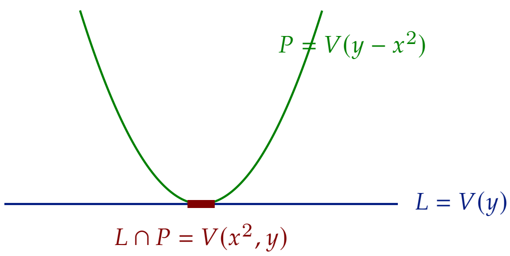
$$I(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } K \in \mathsf{Field}_k^{\mathrm{fin}}, x \in X(K)\}.$$

The key observation is that for any $f \in k[x_1, \dots, x_n]$, we have $f \in I(V(I))$ if and only if f vanishes on $V(I)(K)$ for all finite extensions K/k. This follows from the classical Nullstellensatz applied to the algebraic closure $\overline{k}$, since any zero of f in $\overline{k}$ lies in a finite extension of k. □

Example 1.5.6 (Non-reduced Intersections). Even in the context of algebraic geometry over an algebraically closed field k, there are geometric phenomena that are not captured by only considering solutions in k. Consider for example the vanishing loci $L = V(y)$ and $P = V(y - x^2)$ in $\mathbb{A}_k^2 = \operatorname{Spec}(k[x, y])$.

Since the k-algebras $k[x, y]/(y) \simeq k[t]$ and $k[x, y]/(y - x^2) \simeq k[t]$ are reduced, these affine k-schemes are determined by the algebraic sets $L(k)$ and $P(k)$ in k^2 (by the Nullstellensatz). The intersection $L(k) \cap P(k)$ is the algebraic set $\{(0, 0)\} \subset k^2$, which in turn corresponds to the subfunctor $V(x, y) \subset \mathbb{A}_k^2$.

However, the functorial intersection $L \cap P \subset \mathbb{A}_k^2$ is the subfunctor $V(x^2, y)$, which is isomorphic to $\operatorname{Spec}(k[t]/(t^2))$. One can think of $V(x^2, y)$ as a first-order infinitesimal neighborhood of the origin $V(x, y)$ along the x-axis; this captures the fact that the line L is tangent (to first order) to the parabola P, so that they both contain the same infinitesimal horizontal segment at the origin. This residual tangency information in the intersection can only be seen by evaluating the functor $L \cap P$ on non-reduced k-algebras.



This also resolves another issue in classical algebraic geometry, which is that intersections do not vary nicely in families. Consider for example the family of horizontal lines $L_a = V(y - a)$ for $a \in k$. The intersection $L_a(k) \cap P(k)$ has exactly two points for any $a \neq 0$ (since k is algebraically closed), but only a single point when $a = 0$. On the other hand, the scheme-theoretic intersection $L_a \cap P$ is given by a 2-dimensional k-algebra for all $a \in k$, namely $k \times k$ when $a \neq 0$ and $k[t]/(t^2)$ when $a = 0$.

Example 1.5.7 (Geometry in Mixed Characteristic). Another aspect that is not captured by classical algebraic geometry over fields is algebraic geometry in mixed characteristic, i.e., involving rings R that do not contain any field. This is especially relevant in number theory, which studies rings of integers in finite extensions of $\mathbb{Q}$. Such rings can map to fields with different characteristics, which sometimes allows us to transport results from one characteristic to another.

As a very basic example, consider the following proof that $\sqrt{2}$ is irrational (which is a reformulation of the usual argument). A positive rational number x such that $x^2 = 2$ is the same thing as an element of $X(\mathbb{Z})$ where $X \subset \mathbb{P}^1$ is the solution functor to the homogeneous polynomial equation $x^2 = 2y^2$. Since X is a functor, the ring map $\mathbb{Z} \to \mathbb{Z}/4$ induces a map $X(\mathbb{Z}) \to X(\mathbb{Z}/4)$. Since the squares in $\mathbb{Z}/4$ are 0 and 1, none of the six elements of $\mathbb{P}^1(\mathbb{Z}/4)$ satisfy the equation $x^2 = 2y^2$, so that $X(\mathbb{Z}/4)$ is empty. It follows that $X(\mathbb{Z})$ is also empty, i.e., that there does not exist $x \in \mathbb{Q}$ with $x^2 = 2$.

Looking Ahead

This chapter has built affine geometry from the ground up: algebraic functors as the ambient arena, affine schemes as the representable ones, closed and open subfunctors classified by (radical) ideals, Zariski descent as the gluing mechanism, and the functorial Nullstellensatz as the dictionary translating between algebra and geometry.

Affine geometry is powerful but local. Many objects of central interest — projective curves, complete varieties, moduli spaces of compact objects — are not affine, and their natural home is the projective setting. The next chapter builds exactly the parallel theory: $\mathbb{N}$-graded rings replace commutative rings, the Proj construction replaces Spec, saturated radical homogeneous ideals replace radical ideals, and the functorial projective Nullstellensatz replaces its affine predecessor. The entire web of correspondences will have to be reconstructed, but the guiding principle — “everything is a functor, everything is sheaf-theoretic” — carries across without change.

Chapter 2.

Projective Geometry

The goal of this chapter is to introduce projective geometry and transfer certain results from affine geometry to the projective setting. Specifically, we establish the following analogy:

Affine Geometry	*Projective Geometry*
Affine n-space $\mathbb{A}^n$	Projective n-space $\mathbb{P}^n$
Affine schemes	Projective schemes
Rings	Graded rings
$\mathrm{Spec}\colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$	$\mathrm{Proj}\colon (\mathsf{CAlg}^{\mathbb{N},\mathrm{es}})^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$
Classify open/closed subfunctors of $\mathrm{Spec}(A)$ via ideals in A	Classify open/closed subfunctors of $\mathrm{Proj}(A)$ via homogeneous ideals in A

***Remark* 2.0.1** (Spec vs. Proj)**.** Spec induces an anti-equivalence of categories between rings and affine schemes. By contrast, Proj is *not* fully faithful: many $\mathbb{N}$-graded rings can yield the same Proj (e.g. a ring and any of its Veronese subrings). The classification of closed subfunctors of $\mathrm{Proj}(A)$ is correspondingly subtler: it matches *saturated* radical homogeneous ideals rather than arbitrary ones.

2.1. Projective Spaces

Classically, projective space $\mathbb{P}^n(k)$ over a field is the set of lines through the origin in k^{n+1}. The functor-of-points version replaces "lines" by "line quotients": an R-point of $\mathbb{P}^n$ is a quotient R-line of R^{n+1}. This presentation is manifestly functorial, requires no choice of homogeneous coordinates, and extends the naive picture from fields to arbitrary rings.

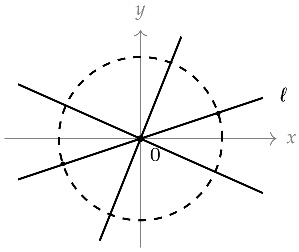
2.1.1. Projective Spaces over Fields

Projective space has two complementary descriptions: classically, as the set of lines through the origin; functorially, as the set of quotient lines. The two perspectives are linear-algebra duals of each other, and the move from "subspaces" to "quotients" is precisely what lets the definition extend from fields to arbitrary commutative rings, where "1-dimensional subspace" is an ill-defined notion but "quotient R-line" remains meaningful. We begin with the classical picture and transition to the functorial one at the end of this section.

Let k be a field and $n \geq -1$. The *projective n-space over k* is the set of all lines through the origin in k^{n+1}:

$$\mathbb{P}^n(k) = \{\text{one-dimensional subspaces of } k^{n+1}\}.$$

Concretely for $n = 1$ and $k = \mathbb{R}$, each point of $\mathbb{P}^1(\mathbb{R})$ corresponds to a line through the origin in the plane; each such line meets the unit circle in exactly two antipodal points, so $\mathbb{P}^1(\mathbb{R}) \simeq S^1/(\pm 1) \simeq S^1$.



Three lines through the origin, each meeting the unit circle in a pair of antipodal points; a single point of $\mathbb{P}^1(\mathbb{R})$ is a full line.

Given a nonzero $(n+1)$-tuple $(a_0, \ldots, a_n) \in k^{n+1} - \{0\}$, we can consider the one-dimensional subspace it generates in k^{n+1}, denoted $[a_0 : \cdots : a_n]$. That is,

$$[a_0 : \cdots : a_n] = [b_0 : \cdots : b_n] \Leftrightarrow \exists \lambda \in k^\times \text{ such that } \lambda(a_0, \ldots, a_n) = (b_0, \ldots, b_n).$$

In this way, we can write $\mathbb{P}^n(k)$ as the set of orbits of the free $k^\times$-action on $k^{n+1} - \{0\}$:

$$(k^{n+1} - \{0\})/k^\times \xrightarrow{\sim} \mathbb{P}^n(k), \quad (a_0, \ldots, a_n) \mapsto [a_0 : \cdots : a_n].$$

In fact, we can view $\mathbb{P}^n(k)$ as the union of $n+1$ subsets $U_0, \ldots, U_n$, where

$$U_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n(k) \mid a_i \text{ is invertible}\}.$$

By the definition of $[a_0 : \cdots : a_n]$, we can assume $a_i = 1$. Therefore, we can identify U_i via:

$$\begin{aligned} U_i &\simeq \mathbb{A}^n(k), \\ [a_0 : \cdots : a_n] &\mapsto \left(\frac{a_0}{a_i}, \ldots, \widehat{\frac{a_i}{a_i}}, \ldots, \frac{a_n}{a_i} \right), \end{aligned}$$

with inverse given by $(a_0, \ldots, \widehat{a_i}, \ldots, a_n) \mapsto [a_0 : \cdots : 1 : \cdots : a_n]$.

The complement $H_i = \mathbb{P}^n(k) - U_i$ is given by

$$H_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n(k) \mid a_i = 0\}.$$

According to the above characterization, we have

$$H_i \xrightarrow{\sim} \mathbb{P}^{n-1}(k), \quad [a_0 : \cdots : a_n] \mapsto [a_0 : \cdots : \widehat{a_i} : \cdots : a_n].$$

Therefore, we can summarize:

- $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$.
- For each $0 \leq i \leq n$, $\mathbb{P}^n(k) = U_i \sqcup H_i$ (disjoint as sets) with $U_i \simeq \mathbb{A}^n(k) = k^n$ and $H_i \simeq \mathbb{P}^{n-1}(k)$. Iterating, we get a stratification

$$\mathbb{P}^n(k) = k^n \sqcup k^{n-1} \sqcup \cdots \sqcup k \sqcup k^0.$$

- Taking $i = 0$, we can view $\mathbb{P}^n(k)$ as the space obtained by compactifying $U_0 = \mathbb{A}^n(k)$ by adding a "point at infinity" for each line through the origin:

  $$[0 : a_1 : \cdots : a_n] = \lim_{\lambda \to \infty} (\lambda a_1, \ldots, \lambda a_n),$$

  with

  $$(\lambda a_1, \ldots, \lambda a_n) = [1 : \lambda a_1 : \cdots : \lambda a_n] = [\lambda^{-1} : a_1 : \cdots : a_n].$$

Some additional observations:

- Choosing a basis identifies k^n with its dual $(k^n)^\vee$. Two natural bijections then arise from a d-dimensional subspace $V \subset k^n$:
  - the orthogonal complement $V^\perp \subset k^n$ (an $(n-d)$-dimensional subspace);
  - the quotient k^n/V (an $(n-d)$-dimensional vector space).

  Composing them gives the bijection

  $$\{d\text{-dim subspaces of } k^n\} \xrightarrow{\sim} \{d\text{-dim quotients of } k^n\}, \quad V \mapsto k^n/V^\perp.$$

  Specializing to lines: $\mathbb{P}^n(k)$ admits two equivalent descriptions, as either the set of 1-dimensional subspaces of k^{n+1} or the set of 1-dimensional quotients of k^{n+1}. Both send a representative $(a_0, \ldots, a_n) \in k^{n+1} - \{0\}$ to the same point $[a_0 : \cdots : a_n]$.

The quotient-line description is the right one to generalize to arbitrary rings, since taking duals is well-behaved only for projective modules.

2.1.2. Vector Spaces and Lines over Rings

Proposition 2.1.1 (Characterization of Vector Spaces). *Let R be a ring and $M \in \mathrm{Mod}_R$ a module. The following conditions are equivalent:*

(i) M is a finitely generated projective module.

(ii) M is a finitely presented flat module.

(iii) There exists $n \in \mathbb{N}$ such that M is a direct summand of R^n.

(iv) There exist $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and each M_{f_i} is a finitely generated free R_{f_i}-module.

(v) M is a dualizable module, i.e., there exists $M^\vee \in \mathrm{Mod}_R$ and linear maps $\mathrm{ev}\colon M^\vee \otimes M \to R$ and $\mathrm{coev}\colon R \to M \otimes M^\vee$ such that the following diagrams commute:

$$\begin{array}{ccccc} & & M \otimes M^\vee \otimes M & & \\ & \overset{\mathrm{coev}\otimes\mathrm{id}}{\nearrow} & & \overset{\mathrm{id}\otimes\mathrm{ev}}{\searrow} & \\ M & & \xlongequal{\quad\quad\quad} & & M \end{array}$$

and

$$\begin{array}{ccccc} & & M^\vee \otimes M \otimes M^\vee & & \\ & \overset{\mathrm{id}\otimes\mathrm{coev}}{\nearrow} & & \overset{\mathrm{ev}\otimes\mathrm{id}}{\searrow} & \\ M^\vee & & \xlongequal{\quad\quad\quad} & & M^\vee \end{array}$$

***Remark* 2.1.2.** In practice, condition (4) is most commonly used, as it is closely related to Zariski descent.

Proof of Proposition 2.1.1.

$(1 \Leftrightarrow 3 \Rightarrow 2)$ Let M be a finitely generated projective module. We must show that there exists $n \in \mathbb{N}$ such that M is a retract of R^n.

Since M is finitely generated, there exists $n \in \mathbb{N}$ and a surjection $\pi\colon R^n \twoheadrightarrow M$. We now construct a section.

Since M is projective, $\mathrm{Hom}_R(M, -)$ is an exact functor, hence preserves surjections. Therefore:

$$\mathrm{Hom}_R(M, R^n) \twoheadrightarrow \mathrm{Hom}_R(M, M).$$

Taking a preimage of id_M gives the desired section, so M is a retract of R^n.

Conversely, projective modules are closed under retracts, and R^n is obviously projective; this gives $(2 \Rightarrow 1)$. The implication $(1) \Rightarrow (2)$ is immediate as well: a retract of R^n is finitely presented and flat.

$(2 \Rightarrow 4)$ We first establish a local-freeness claim.

Claim 2.1.3. *If M is a finitely presented flat module, then for any maximal ideal $\mathfrak{m} \subset R$, $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module.*

Proof of claim. Take a maximal ideal $\mathfrak{m} \subset R$. We have $M/\mathfrak{m}M$ is an $R/\mathfrak{m}$-module. Since $\mathfrak{m}$ is maximal, $R/\mathfrak{m}$ is a field. Therefore there exists $n \in \mathbb{N}$ such that $(R/\mathfrak{m})^n \simeq M/\mathfrak{m}M$.

Consider the commutative diagram:

$$\begin{array}{ccc} (R/\mathfrak{m})^n & \xrightarrow{\sim} & M/\mathfrak{m}M \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ R_\mathfrak{m}^n & \xrightarrow{f} & M_\mathfrak{m} \end{array}$$

Surjectivity is immediate from the construction; we turn to injectivity.

By flatness of M, we have $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$. Therefore, the sequence

$$0 \to \ker(f) \otimes_R R/\mathfrak{m} \to (R/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0$$

is exact.

Since M is finitely presented over R, $M_\mathfrak{m}$ is finitely presented over $R_\mathfrak{m}$. Consider the exact sequence:

$$R^m \to R^n \to M \to 0.$$

Localizing gives:

$$R_\mathfrak{m}^m \to R_\mathfrak{m}^n \to M_\mathfrak{m} \to 0,$$

which is also exact. Thus $\ker(f)$ is a finitely generated $R_\mathfrak{m}$-module.

Therefore:

$$\ker(f) \otimes_{R_\mathfrak{m}} (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) \simeq \ker(f) \otimes_R R/\mathfrak{m} = 0.$$

By Nakayama's lemma over $R_\mathfrak{m}$, we have $\ker(f) = 0$. □

Now we use the claim to prove (4). Recall that $M_\mathfrak{m} = M[(R - \mathfrak{m})^{-1}]$, so:

$$M_\mathfrak{m} = \mathrm{colim}_{f \notin \mathfrak{m}} M_f.$$

Since this is a filtered colimit, a basis of $M_\mathfrak{m}$ lifts to M_f for some $f \notin \mathfrak{m}$. As $\mathfrak{m}$ ranges over all maximal ideals, the chosen elements f collectively avoid every maximal ideal, hence generate the unit ideal of R; finitely many of them already suffice (since M is finitely presented). This produces the desired covering family.

$(4 \Rightarrow 1)$ Use that finite generation and projectivity are Zariski local properties (see Proposition 1.3.9), together with the fact that free modules are automatically projective.

$(4 \Rightarrow 5)$ Recall that M is dualizable iff the canonical map $N \otimes_R M^\vee \to \mathrm{Hom}_R(M, N)$ is an isomorphism for all N. This is a Zariski-local condition on R, so by hypothesis we may pass to each R_{f_i}, where M_{f_i} is free of finite rank — and free modules are visibly dualizable.

$(5 \Rightarrow 1)$ Use that $\mathrm{Hom}_R(M, -)$ preserves colimits when M is dualizable.

□

The equivalent characterizations above justify treating dualizable R-modules as the algebro-geometric analogue of finite-dimensional vector spaces.

Definition 2.1.4 (Vector Spaces over Rings)**.** Let R be a ring. We say that an R-module M is a *vector space* if M satisfies the equivalent conditions in Proposition 2.1.1. We denote by

$$\mathsf{Vect}_R \subset \mathrm{Mod}_R$$

the full subcategory spanned by all R-vector spaces.

***Remark* 2.1.5.** Note that when $R = k$ is a field, the resulting vector spaces are precisely the *finite-dimensional* vector spaces over k.

Over a field, all bases of a vector space have the same cardinality, called its dimension or rank. To extend this notion over a general ring R, we test against *R-fields* — fields equipped with a ring homomorphism from R.

Definition 2.1.6 (Rank)**.** Let R be a ring, M an R-module, and κ an R-field. The *rank* $\mathrm{rk}_\kappa(M)$ of M is the rank of $M \otimes_R \kappa$.

We say that M has *constant rank* if there exists r such that for every R-field κ, we have $\mathrm{rk}_\kappa(M) = r$.

***Remark* 2.1.7.** We have the following relations:

- $\mathrm{rk}_\kappa(M \oplus N) = \mathrm{rk}_\kappa(M) + \mathrm{rk}_\kappa(N)$.
- $\mathrm{rk}_\kappa(M \otimes N) = \mathrm{rk}_\kappa(M) \cdot \mathrm{rk}_\kappa(N)$.
- $\mathrm{rk}_\kappa(R^n) = n$.

Proposition 2.1.8 (Properties of Rank)**.** *Let R be a ring.*

(i) For an R-vector space V, we have $V = 0$ if and only if V has constant rank 0.

(ii) For a short exact sequence in Vect_R:

$$0 \to U \to V \to W \to 0,$$

and any R-field κ, we have $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(U) + \mathrm{rk}_\kappa(W)$.

Proof. (i) The direction $V = 0 \Rightarrow \mathrm{rk}_\kappa(V) = 0$ is obvious. Now take V with constant rank 0. By Proposition 2.1.1, choose $f_1, \ldots, f_n$ such that $(f_1, \ldots, f_n) = R$. Then for any κ:

$$\mathrm{rk}_\kappa(V_{f_i}) = 0,$$

which means $V_{f_i} = 0$. By Zariski descent, $V = 0$.

(ii) By Proposition 2.1.1, vector spaces are projective modules. Therefore the short exact sequence

$$0 \to U \to V \to W \to 0$$

splits, reducing to Remark 2.1.7.

□

***Remark* 2.1.9.** Note that having constant rank 0 implies $V = 0$ only for R-vector spaces. For general modules, there are counterexamples. For instance, take $R = \mathbb{Z}$; the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ has rank 0.

Recall that the *residue field* at a prime ideal $\mathfrak{p} \subset R$ is the R-field

$$\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \simeq \mathrm{Frac}(R/\mathfrak{p}).$$

The residue fields are precisely the initial objects of the connected components of Field_R: for any R-field κ, the kernel $\mathfrak{p} := \ker(R \to \kappa)$ is prime, and the map $R \to \kappa$ factors uniquely through the residue field as

$$R \to \kappa(\mathfrak{p}) \hookrightarrow \kappa.$$

In other words, Field_R decomposes as

$$\mathsf{Field}_R = \coprod_{\mathfrak{p} \subseteq R} \mathsf{Field}_{\kappa(\mathfrak{p})}.$$

Using the fact that scalar extension between fields preserves dimension, we deduce that for κ lying in $\mathsf{Field}_{\kappa(\mathfrak{p})}$, we have $\mathrm{rk}_\kappa(M) = \mathrm{rk}_{\kappa(\mathfrak{p})}(M)$.

Next, we discuss what "one-dimensional" vector spaces are. Rather than thinking of "one-dimensional" in terms of dimension, we prefer to think of it as something invertible.

Proposition 2.1.10 (Characterization of Lines). *Let R be a ring. For any R-module L, the following conditions are equivalent:*

(i) L is a vector space with constant rank 1.

(ii) L is an invertible module, i.e., there exists an R-module $L^\vee$ such that $L \otimes_R L^\vee \simeq R$.

***Remark* 2.1.11.** An invertible module is automatically a vector space — as is its inverse.

Proof of Proposition 2.1.10. (1 ⇒ 2) Let L be an R-vector space. For $f \in R$, we show that $(L^\vee)_f = (L_f)^\vee$.

By Proposition 2.1.1, L is finitely presented. Therefore we have a right exact sequence:

$$R^m \to R^n \to L \to 0.$$

Applying $\mathrm{Hom}_R(-, R)$ gives a left exact sequence:

$$0 \to \mathrm{Hom}_R(L, R) \to \mathrm{Hom}_R(R^n, R) \to \mathrm{Hom}_R(R^m, R).$$

Since R_f is flat over R, we obtain a left exact sequence:

$$0 \to \mathrm{Hom}_R(L, R)_f \to \mathrm{Hom}_R(R^n, R)_f \to \mathrm{Hom}_R(R^m, R)_f.$$

Localizing the right exact sequence first and then applying $\mathrm{Hom}_{R_f}(-, R_f)$ gives:

$$0 \to \mathrm{Hom}_{R_f}(L_f, R_f) \to \mathrm{Hom}_{R_f}(R_f^n, R_f) \to \mathrm{Hom}_{R_f}(R_f^m, R_f).$$

Since $\mathrm{Hom}_R(R^n, R) \simeq R^n$ for any ring R, using the five lemma gives the isomorphism:

$$\mathrm{Hom}_R(L, R)_f \simeq \mathrm{Hom}_{R_f}(L_f, R_f).$$

Since $L^\vee = \mathrm{Hom}_R(L, R)$, the localization $(L^\vee)_f \simeq (L_f)^\vee$ commutes with $(-)^\vee$. By Proposition 2.1.1(4), choose $f_1, \ldots, f_n$ with $(f_1, \ldots, f_n) = R$ and each L_{f_i} free of rank 1 over R_{f_i}. Then $L_{f_i} \otimes_{R_{f_i}} L_{f_i}^\vee \simeq R_{f_i}$, and Zariski descent (Proposition 2.1.1) glues these isomorphisms to $L \otimes_R L^\vee \simeq R$.

$(2 \Rightarrow 1)$ Invertible modules are automatically dualizable, so it remains to check the rank. By multiplicativity of rank,

$$\mathrm{rk}_\kappa(L) \cdot \mathrm{rk}_\kappa(L^\vee) = \mathrm{rk}_\kappa(L \otimes_R L^\vee) = \mathrm{rk}_\kappa(R) = 1.$$

Since ranks are non-negative integers, both factors equal 1.

□

Therefore, we establish the principle: in R-vector spaces, "one-dimensional" = invertible.

Definition 2.1.12 (Lines). Let R be a ring. A *line over* R is an invertible R-module M. We denote by

$$\mathsf{Line}_R \subset \mathsf{Vect}_R$$

the full subcategory spanned by all R-lines.

Lines over R need not be trivial in general:

Example 2.1.13 (Non-trivial Line). Let $R = \mathbb{Z}[\sqrt{-5}] \subseteq \mathbb{Q}(\sqrt{-5}) = \mathbb{Q} \oplus \mathbb{Q}\sqrt{-5} \subseteq \mathbb{C}$. This is a classical example of a non-UFD, since:

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3.$$

Consider the ideal:

$$I = (2, 1 + \sqrt{-5}) = (2, 1 - \sqrt{-5}) \subseteq R.$$

We can construct an R-linear map:

$$I \otimes_R I \longrightarrow (2) \subseteq R, \quad (i, j) \mapsto i \cdot j.$$

Since (2) is principal, $(2) \simeq R$. We claim the above map is an isomorphism. Consider the map:

$$(2) \longrightarrow I \otimes_R I, \quad 2 \mapsto (1 + \sqrt{-5}) \otimes (1 - \sqrt{-5}) - 2 \otimes 2.$$

First, the composition $(2) \to (2)$ is the identity. For the composition $I \otimes_R I \to I \otimes_R I$, write $I \otimes_R I = (2, 1 + \sqrt{-5}) \otimes (2, 1 - \sqrt{-5})$ and check on the four generators:

$$2 \otimes 2, \quad 2 \otimes (1 - \sqrt{-5}), \quad (1 + \sqrt{-5}) \otimes 2, \quad (1 + \sqrt{-5}) \otimes (1 - \sqrt{-5}).$$

To show I is non-trivial, we show it is not principal. If $a + b\sqrt{-5}$ divides 2, then so does its conjugate $a - b\sqrt{-5}$. Thus $a^2 + 5b^2$ divides 4. The only possibilities are $b = 0$ and $a = \pm 2$ or ± 1. None of these combinations generate I.

2.1.3. Maps Between Vector Spaces

Next, we discuss morphisms between vector spaces. We want to understand what injectivity and surjectivity mean for maps between vector spaces.

In fact, for vector spaces, surjectivity remains surjectivity. Let M and N be R-vector spaces, and $f: M \twoheadrightarrow N$ a surjection. We have an exact sequence:

$$0 \to \ker(f) \to M \to N \to 0.$$

Since N is a vector space, hence projective, the sequence splits. Thus $\ker(f)$ is a retract of the projective module M, hence also projective, thus a vector space.

However, if f is injective, $\operatorname{coker}(f)$ need not be a vector space. For example, $\mathbb{Z} \overset{2}{\hookrightarrow} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$ is not a vector space.

Concretely, we need a refined notion of injectivity dual to surjectivity — one that ensures the cokernel remains a vector space.

Definition 2.1.14 (Universal Injectivity). Let R be a ring, and M, N be R-modules. We say that an R-linear map $\varphi: M \to N$ is *universally injective* if for any ring homomorphism $R \to S$, the map $\varphi \otimes_R S$ is injective.

***Remark* 2.1.15.** The prefix "universally" generally signals that a property is preserved under arbitrary base change.

Universal injectivity is the right notion of injectivity in the context of vector spaces, as the propositions below confirm.

***Remark* 2.1.16.**

- If $f: M \hookrightarrow N$ is universally injective, it is automatically injective (apply $\otimes_R R$). In fact, for any R-module P, $f \otimes_R P$ is injective. This follows by considering the symmetric algebra $\operatorname{Sym}_R(P)$ and the canonical map $R \to \operatorname{Sym}_R(P)$.
- Due to right exactness of tensor products, there is no need to consider "universal surjectivity."

Example 2.1.17 (Universal Injectivity Examples).

(i) Any map with a retraction is universally injective.

(ii) Maps without retractions can also be universally injective, e.g., $\bigoplus_{\mathbb{N}} \mathbb{Z} \hookrightarrow \prod_{\mathbb{N}} \mathbb{Z}$.

(iii) Let $f \in R$. The map $R \xrightarrow{\cdot f} R$ is injective if and only if f is not a zero divisor. Universal injectivity requires f to be a unit (invertible element), since you can always base change to $R/(f)$ to make f a zero divisor.

Finite presentation, finite generation, and projectivity are all Zariski-local conditions, so being an R-vector space (resp. an R-line) is itself Zariski-local. Since exactness of module sequences is also Zariski-local, the same holds for universal injectivity.

Definition 2.1.18 (Subspaces and Quotient Spaces). Let R be a ring and M an R-module.

- We say that a module N is a *subspace* of M if $N \subset M$ is a submodule, N is a vector space, and the embedding $N \hookrightarrow M$ is universally injective.
- We say that a module N is a *quotient space* of M if N is a quotient module of M and N is a vector space.

The following proposition characterizes subspaces and quotient spaces.

Proposition 2.1.19 (Characterization of Subspaces). *Let R be a ring, and U, V be R-vector spaces. For an R-linear map $f : U \to V$, the following are equivalent:*

(i) f is universally injective.

(ii) For any R-field κ, the base change $f \otimes_R \kappa$ is injective.

(iii) f is injective and $\operatorname{coker}(f)$ is a vector space.

(iv) f has a retraction.

(v) The dual map $f^\vee : V^\vee \to U^\vee$ is surjective.

Proof.

- First, by Example 2.1.17, we immediately get $(4 \Rightarrow 1)$.
- By definition, $(1 \Rightarrow 2)$.
- Since $\operatorname{coker}(f)$ is a vector space, the exact sequence

$$0 \to U \to V \to \operatorname{coker}(f) \to 0$$

splits, so f has a retraction, giving $(3 \Rightarrow 4)$.

- If f has a retraction, this means the exact sequence

$$0 \to U \to V \to \operatorname{coker}(f) \to 0$$

splits, so $\operatorname{coker}(f)$ is also a retract of V, hence $\operatorname{coker}(f)$ is a vector space, proving $(4 \Rightarrow 3)$.

- If the dual map $f^\vee : V^\vee \to U^\vee$ is surjective, then for any R-module M, $- \otimes_R M$ preserves surjections, so $\operatorname{Hom}_R(V, N) \twoheadrightarrow \operatorname{Hom}_R(U, N)$ is surjective. Taking $N = U$ and a preimage of $\mathrm{id}_U \in \operatorname{Hom}_R(U, U)$ gives the retraction, so $(5 \Rightarrow 4)$.
- For any R-field κ, $f \otimes_R \kappa$ is automatically a κ-linear map, so it is injective if and only if

$$(f \otimes_R \kappa)^\vee = f^\vee \otimes_R \kappa$$

is surjective. For a maximal ideal $\mathfrak{m}$, take $\kappa(\mathfrak{m}) = R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. Then:

$$f^\vee \otimes_R \kappa(\mathfrak{m}) = f_\mathfrak{m}^\vee \otimes_{R_\mathfrak{m}} (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) = V_\mathfrak{m}^\vee/\mathfrak{m}V_\mathfrak{m}^\vee \to U_\mathfrak{m}^\vee/\mathfrak{m}U_\mathfrak{m}^\vee$$

is surjective. Therefore $\operatorname{coker}(f_\mathfrak{m}^\vee) \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} = 0$. Since $V_\mathfrak{m}$ is finitely generated over $R_\mathfrak{m}$, by Nakayama's lemma, $f_\mathfrak{m}^\vee$ is surjective.

Since localizing at $\mathfrak{m}$ is the filtered colimit of localizations at $g \notin \mathfrak{m}$, there exists $g \notin \mathfrak{m}$ such that $f^\vee \otimes_R R_g$ is surjective (more precisely, one should take stalks in the module sheaf). Using that surjectivity is a Zariski local property gives $(2 \Rightarrow 5)$.

□

Dually, we can give the quotient space version.

Proposition 2.1.20 (Characterization of Quotient Spaces). *Let R be a ring, and U, V be R-vector spaces. For an R-linear map $f : U \to V$, the following are equivalent:*

(i) f is surjective.

(ii) For any R-field κ, the base change $f \otimes_R \kappa$ is surjective.

(iii) f is surjective and $\ker(f)$ is a vector space.

(iv) f has a section.

(v) The dual map $f^\vee : V^\vee \to U^\vee$ is universally injective.

We immediately obtain the following corollaries.

Corollary 2.1.21 (Isomorphisms Detected on Fields). *Let R be a ring, and V, W be R-vector spaces. Then $f: V \to W$ is an isomorphism if and only if for any R-field κ, $f \otimes_R \kappa$ is an isomorphism.*

Proof. Note that universal injectivity + surjectivity = isomorphism. □

Corollary 2.1.22 (Pigeonhole for Vector Spaces). *Let R be a ring, and V, W be R-vector spaces with the same rank. For an R-linear map $f: V \to W$, the following are equivalent:*

(i) f is an isomorphism.

(ii) f is surjective.

(iii) f is universally injective.

Proof. Since V and W have the same rank, for all R-fields κ, we have $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(W)$. If $f \otimes_R \kappa$ is injective/surjective, by the pigeonhole principle, f is an isomorphism. □

Corollary 2.1.23 (Duality Correspondence). *Let R be a ring and V an R-vector space. Then dualization induces bijections*

$$\{\textit{subspaces of } V\} \xrightarrow{\sim} \{\textit{quotient spaces of } V^\vee\}, \qquad (U \hookrightarrow V) \mapsto (V^\vee \twoheadrightarrow U^\vee),$$
$$\{\textit{quotient spaces of } V\} \xrightarrow{\sim} \{\textit{subspaces of } V^\vee\}, \qquad (V \twoheadrightarrow W) \mapsto (W^\vee \hookrightarrow V^\vee).$$

Proof. By Definition 2.1.18, a subspace of V is a universally injective map $U \hookrightarrow V$ from an R-vector space, and a quotient space is a surjection $V \twoheadrightarrow W$ onto an R-vector space. Proposition 2.1.19(v) and Proposition 2.1.20(v) tell us that dualization swaps these two classes:

$$\text{universally injective } f \iff f^\vee \text{ surjective}, \qquad f \text{ surjective} \iff f^\vee \text{ universally injective}.$$

Vector spaces are reflexive — by Proposition 2.1.1, the canonical map $V \to V^{\vee\vee}$ is an isomorphism — so applying $(-)^\vee$ twice recovers the original inclusion or surjection. The two assignments are therefore mutual inverses. □

2.1.4. Projective Space

Now we can define projective space.

Definition 2.1.24 (Projective Space). Let k be a ring and I a set. The *projective I-space over k* is the algebraic k-functor:

$$\mathbb{P}^I_k : \mathsf{CAlg}_k \to \mathsf{Set}, \quad R \mapsto \{\text{quotient lines of } R^{(I)}\}.$$

When $k = \mathbb{Z}$, we simply write $\mathbb{P}^I$. For $n \geq -1$, the *projective n-space over k* is $\mathbb{P}^n_k = \mathbb{P}^{\{0,\dots,n\}}_k$. When $n = 1$, we also call it the *projective line*; when $n = 2$, the *projective plane*.

***Remark* 2.1.25.** (i) $\mathbb{P}^{-1}_k = \mathbb{P}^{\emptyset}_k$ is the empty k-scheme $\emptyset$ (note that it is not the initial object of $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$).

(ii) $\mathbb{P}^0_k = \mathrm{Spec}(k)$ is the terminal k-scheme.

(iii) If I is finite, by Corollary 2.1.23, we have:

$$\mathbb{P}^I_k(R) \simeq \{\text{subspace lines of } R^I\}.$$

However, when I is infinite, $R^{(I)}$ is no longer a vector space, so the definition must use quotient lines.

Notation 2.1.26 (Projective Coordinates). Let R be a ring.

- Let L be a line over R.
- Let $(a_0, \dots, a_n) \in L^{n+1}$ be a family of elements in L that generate the module.

This is equivalent to saying that the R-linear map induced by the a_i:

$$R^{n+1} \to L, \quad e_i \mapsto a_i$$

is surjective. The *coimage* defines a *quotient line* of R^{n+1}, which we denote:

$$[a_0 : \cdots : a_n] \in \mathbb{P}^n(R).$$

For another line M and generating family $(b_0, \dots, b_n) \in M^{n+1}$, we define:

$$[a_0 : \cdots : a_n] = [b_0 : \cdots : b_n]$$

if and only if there exists a unique isomorphism $\varphi\colon L \xrightarrow{\sim} M$ such that $\varphi(a_i) = b_i$ for all $0 \le i \le n$.

2.1.5. Relationship Between Affine and Projective Spaces

We now relate affine I-space and projective I-space. The bridge between them is the *punctured affine I-space* $\mathbb{A}^I_k - 0$.

Recall that the *punctured affine I-space* is the functor defined by:

$$(\mathbb{A}^I_k - 0)\colon R \mapsto \{a \in R^I \mid (a) = R\}.$$

Here (a) denotes the ideal in R generated by all components of $a = (a_i)_i \in R^I$.

Clearly $\mathbb{A}^I_k - 0$ is a subfunctor of $\mathbb{A}^I_k$. By duality, an I-tuple in R generating the unit ideal is the same datum as a surjection $R^{(I)} \twoheadrightarrow R$, and remembering only the kernel produces a quotient line of $R^{(I)}$. This gives a canonical morphism

$$\mathbb{A}^I_k - 0 \longrightarrow \mathbb{P}^I_k.$$

In summary, we have the following relationship diagram:

$$\mathbb{A}^I_k \hookleftarrow (\mathbb{A}^I_k - 0) \to \mathbb{P}^I_k.$$

In particular, for $n \ge -1$, we have:

$$\mathbb{A}^{n+1}_k \hookleftarrow (\mathbb{A}^{n+1}_k - 0) \to \mathbb{P}^n_k.$$

Proposition 2.1.27 (Trivial Quotient Lines). *Let I be a set and R a ring. Then the morphism $\mathbb{A}^I - 0 \to \mathbb{P}^I$ induces an injection:*

$$(\mathbb{A}^I - 0)(R)/R^\times \hookrightarrow \mathbb{P}^I(R).$$

Here $R^\times$ acts on $(\mathbb{A}^I - 0)(R)$ by scalar multiplication. The image of this map is precisely the trivial quotient lines *of $R^{(I)}$ (i.e., quotient lines isomorphic to R).*

In particular, if all lines over R are trivial (e.g., R is a local ring or a principal ideal domain), then this map is a bijection.

Proof. Elements of $(\mathbb{A}^I - 0)(R)$ correspond to surjections $u\colon R^{(I)} \twoheadrightarrow R$, and the morphism $\mathbb{A}^I - 0 \to \mathbb{P}^I$ sends such a u to its quotient line $[u] \in \mathbb{P}^I(R)$ — manifestly the trivial line R.

Injectivity: We need to characterize when two surjections $a, b\colon R^{(I)} \twoheadrightarrow R$ define the same point in $\mathbb{P}^I(R)$. By definition, $[a] = [b]$ if and only if there exists an isomorphism $\varphi\colon R \xrightarrow{\sim} R$ making the following diagram commute:

$$\begin{array}{ccc} R^{(I)} & \xrightarrow{\;a\;} & R \\ & {\scriptstyle b}\searrow \quad \swarrow{\scriptstyle \varphi,\ \simeq} & \\ & R & \end{array}$$

Note that $\mathrm{Aut}_R(R) \simeq R^\times$, so φ must be multiplication by some unit $\lambda \in R^\times$. Therefore, $[a] = [b]$ if and only if there exists $\lambda \in R^\times$ such that $b = \lambda a$ in $(\mathbb{A}^I - 0)(R)$. This is precisely the equivalence relation in $(\mathbb{A}^I - 0)(R)/R^\times$.

Image: Clearly, any element from $(\mathbb{A}^I - 0)(R)$ gives a quotient line with target R, i.e., a trivial quotient line. Conversely, if some quotient line L of $R^{(I)}$ is trivial, then there exists an isomorphism $L \simeq R$. Composing $R^{(I)} \twoheadrightarrow L \xrightarrow{\sim} R$ gives an element of $(\mathbb{A}^I - 0)(R)$. □

***Remark* 2.1.28** ($\mathbb{G}_m$-Action and Quotient Functors)**.** The scalar action of $R^\times$ on $(\mathbb{A}^I - 0)(R)$ is functorial in R, so it upgrades to an action of the multiplicative group scheme $\mathbb{G}_m$ on $\mathbb{A}^I - 0$. The previous proposition then promotes the comparison map to an injection of algebraic functors

$$(\mathbb{A}^I - 0)/\mathbb{G}_m \hookrightarrow \mathbb{P}^I,$$

which becomes an isomorphism on $\mathrm{Spec}(R)$ whenever every line over R is trivial.

By Proposition 2.1.1, every quotient line L over R becomes free after passing to a Zariski cover $\{R \to R_{f_i}\}_{i=1}^n$ with $(f_1, \dots, f_n) = R$. Consequently, $(\mathbb{A}^I - 0)/\mathbb{G}_m \hookrightarrow \mathbb{P}^I$ is surjective Zariski-locally, and after sheafification we obtain an isomorphism

$$(\mathbb{A}^I - 0)/\mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^I$$

of Zariski sheaves.

***Remark* 2.1.29** (Functoriality and Topological Property Differences)**.** (i) *Functoriality.* The assignment $I \mapsto \mathbb{A}^I$ upgrades to a contravariant functor $\mathbb{A}^-\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$, but $I \mapsto (\mathbb{A}^I - 0)$ and $I \mapsto \mathbb{P}^I$ do not extend that far: they are contravariant only on the subcategory $\mathsf{Set}^{\mathrm{surj}}$ of sets and surjections. Indeed, pulling back a quotient line of $R^{(I)}$ to a quotient line of $R^{(J)}$ requires a surjection $J \twoheadrightarrow I$ to begin with.

(ii) *Quasi-compactness.* Topologically, $\mathbb{A}^I$ and $\mathbb{P}^I$ behave very differently: $\mathbb{A}^I$ is quasi-compact for every set I, even infinite, whereas $\mathbb{P}^I$ (and $\mathbb{A}^I - 0$) is quasi-compact if and only if I is finite. Quasi-compactness and quasi-separatedness are the central finiteness conditions in algebraic geometry — the six-functor formalism for étale cohomology, for instance, is typically developed for qcqs schemes — so most classical results in projective geometry are stated only in the finite case.

2.2. Graded Rings

Just as affine schemes are represented by rings, projective schemes will be represented by $\mathbb{N}$-graded rings. This section gathers the basic vocabulary — homogeneous polynomials, Veronese subrings, localisations of graded rings, and eventually surjective maps — all tools we will need to define Proj properly.

Definition 2.2.1 (Homogeneous Polynomials). Let k be a ring, I a set, and $d \in \mathbb{N}$. We say that a polynomial $f \in k[x_i \mid i \in I]$ is a *homogeneous polynomial of degree d* if it is a k-linear combination of monomials of the form $\prod_{i\in I} x_i^{n_i}$ with $\sum_{i\in I} n_i = d$. We denote by $k[x_i \mid i \in I]_d \subset k[x_i \mid i \in I]$ the k-submodule spanned by all homogeneous polynomials of degree d.

Every polynomial decomposes uniquely as a sum of homogeneous components,

$$k[x_i \mid i \in I] = \bigoplus_{d\in\mathbb{N}} k[x_i \mid i \in I]_d,$$

and multiplication respects degrees: a degree-d polynomial times a degree-e polynomial is homogeneous of degree $d + e$.

This property motivates the concept of graded rings.

Definition 2.2.2 (Graded Rings). Let $(\Gamma, +, 0)$ be a commutative monoid (e.g., $\mathbb{N}$ or $\mathbb{Z}$). A Γ-*graded ring* is a ring A together with, for each $\gamma \in \Gamma$, a subgroup $A_\gamma \subset A$ such that:

- A decomposes as a direct sum: $\bigoplus_{\gamma\in\Gamma} A_\gamma \xrightarrow{\sim} A$.
- $1 \in A_0$ and for $\gamma, \delta \in \Gamma$, we have $A_\gamma A_\delta \subset A_{\gamma+\delta}$.

For $\gamma \in \Gamma$, we call elements of A_γ *homogeneous elements of degree γ*. For a ring k, a Γ-*graded k-algebra* is a Γ-graded ring A with a ring homomorphism $k \to A$ whose image lies in A_0.

Definition 2.2.3 (Graded Modules). Let A be a Γ-graded ring. A Γ-*graded A-module* is an A-module M together with, for each $\gamma \in \Gamma$, a subgroup $M_\gamma \subset M$ such that:

- M decomposes as a direct sum: $\bigoplus_{\gamma\in\Gamma} M_\gamma \xrightarrow{\sim} M$.
- For all $\gamma, \delta \in \Gamma$, we have $A_\gamma M_\delta \subset M_{\gamma+\delta}$.

An ideal $I \subset A$ is *homogeneous* if $I = \bigoplus_{\gamma\in\Gamma}(I \cap A_\gamma)$ — equivalently, if I is a graded A-submodule of A.

Remark **2.2.4** (Properties of Graded Rings and Modules). Let A be a Γ-graded ring and M a Γ-graded A-module.

(i) A_0 is a subring of A, each A_γ is an A_0-module, and each M_γ is also an A_0-module.

(ii) Let $\Gamma \to \Delta$ be a homomorphism of commutative monoids. Then A can be made into a Δ-graded ring by: for $\delta \in \Delta$, defining

$$A_\delta := \bigoplus_{\gamma\mapsto\delta} A_\gamma.$$

Similarly, M can be made into a Δ-graded A-module in the same way. For example, an $\mathbb{N}$-graded ring A can be viewed as a $\mathbb{Z}$-graded ring by taking $A_n = 0$ for $n < 0$.

(iii) An ideal $I \subset A$ is homogeneous if and only if it is generated by homogeneous elements.

(iv) For a homogeneous ideal $I \subset A$, the quotient ring A/I inherits a unique Γ-graded ring structure making $A \twoheadrightarrow A/I$ a graded map. Moreover, the A/I-module M/IM has a unique Γ-graded structure making $M \to M/IM$ a graded map.

(v) Let $S \subset A$ be the set of homogeneous elements whose degrees are invertible in Γ. Then the localization $A[S^{-1}]$ inherits a unique Γ-graded structure making $A \to A[S^{-1}]$ a graded map. Moreover, the $A[S^{-1}]$-module $M[S^{-1}]$ inherits a unique Γ-graded structure making $M \to M[S^{-1}]$ a graded map.

Example 2.2.5 (Graded Rings). Let k be a ring.

- As mentioned earlier, for any set I, the polynomial ring $k[x_i \mid i \in I]$ is a $\mathbb{Z}$-graded ring.

- The Laurent polynomial ring $k[x^{\pm 1}]$ is a $\mathbb{Z}$-graded ring.
- Let M be a k-module. Then the symmetric algebra $\mathrm{Sym}_k(M)$ has a canonical $\mathbb{N}$-graded structure:

$$\mathrm{Sym}_k(M) = \bigoplus_{d\in\mathbb{N}} \mathrm{Sym}_k^d(M),$$

  where $\mathrm{Sym}_k^d(M)$ is the d-th symmetric power of M.
- Let L be a line over k. Then $\bigoplus_{d\in\mathbb{Z}} L^{\otimes d}$ is automatically a $\mathbb{Z}$-graded ring. In fact, we have $L^{\otimes d} = \mathrm{Sym}_k^d(L)$ and $L^{\otimes -1} = L^\vee$.

Notation 2.2.6 (Shifts, Veronese subrings, and homogeneous localization)**.** Let A be a Γ-graded ring, M a Γ-graded A-module, and $\gamma \in \Gamma$.

(i) We denote by $M(\gamma)$ the Γ-graded A-module defined as follows: its underlying A-module is M, and for $\delta \in \Gamma$, its graded component is:

$$(M(\gamma))_\delta = M_{\gamma+\delta}.$$

The assignment $M \mapsto M(\gamma)$ defines an endofunctor on graded modules. When $\gamma \in \Gamma$ is invertible, it is an equivalence of categories — the *shift by* γ.

(ii) We denote by $A^{(\gamma)}$ the $\mathbb{N}$-graded ring $\bigoplus_{n\in\mathbb{N}} A_{n\gamma}$, where $(A^{(\gamma)})_n = A_{n\gamma}$. Similarly, we define $M^{(\gamma)}$ to be the $\mathbb{N}$-graded $A^{(\gamma)}$-module.

(iii) Suppose $\gamma \in \Gamma$ is invertible and $f \in A_\gamma$. By Remark 2.2.4(5), the localization A_f inherits a Γ-grading; write $A_{(f)} := (A_f)_0$ for its degree-zero part, called the *homogeneous localization*, and similarly $M_{(f)} := (M_f)_0$. There is a canonical isomorphism of $\mathbb{Z}$-graded rings

$$A_{(f)}[x^{\pm 1}] \xrightarrow{\sim} A_f^{(\gamma)}, \quad x \mapsto f.$$

2.2.1. Projective Schemes

Now we can define projective schemes.

Construction 2.2.7 (Evaluation Map for Homogeneous Polynomials)**.** Let R be a k-algebra, $L \in \mathsf{Line}_R$ an R-line, I a set, and $d \in \mathbb{N}$. We define the *evaluation map*

$$k[x_i \mid i \in I]_d \times L^I \to L^{\otimes d}, \quad (f, a) \mapsto f(a).$$

Construction. An element $a \in L^I$ is the same datum as an R-linear map $a\colon R^{(I)} \to L$, uniquely determined by the images of the basis vectors. The map $(f, a) \mapsto f(a)$ is then defined as the composite

$$k[x_i \mid i \in I]_d = \mathrm{Sym}_k^d(k^{(I)}) \to \mathrm{Sym}_R^d(R^{(I)}) \xrightarrow{\mathrm{Sym}_R^d(a)} \mathrm{Sym}_R^d(L) \simeq L^{\otimes d}.$$

(For a rank-1 module L, the canonical map $\mathrm{Sym}^d(L) \simeq L^{\otimes d}$ is an isomorphism.)

Concretely, this is a k-linear map uniquely determined by the images of monomials:

$$\prod_i x_i^{n_i} \mapsto \bigotimes_i a_i^{\otimes n_i} \in L^{\otimes d}.$$

In particular, when $L = R$ (i.e., L is a trivial R-line), we have $L^{\otimes d} \simeq R$, which recovers the usual polynomial evaluation.

Recall that we defined affine schemes as loci of polynomial systems. For projective schemes, we naturally need to use homogeneous polynomials.

Definition 2.2.8 (System of Homogeneous Polynomials). Let k be a ring, and I, J be sets. A *system of J homogeneous polynomials in I variables over k* is a J-tuple $\Sigma = (f_j)_{j\in J}$, where each f_j is a *homogeneous polynomial* in the polynomial ring $k[x_i \mid i \in I]$.

We denote by (Σ) the homogeneous ideal generated by Σ in $k[x_i \mid i \in I]$, and write

$$k[\Sigma] = k[x_i \mid i \in I]/(\Sigma)$$

for the corresponding $\mathbb{N}$-graded k-algebra.

Definition 2.2.9 (Projective Vanishing Locus). Let $F \subset k[x_i \mid i \in I]$ be a family of homogeneous polynomials. The *vanishing locus of F in $\mathbb{P}^I_k$* is the subfunctor $V(F) \subset \mathbb{P}^I_k$ defined by:

$$V(F)(R) = \left\{a\colon R^{(I)} \twoheadrightarrow L \,\middle|\, \text{for all } f \in F \cap k[x_i]_d, \text{ we have } f(a) = 0 \in L^{\otimes d}\right\}.$$

Verification as Subfunctor. We verify that for any k-algebra homomorphism $\varphi\colon R \to S$, the induced map $\mathbb{P}^I(\varphi)$ maps $V(F)(R)$ into $V(F)(S)$.

Let $a \in V(F)(R)$ be a quotient line $R^{(I)} \twoheadrightarrow L$ satisfying $f(a) = 0$. Let a_S be the base change of a along φ, i.e., the quotient line:

$$S^{(I)} \simeq R^{(I)} \otimes_R S \xrightarrow{a\otimes 1} L \otimes_R S =: L_S.$$

We must show $f(a_S) = 0 \in L_S^{\otimes d}$.

This follows from the *commutativity of evaluation with base change*. Consider the following commutative diagram (for $d = \deg f$):

$$\begin{array}{ccc} \operatorname{Sym}^d_k(k^{(I)}) & \xrightarrow{\;f(a)\;} & L^{\otimes d} \\ & \searrow^{f(a_S)} & \downarrow{\scriptstyle \varphi_{L^{\otimes d}}} \\ & & L^{\otimes d} \otimes_R S \simeq (L \otimes_R S)^{\otimes d} \end{array}$$

where $\varphi_{L^{\otimes d}}$ is the canonical map sending x to $x \otimes 1$. Since $a \in V(F)(R)$, we have $f(a) = 0_R$ (the zero element in $L^{\otimes d}$). Since $\varphi_{L^{\otimes d}}$ is an additive group homomorphism, it maps 0_R to 0_S (the zero element in $L_S^{\otimes d}$). By commutativity of the diagram, $f(a_S) = 0_S$. Therefore $a_S \in V(F)(S)$. □

The subfunctor $V(F) \subset \mathbb{P}^I_k$ manifestly depends only on the homogeneous ideal (F). A subtlety distinguishes the projective case from the affine one: distinct homogeneous ideals can produce the same vanishing locus. We return to this point in Definition 2.3.20.

Now we can define the solution set functor.

Definition 2.2.10 (Homogeneous Solution Functor). Let $\Sigma = (f_j)_{j\in J}$ be a system of J homogeneous polynomials in I variables over k. Its *solution set functor* $\mathrm{hSol}_\Sigma\colon \mathsf{CAlg}_k \to \mathsf{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{P}^I_k$, i.e.,

$$\mathrm{hSol}_\Sigma = V(\{f_j \mid j \in J\}) \subset \mathbb{P}^I_k.$$

***Remark* 2.2.11.** The "h" in hSol stands for "homogeneous".

As in the affine case, we define projective schemes as algebraic functors isomorphic to solution set functors.

Definition 2.2.12 (Projective Schemes). Let k be a ring. We say that a k-algebraic functor $X\colon \mathsf{CAlg}_k \to \mathsf{Set}$ is a *projective k-scheme* if there exists a system of homogeneous polynomials in *finitely many variables* Σ such that $X \simeq \mathrm{hSol}_\Sigma$.

We denote by $\mathrm{Proj}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ the full subcategory spanned by all projective k-schemes.

***Remark* 2.2.13.** The finiteness assumption is a convenience: it ensures that projective k-schemes are automatically of finite type and proper.

We now seek a projective analogue of the affine identity $\mathrm{Spec}(k[\Sigma]) = \mathrm{Sol}_\Sigma$. Not every $\mathbb{N}$-graded k-algebra A has the form $k[\Sigma]$: this happens precisely when A is generated by A_1, i.e., when $\mathrm{Sym}_k(A_1) \twoheadrightarrow A$ is surjective. We restrict to this case for the remainder of this section, deferring the general construction to §Section 2.3.

Construction 2.2.14 (The Proj Functor - Preliminary Version)**.** Let k be a ring and A an $\mathbb{N}$-graded k-algebra generated by A_1. Define the algebraic k-functor $\mathrm{Proj}(A)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ by:

$$\mathrm{Proj}(A)(R) = \left\{\text{quotient line } A_1 \otimes_k R \twoheadrightarrow L \,\middle|\, \mathrm{Sym}_k(A_1) \to \mathrm{Sym}_R(L) \text{ factors through } A\right\}.$$

Proposition 2.2.15 (Comparison with Solution Functors)**.** *Let Σ be a system of homogeneous polynomials over k. Then there is a canonical isomorphism:*

$$\mathrm{hSol}_\Sigma \simeq \mathrm{Proj}(k[\Sigma])\colon \mathsf{CAlg}_k \to \mathsf{Set}.$$

Proof. We construct the bijection for any k-algebra R.

$\Rightarrow$: Let $a \in \mathrm{hSol}_\Sigma(R)$. By definition, a is a quotient line $R^{(I)} \twoheadrightarrow L$ satisfying $f(a) = 0$ for all $f \in \Sigma$. According to Construction 2.2.7 and the isomorphism $\mathrm{Sym}_R(R^{(I)}) \simeq R[x_i \mid i \in I]$, giving a is equivalent to giving a graded surjection of $\mathbb{N}$-graded R-algebras:

$$\varphi_a\colon\ \mathrm{Sym}_R(R^{(I)}) \twoheadrightarrow \mathrm{Sym}_R(L).$$

The condition "$f(a) = 0$ for all $f \in \Sigma$" is equivalent to saying that the kernel of φ_a contains the ideal $(\Sigma) \otimes_k R$. This means φ_a factors through the quotient algebra $k[\Sigma] \otimes_k R$, i.e., there exists a unique factorization:

$$\begin{array}{ccccc}
\mathrm{Sym}_R(R^{(I)}) & \twoheadrightarrow & \mathrm{Sym}_R(k[\Sigma]_1 \otimes_k R) & \twoheadrightarrow & k[\Sigma] \otimes_k R \\
& {\scriptstyle \mathrm{Sym}_R(a)} \searrow & \downarrow & \swarrow & \\
& & \mathrm{Sym}_R(L) & &
\end{array}$$

This corresponds precisely to an element of $\mathrm{Proj}(k[\Sigma])(R)$.

$\Leftarrow$: Conversely, given an element of $\mathrm{Proj}(k[\Sigma])(R)$, i.e., given $k[\Sigma]_1 \otimes_k R \twoheadrightarrow L$ such that the induced Sym map factors through $k[\Sigma] \otimes_k R$. Since $k[\Sigma]$ is generated by its degree 1 part (as a quotient of $k[x_i]$), we can pull this back to $R^{(I)} \twoheadrightarrow k[\Sigma]_1 \otimes_k R \twoheadrightarrow L$. Clearly this composition satisfies the equations in Σ. □

Recall that for a k-module M, we can consider its symmetric algebra $\mathrm{Sym}_k(M)$.

Example 2.2.16 (Projective Space as $\mathbb{P}(M)$)**.** Let M be a k-module. Define $\mathbb{P}(M)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ by:

$$\mathbb{P}(M)(R) = \{\text{quotient line } M \otimes_k R \twoheadrightarrow L\}.$$

Comparing constructions yields the identification $\mathbb{P}(M) \simeq \mathrm{Proj}(\mathrm{Sym}_k(M))$; when M is finitely generated, $\mathbb{P}(M)$ is a projective k-scheme. For a free module $M = k^{(I)}$ we recover the earlier notation $\mathbb{P}^I_k \simeq \mathbb{P}(k^{(I)})$.

Similarly, we can consider the dual functor $\mathbb{P}^\vee(M)\colon \mathsf{CAlg}_k \to \mathsf{Set}$ defined by:

$$\mathbb{P}^\vee(M)(R) = \{\text{subspace line } L \hookrightarrow M \otimes_k R\}.$$

If M is a vector space, we have $\mathbb{P}^\vee(M^\vee) \simeq \mathbb{P}(M)$. However, in general, $\mathbb{P}^\vee(M)$ is not a scheme.

***Remark* 2.2.17.** If A is an $\mathbb{N}$-graded k-algebra generated by A_1, then $\mathrm{Proj}(A)$ is a subfunctor of $\mathbb{P}(A_1)$.

2.2.2. Loci of Linear Maps

Consider the projective n-space $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[x_0, \ldots, x_n])$, whose R-points are quotient lines $a\colon R^{\{0,\ldots,n\}} \twoheadrightarrow L$.

For a homogeneous polynomial $f \in \mathbb{Z}[x_0, \ldots, x_n]_d = \mathrm{Sym}^d(\mathbb{Z}^{\{0,\ldots,n\}})$, the *non-vanishing locus* $D(f) \subset \mathbb{P}^n$ is the subfunctor whose R-points are quotient lines a for which $f(a)$ generates $L^{\otimes d}$ — equivalently, for which the composite

$$s_f\colon R \xrightarrow{f} \mathrm{Sym}^d_R(R^{\{0,\ldots,n\}}) \xrightarrow{\mathrm{Sym}^d(a)} L^{\otimes d}$$

is an R-module isomorphism. This expresses the geometric condition "$f(a)$ is non-degenerate" functorially: it is equivalent to $f(a) \neq 0$ in $L^{\otimes d} \otimes_R \kappa$ for every R-field κ.

Recall the affine analogue: for a k-algebra A and $f \in A$,

$$D(f)(R) = \{x \in \mathrm{Hom}(A, R) \mid (x(f)) = R\}.$$

We can rephrase this in line with the projective case: at a point x, the "function" f gives an R-linear map

$$R \xrightarrow{\cdot x(f)} R, \quad 1 \mapsto x(f),$$

and $(x(f)) = R$ — i.e. $x(f)$ is a unit — exactly when this map is an isomorphism.

Summary (Unified Description of Non-vanishing Loci). In both the affine and projective settings, the non-vanishing locus $D(f)$ consists of the points x at which the evaluation

$$R \xrightarrow{f(x)} \mathcal{L}$$

is an isomorphism. The target line is $\mathcal{L} = R$ (trivial) in the affine case and $\mathcal{L} = L^{\otimes d}$ in the projective case.

There is a subtlety, however: $f \in \mathbb{Z}[x_0, \ldots, x_n]_d$ is not a function on $\mathbb{P}^n$ but a section of the line bundle $\mathcal{O}(d)$ (Definition 3.3.14). Openness of $D(f) \subset \mathbb{P}^n$ is therefore not immediate. The goal of this section is to show that $D(f)$ is open and even *relatively affine* — every pullback along an affine morphism is affine (Proposition 3.2.14).

To do this, consider any A-point $x\colon \mathrm{Spec}(A) \to \mathbb{P}^n$ (i.e., a quotient line $A^{n+1} \twoheadrightarrow L$ over A). Consider the pullback diagram:

$$\begin{array}{ccc} x^{-1}(D(f)) & \hookrightarrow & \mathrm{Spec}(A) \\ \downarrow & \lrcorner & \downarrow x \\ D(f) & \hookrightarrow & \mathbb{P}^n \end{array}$$

We must show that $x^{-1}(D(f)) \subset \mathrm{Spec}(A)$ is open.

First, note that we can actually characterize $x^{-1}(D(f))$ in the following way: for an A-algebra R, we have:

$$x^{-1}(D(f))(R) = \{\varphi\colon A \to R \mid \varphi^*(s)\colon R \to L^{\otimes d} \otimes_A R \text{ is an isomorphism}\},$$

where s is the linear map over A:

$$s\colon A \xrightarrow{f} \mathrm{Sym}^d_A(A^{\{0,\ldots,n\}}) \xrightarrow{x} L^{\otimes d}.$$

Equivalently, $x^{-1}(D(f))$ is the locus in $\mathrm{Spec}(A)$ where the linear map "s is an isomorphism".

We adopt a broader perspective. Let P be a property of modules stable under base change — e.g., being zero, finitely generated, or flat. For a ring A and an A-module M, the locus where $\varphi^*(M)$ has P defines a subfunctor of $\mathrm{Spec}(A)$:

$$R \mapsto \{\varphi\colon A \to R \mid \text{the } R\text{-module } \varphi^*(M) \text{ has property } P\}.$$

Similarly, for an A-linear map $u\colon M \to N$, we can also define the locus where it has a specific property:

Definition 2.2.18 (Loci of Linear Maps). Let $u\colon M \to N$ be an A-module homomorphism. We define:

- *Vanishing locus* $V(u)$: the locus where u is zero.
- *Surjection locus* $\mathrm{Epi}(u)$: the locus where u is surjective.
- *Injection locus* $\mathrm{Mono}(u)$: the locus where u is universally injective.
- *Isomorphism locus* $\mathrm{Iso}(u)$: the locus where u is an isomorphism.

Returning to the problem of $D(f)$, we find that $x^{-1}(D(f))$ is precisely the *isomorphism locus* $\mathrm{Iso}(s)$ of the linear map $s\colon A \to L^{\otimes d}$.

Proposition 2.2.19 (Loci as Open/Closed Subfunctors). *Let A be a ring and $f\colon M \to N$ an A-linear map. Then:*

(i) If N is a vector space, then $V(f) \subset \mathrm{Spec}(A)$ is a closed subfunctor.

(ii) If N is a vector space, then $\mathrm{Epi}(f) \subset \mathrm{Spec}(A)$ is an open subfunctor.

(iii) If M and N are both vector spaces, then $\mathrm{Mono}(f) \subset \mathrm{Spec}(A)$ is an open subfunctor.

(iv) If M and N are both vector spaces, then $\mathrm{Iso}(f) \subset \mathrm{Spec}(A)$ is an open subfunctor.

***Remark* 2.2.20.** Part (4) immediately implies that $D(f) \subset \mathbb{P}^n$ is open, since $D(f)$ is precisely the isomorphism locus of $s\colon A \to L^{\otimes d}$.

Proof. (i) *Vanishing locus:* f is zero if and only if for any $x \in M$, the composite map $f(x)\colon R \xrightarrow{x} M \xrightarrow{f} N$ is zero. That is, $V(f) = \bigcap_{x\in M} V(f(x))$. Since arbitrary intersections of closed subfunctors remain closed subfunctors, we may assume $M = R$. In this case, $f\colon R \to N$ corresponds to an element $n = f(1) \in N$.

Since N is a vector space, N is reflexive ($N \simeq N^{\vee\vee}$), so $n = 0$ if and only if for all $\lambda \in N^\vee$, we have $\lambda(n) = 0$. Therefore:

$$V(R \xrightarrow{f} N) = V(N^\vee \xrightarrow{f^\vee} R) = V(\mathrm{im}(f^\vee)),$$

where $f^\vee\colon N^\vee \to R$ is the dual map induced by f. For any R-algebra homomorphism $\varphi\colon R \to S$, we have $\varphi \in V(f)$ if and only if $\varphi^*(\mathrm{im}(f^\vee)) = 0$. This is precisely the closed subfunctor $V(\mathrm{im}(f^\vee))$ defined by the ideal $\mathrm{im}(f^\vee) \subset R$.

(ii) *Surjection locus.* Since N is a vector space, it is a direct summand of R^n: pick N' with $N \oplus N' \simeq R^n$. Surjectivity of f is equivalent to surjectivity of $f \oplus \mathrm{id}_{N'}$, so

$$\mathrm{Epi}(f) = \mathrm{Epi}(f \oplus \mathrm{id}_{N'}\colon M \oplus N' \to R^n).$$

Replacing f with $f \oplus \mathrm{id}_{N'}$, we may assume $N = R^n$, so $f\colon M \to R^n$.

Claim 2.2.21. *The surjection locus can be detected by the exterior power functor, i.e.,*

$$\mathrm{Epi}(f) = \mathrm{Epi}(\Lambda^n(f)).$$

Proof of Claim. We test f on n-tuples. For $m_1, \ldots, m_n \in M$, the images $f(m_1), \ldots, f(m_n)$ assemble into an $n \times n$ matrix in R, and $\det\colon R^{n\times n} \to R$ detects whether they form a basis. This is encoded by the commutative diagram

$$\begin{array}{ccccc} M^n & \xrightarrow{f^n} & (R^n)^n \simeq R^{n\times n} & \xrightarrow{\det} & R \\ {\scriptstyle\text{can}}\downarrow & & {\scriptstyle\text{can}}\downarrow & & \| \\ \Lambda^n M & \xrightarrow{\Lambda^n(f)} & \Lambda^n(R^n) & \xrightarrow{\simeq} & R \end{array}$$

- If f is surjective, then the standard basis $e_1, \ldots, e_n$ of R^n all lie in the image of f. There exist m_i such that $f(m_i) = e_i$. Then in the bottom row of the diagram, $\Lambda^n(f)(m_1 \wedge \cdots \wedge m_n) = 1$, so the image of $\Lambda^n(f)$ contains a unit, hence is surjective.
- If $\Lambda^n(f)$ is surjective, pick $\omega \in \Lambda^n M$ with $\Lambda^n(f)(\omega) = 1$. Expanding ω as a finite sum of wedges $m_{i_1} \wedge \cdots \wedge m_{i_n}$, some n-tuple $(m_1, \ldots, m_n)$ already has $\det\big(f(m_1), \ldots, f(m_n)\big) \in R^\times$, so $(f(m_1), \ldots, f(m_n))$ is an invertible matrix in $R^{n\times n}$.

  Equivalently, the column vectors $f(m_1), \ldots, f(m_n)$ form a basis of R^n. Since these vectors all lie in $\mathrm{im}(f)$, we have $\mathrm{im}(f) = R^n$, i.e., f is surjective.

□

Since $\Lambda^n(R^n) \simeq R$, the claim reduces $\mathrm{Epi}(f)$ to a rank-one statement: $\Lambda^n(f)\colon \Lambda^n M \to R$ is surjective iff its image generates the unit ideal. Hence

$$\mathrm{Epi}(f) = \mathrm{Epi}(\Lambda^n(f)) = D(\mathrm{im}(\Lambda^n(f))),$$

an open subfunctor of $\mathrm{Spec}(A)$.

(iii) *Injection locus:* When M, N are both vector spaces, $f : M \to N$ is universally injective if and only if $f^\vee\colon N^\vee \to M^\vee$ is surjective. Since $N^\vee, M^\vee$ are also vector spaces, by (2), $\mathrm{Epi}(f^\vee)$ is an open subfunctor.

(iv) *Isomorphism locus:* f is an isomorphism if and only if it is surjective and universally injective. That is, $\mathrm{Iso}(f) = \mathrm{Epi}(f) \cap \mathrm{Mono}(f)$. The intersection of two open subfunctors is still an open subfunctor.

□

2.2.3. Localization at a Section

We now show that $D(s)$ is affine for any section $s\colon R \to L$.

More generally, given a map of R-lines $s\colon L' \to L$, tensoring with $(L')^\vee$ gives $D(s) = D(s \otimes_R (L')^\vee)$ — this reduces to the case $s\colon R \to L$, equivalently an element $s \in L$. When $L = R$ this is just $D(s) = \mathrm{Spec}(R_s)$ (classical localization); the substance is in the case of a non-trivial line L.

Definition 2.2.22 (Periodic Modules). Let R be a ring, L an R-line, and $s \in L$. We say that an R-module M is *s-periodic* if the morphism:

$$\mathrm{id}_M \otimes s\colon M \otimes_R R \to M \otimes_R L$$

is an isomorphism.

Proposition 2.2.23 (Localization at a Line Section). *Let R be a ring, L an R-line (i.e., an invertible module over R), and $s \in L$ (i.e., a section $s\colon R \to L$). Then:*

(i) There exists an s-periodic R-algebra R_s that is initial among all s-periodic R-algebras.

(ii) The forgetful functor $\mathrm{Mod}_{R_s} \to \mathrm{Mod}_R$ identifies Mod_{R_s} with the full subcategory spanned by s-periodic modules. Therefore, it has a left adjoint $\mathrm{Mod}_R \to \mathrm{Mod}_{R_s}$ sending M to $M_s := M \otimes_R R_s$.

(iii) *Concretely, M_s can be computed as the following colimit in* Mod_R*:*

$$M \xrightarrow{\cdot s} M \otimes_R L \xrightarrow{\cdot s} M \otimes_R L^{\otimes 2} \xrightarrow{\cdot s} \cdots .$$

Here the map $M \otimes L^{\otimes n} \xrightarrow{\cdot s} M \otimes L^{\otimes n+1}$ is defined by $x \mapsto x \otimes s$ (using the isomorphism $M \otimes L^{\otimes n} \simeq M \otimes L^{\otimes n} \otimes R \xrightarrow{1 \otimes s} M \otimes L^{\otimes n} \otimes L$).

Proof. We construct R_s explicitly and verify the three claims simultaneously.

Construction. Define

$$R_s := \mathrm{colim}\big(R \xrightarrow{\cdot s} L \xrightarrow{\cdot s} L^{\otimes 2} \xrightarrow{\cdot s} \cdots\big)$$

as an R-module, where each transition map $L^{\otimes n} \to L^{\otimes(n+1)}$ sends $\xi \mapsto \xi \otimes s$. Because L is invertible, the natural isomorphisms $L^{\otimes m} \otimes_R L^{\otimes n} \xrightarrow{\sim} L^{\otimes(m+n)}$ are compatible with the transitions, and they assemble into an associative, unital multiplication

$$R_s \otimes_R R_s \to R_s,$$

making R_s a commutative R-algebra. The unit is the class of $1 \in R$.

R_s is s-periodic. The structure map $R \to R_s$ identifies $1 \in R$ with the class of 1 in the 0-th term of the colimit, while $s \otimes 1 \in L \otimes_R R_s$ identifies with the class of s in the 1-st term, shifted up by one through the structure isomorphism $L \otimes_R R_s \simeq R_s$ given by the transition maps. Concretely, multiplication by s on R_s corresponds to the shift $L^{\otimes n} \to L^{\otimes(n+1)}$, which is an isomorphism on the colimit. Hence $\mathrm{id}_{R_s} \otimes s : R_s \to R_s \otimes_R L$ is an isomorphism, so R_s is s-periodic.

Universal property. Let A be any s-periodic R-algebra. We must show that $\mathrm{Hom}_{\mathsf{CAlg}_R}(R_s, A)$ has exactly one element. Given $\varphi : R \to A$ in CAlg_R, the s-periodicity of A means that $\cdot s : A \to A \otimes_R L$ is an isomorphism, with inverse $\sigma : A \otimes_R L \to A$. Iterating gives compatible maps $A \otimes_R L^{\otimes n} \to A$ that match the transitions in the colimit defining R_s, hence a unique R-algebra map $R_s \to A$ extending φ.

Module-theoretic version (parts 2 and 3). Repeat the construction with M in place of R: the colimit $M_s := \mathrm{colim}(M \to M \otimes L \to M \otimes L^{\otimes 2} \to \cdots)$ exhibits M_s as the universal s-periodic R-module receiving a map from M. By the universal property, $M_s \simeq M \otimes_R R_s$, and the assignment $M \mapsto M_s$ defines a left adjoint to the forgetful functor $\mathrm{Mod}_{R_s} \to \mathrm{Mod}_R$. Conversely, any s-periodic R-module N admits a unique R_s-action extending its R-action — built by inverting the s-action — so the forgetful functor is fully faithful onto s-periodic modules. ◻

***Remark* 2.2.24** (Classical Localization as Special Case). When $M = R$ and $L = R$ (trivial line), s corresponds to an element $f \in R$. In this case, the above colimit is $R \xrightarrow{f} R \xrightarrow{f} R \cdots$, whose colimit is precisely the classical localization $R_f = R[1/f]$.

For a general line bundle L one cannot literally write "$1/s$" — s is not an element of a ring. The construction above instead universally inverts $\cdot s$, making it an isomorphism. This succeeds because L is invertible: tensoring with L is an autoequivalence of Mod_R, so it behaves formally like multiplication by an invertible scalar.

Therefore, we immediately obtain the following corollary:

Corollary 2.2.25 (Non-vanishing Locus is Affine)**.** *Let R be a ring, L an R-line, and $s \in L$. Then $D(s) = \mathrm{Epi}(R \xrightarrow{s} L) \simeq \mathrm{Spec}(R_s)$.*

Proof. For an R-algebra $\varphi : R \to R'$, the pullback section $\varphi^*(s) \in L \otimes_R R'$ generates $L \otimes_R R'$ as an R'-module if and only if $\cdot\varphi^*(s) : R' \to L \otimes_R R'$ is surjective; since $L \otimes_R R'$ is a line over R' and a surjection between line bundles is automatically an isomorphism (Corollary 2.1.22), this is equivalent to R' being

s-periodic as an R-module. By Proposition 2.2.23(i), this in turn is equivalent to φ factoring uniquely through $R \to R_s$. Hence

$$D(s)(R') = \mathrm{Hom}_{\mathsf{CAlg}_R}(R_s, R') = \mathrm{Spec}(R_s)(R'),$$

naturally in R'. □

Summary (Relative Affineness of $D(f)$). Therefore, for $f \in \mathbb{Z}[x_0, \dots, x_n]_d$, we have that $D(f) \subset \mathbb{P}^n$ is an open subfunctor. Moreover, for any $x\colon \mathrm{Spec}(R) \to \mathbb{P}^n$ corresponding to a quotient line $a\colon R^{n+1} \twoheadrightarrow L$, we have $x^{-1}D(f) = D(s)$, where $s = f(a) \in L^{\otimes d}$. Therefore, $D(f) \hookrightarrow \mathbb{P}^n$ is relatively affine.

2.2.4. Generalized Zariski Descent

The localization-at-a-section construction generalizes classical Zariski descent.

Proposition 2.2.26 (Generalized Zariski Descent). *Let R be a ring, L an R-line, and $(s_i)_{i\in I}$ a generating set for L. Then for any R-module M, the diagram:*

$$M \longrightarrow \prod_{i\in I} M_{s_i} \rightrightarrows \prod_{i,j\in I} M_{s_i s_j}$$

is an equalizer, where $s_i s_j = s_i \otimes s_j \in L^{\otimes 2}$.

Proof. The strategy is to reduce the descent over the line section s_i's to standard Zariski descent over a suitable set of ring elements.

Step 1: From line sections to ring elements. For a single section $s\colon R \to L$, dualizing gives

$$s^\vee\colon L^\vee \;=\; R \otimes_R L^\vee \xrightarrow{s\otimes \mathrm{id}} L \otimes_R L^\vee \;=\; R.$$

Fix a generating set $F \subset L^\vee$ and set $F_s := s^\vee(F) \subset R$. We claim that (F_s) generates the unit ideal in R_s.

Indeed, R_s is s-periodic, so $\mathrm{id}_{R_s} \otimes s\colon R_s \to R_s \otimes_R L$ is an isomorphism. Tensoring its dual $s^\vee$ with R_s inherits this isomorphism property by base change, hence $s^\vee \otimes_R R_s\colon (L^\vee)_s \to R_s$ is surjective. Since F generates $L^\vee$, the image F_s generates R_s. The same argument applied to $st := s \otimes t \in L^{\otimes 2}$ shows that $F_s F_t$ generates R_{st} for any second section t.

Step 2: Translation to a Zariski cover of R. Apply Step 1 to the family $(s_i)_{i\in I}$ generating L. The dual sections $(s_i^\vee)_{i\in I}$ then generate $L^\vee$, so the union

$$\mathcal{F} := \bigcup_{i\in I} F_{s_i} \subset R$$

generates the unit ideal of R, and similarly $F_{s_i}F_{s_j}$ generates $R_{s_i s_j}$ for each pair i, j. Standard Zariski descent for M along the cover $\mathcal{F}$ yields the equalizer

$$M \to \prod_{f\in\mathcal{F}} M_f \rightrightarrows \prod_{f,g\in\mathcal{F}} M_{fg}, \tag{2.1}$$

and within each $\mathrm{Mod}_{R_{s_i}}$, descent along F_{s_i} exhibits M_{s_i} as an equalizer

$$M_{s_i} \to \prod_{f\in F_{s_i}} (M_{s_i})_f \rightrightarrows \prod_{f,g\in F_{s_i}} (M_{s_i})_{fg}. \tag{2.2}$$

Step 3: The two descent diagrams agree. Abbreviate $F_i := F_{s_i}$. Combining (2.2) for each i (and similarly for $M_{s_i s_j}$) gives the columns of

$$\begin{array}{ccccc}
M & \longrightarrow & \prod_i M_{s_i} & \rightrightarrows & \prod_{i,j} M_{s_i s_j} \\
\| & & \downarrow & & \downarrow \\
M & \longrightarrow & \prod_{i, f \in F_i} M_f & \rightrightarrows & \prod_{i,j,\, fg \in F_i F_j} M_{fg} \\
\| & & \downdownarrows & & \downdownarrows \\
M & \longrightarrow & \prod_{i, f, g \in F_i} M_{fg} & \rightrightarrows & \prod_{i,j,\, fg, f'g' \in F_i F_j} M_{fg \cdot f'g'}
\end{array}$$

The columns are equalizers (each is an instance of (2.2)), and limits commute, so the equalizer of the top row equals the equalizer of the equalizers of the lower rows. The latter is the descent diagram for M along $\mathcal{F} \cup (\mathcal{F} \cdot \mathcal{F})$, which by (2.1) computes M.

Finally, the lower descent only involves the products $F_i \cdot F_j$: every pair (f, g) with $f \in \mathcal{F}$ and $g \in \mathcal{F}$ has the form $f \in F_i$, $g \in F_j$ for some i, j, so the pairwise-compatibility conditions over $\mathcal{F}$ are exactly those recorded in $\prod_{i,j} \prod_{F_i F_j}$. Thus the top row is an equalizer, as claimed. □

Example 2.2.27 (Projective completions of affine spaces)**.** Let k be a ring and M a k-module. Recall the algebraic k-functors $\mathbb{A}(M)$ (Example 1.2.24) and $\mathbb{P}(M)$ (Example 2.2.16). There is a canonical embedding

$$\mathbb{A}(M) \hookrightarrow \mathbb{P}(M \oplus k), \quad (a\colon M \otimes_k R \to R) \mapsto ((a, \mathrm{id}_R)\colon (M \otimes_k R) \oplus R \twoheadrightarrow R).$$

Claim 2.2.28. *The canonical embedding above is an open immersion.*

Proof of Claim. Let $x\colon \mathrm{Spec}(R) \to \mathbb{P}(M \oplus k)$ classify a quotient line $(M \otimes_k R) \oplus R \twoheadrightarrow L$ (Example 2.2.16). The pullback

$$\begin{array}{ccc}
x^{-1}(\mathbb{A}(M)) & \longrightarrow & \mathrm{Spec}(R) \\
\downarrow & & \downarrow \\
\mathbb{A}(M) & \hookrightarrow & \mathbb{P}(M \oplus k)
\end{array}$$

is the locus in $\mathrm{Spec}(R)$ where the composite $R \hookrightarrow (M \otimes_k R) \oplus R \twoheadrightarrow L$ is an isomorphism, namely

$$x^{-1}(\mathbb{A}(M)) = \mathrm{Iso}\big(R \hookrightarrow (M \otimes_k R) \oplus R \twoheadrightarrow L\big),$$

which is open in $\mathrm{Spec}(R)$ by Proposition 2.2.19(iv). □

Taking $M = k^{(I)}$, we get a canonical open immersion

$$\mathbb{A}^I \hookrightarrow \mathbb{P}^{I \sqcup \{0\}},$$

identifying $\mathbb{A}^I$ with $D(x_0) \subset \mathbb{P}^{I \sqcup \{0\}}$. For $I = \{1, \cdots, n\}$, this is an open immersion $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$.

2.3. The Proj Construction

We now extend the Proj functor from rings generated in degree 1 to arbitrary $\mathbb{N}$-graded rings. The construction is a colimit over the divisibility poset of Veronese subrings $A^{(d)}$; the structure theorem then says this colimit is glued from affine opens $D(f) \simeq \mathrm{Spec}(A_{(f)})$ as f ranges over homogeneous elements.

Notation 2.3.1 (Graded Ring Notation). Let A be an $\mathbb{N}$-graded ring and $d \in \mathbb{N}$ a natural number.

- $A^{(d)}$ denotes the $\mathbb{N}$-graded ring $\bigoplus_{n\in\mathbb{N}} A_{nd}$ (Veronese subring), where $(A^{(d)})_n = A_{nd}$.
- $A_+ = \bigoplus_{d\geq 1} A_d$ is the irrelevant ideal.
- For a homogeneous element $f \in A$, A_f is a $\mathbb{Z}$-graded ring, and $A_{(f)} := (A_f)_0$ is its degree 0 part.

We can now construct the general Proj functor — extending the preliminary version (Construction 2.2.14) to all $\mathbb{N}$-graded rings, not only those generated in degree 1.

Definition 2.3.2 (Eventually Surjective). Let A, B be $\mathbb{N}$-graded rings. We say that a graded map $A \to B$ is *eventually surjective* if for all $d \geq 1$ and every homogeneous element $b \in B_d$, there exists $n \geq 1$ such that b^n lies in the image of the map $A_{nd} \otimes_{A_0} B_0 \to B_{nd}$.

We denote by $\mathsf{CAlg}^{\mathbb{N},\mathrm{es}}$ the category of $\mathbb{N}$-graded rings and eventually surjective maps between them.

Construction 2.3.3 (The Proj Functor). We construct the functor:

$$\mathrm{Proj}\colon (\mathsf{CAlg}^{\mathbb{N},\mathrm{es}})^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}).$$

The construction also produces a natural transformation $\mathrm{Proj}(-) \Rightarrow \mathrm{Spec}((-)_0)$.

Step 1: Define $\mathrm{Proj}_1(A)$. We define $\mathrm{Proj}_1(A)$ as an object in $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{Spec}(A_0)} \simeq \mathsf{Fun}(\mathsf{CAlg}_{A_0}, \mathsf{Set})$. Given an A_0-algebra R, the set $\mathrm{Proj}_1(A)(R)$ consists of pairs (L, φ) where:

- L is a quotient line of $A_1 \otimes_{A_0} R$, i.e., a surjective R-module homomorphism $A_1 \otimes_{A_0} R \twoheadrightarrow L$;
- $\varphi\colon A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ is a graded R-algebra morphism that extends the above quotient map (i.e., φ restricted to degree 1 is precisely $A_1 \otimes R \to L$).

Surjectivity of φ is automatic from surjectivity in degree 1 together with the fact that both source and target are generated as algebras by their degree-1 parts. Equivalently, $\mathrm{Proj}_1(A)(R)$ is the set of graded quotients of $A \otimes_{A_0} R$ isomorphic to the symmetric algebra of a line.

Step 2: Define $\mathrm{Proj}_d(A)$ *and the colimit*. For $d \in \mathbb{N}$, define $\mathrm{Proj}_d(A) = \mathrm{Proj}_1(A^{(d)})$. For $n \in \mathbb{N}$, there exists a canonical morphism:

$$\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A), \quad (L, \varphi) \mapsto (L^{\otimes n}, \varphi^{(n)}),$$

where $\varphi^{(n)}$ is the restriction of φ to the Veronese subring (note that $\mathrm{Sym}_R(L)^{(n)} \simeq \mathrm{Sym}_R(L^{\otimes n})$).

This defines a functor:

$$\mathbb{N}^{\mathrm{div}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}), \quad d \mapsto \mathrm{Proj}_d(A),$$

where $\mathbb{N}^{\mathrm{div}}$ is the poset ordered by divisibility. We define $\mathrm{Proj}(A)$ as its colimit:

$$\mathrm{Proj}(A) = \mathrm{colim}_{d\in\mathbb{N}^{\mathrm{div}}_{>0}} \mathrm{Proj}_d(A).$$

Since 1 divides every d, we also have $\mathrm{Proj}(A) \simeq \mathrm{colim}_{d\to\infty} \mathrm{Proj}_1(A^{(d)})$.

Since 0 is the terminal object of $\mathbb{N}^{\mathrm{div}}$, the colimit comes equipped with a canonical structure map

$$\mathrm{Proj}(A) \to \mathrm{Proj}_0(A) = \mathrm{Proj}_1(A^{(0)}) = \mathrm{Proj}_1(A_0[x]) \simeq \mathrm{Spec}(A_0).$$

Step 3: Functoriality. Let $\alpha\colon A \to B$ be a graded map and (L, φ) an R-point of $\mathrm{Proj}_d(B)$. Choose finitely many $b_1, \ldots, b_r \in B_d$ whose images under φ_d generate L. Whenever some $n \geq 1$ makes every b_i^n lie in the image of $\alpha_{nd}\colon A_{nd} \otimes_{A_0} B_0 \to B_{nd}$, the line $L^{\otimes n}$ pulls back to a quotient line of $A^{(nd)}$, yielding the R-point $(L^{\otimes n}, \varphi^{(n)} \circ \alpha^{(nd)})$ of $\mathrm{Proj}_{nd}(A)$.

Therefore, if α is *eventually surjective*, this process is well-defined for any R-point, inducing a morphism:

$$\mathrm{Proj}(\alpha)\colon \mathrm{Proj}(B) \to \mathrm{Proj}(A).$$

***Remark* 2.3.4** (Veronese Isomorphism). For $d \geq 1$, since $d\mathbb{N}^{\mathrm{div}} \subset \mathbb{N}^{\mathrm{div}}$ is cofinal, we have an isomorphism:

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}).$$

***Remark* 2.3.5** (Degree d Maps). We can extend the domain of Proj in the following way.

For $\mathbb{N}$-graded rings A and B and $d \in \mathbb{N}$, a *degree d map* from A to B is a graded ring map $A \to B^{(d)}$.

If it is *eventually surjective* and $d \geq 1$, it induces:

$$\mathrm{Proj}(B) \simeq \mathrm{Proj}(B^{(d)}) \to \mathrm{Proj}(A).$$

This also holds for $d = 0$:

$$\mathrm{Proj}(B) \to \mathrm{Spec}(B_0) = \mathrm{Proj}(B^{(0)}) \to \mathrm{Proj}(A).$$

Composition of a degree-d map with a degree-e map gives a *degree-de map*, so we obtain an extended category — same objects, with $\mathrm{Hom}(A, B) := \coprod_{d \in \mathbb{N}} \mathrm{Hom}_{\mathsf{CAlg}^{\mathbb{N},\mathrm{es}}}(A, B^{(d)})$ — to which Proj extends.

The central result of this section is the following structure theorem.

Theorem 2.3.6 (Proj Structure Theorem). *Let A be an $\mathbb{N}$-graded ring and $d \geq 1$.*

(i) The canonical map $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ *is an* open immersion.

(ii) If A is generated as an A_0-algebra by homogeneous elements whose degrees are divisible by d, *then the canonical map* $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ *is an isomorphism.*

In particular, if A is generated by $A_{\leq 1}$ (i.e., A_0 and A_1), then $\mathrm{Proj}_1(A) \simeq \mathrm{Proj}(A)$.

***Remark* 2.3.7.** Morally, the theorem says $\mathrm{Proj}(A)$ is glued from the open subfunctors $\mathrm{Proj}_d(A)$ as d varies.

As an immediate consequence, every weighted projective scheme embeds into an ordinary projective space.

Corollary 2.3.8 (Projective Schemes are Projective). *If A is finitely generated as an A_0-algebra, then* $\mathrm{Proj}(A)$ *is a* projective A_0-scheme*: it is isomorphic to a closed subscheme of $\mathbb{P}^N_{A_0}$ for some N.*

Proof. Since A is finitely generated as an A_0-algebra and A is an $\mathbb{N}$-graded ring, we can choose finitely many homogeneous generators $x_1, \ldots, x_n$. Let $\gamma_i = \deg(x_i) \in \mathbb{N}_{\geq 1}$.

Choose d to be a *common multiple* of all degrees (e.g., $d = \mathrm{lcm}(\gamma_1, \ldots, \gamma_n)$ or $d = \prod \gamma_i$).

Consider the Veronese subring $A^{(d)} = \bigoplus_{k \geq 0} A_{kd}$. Since $\gamma_i \mid d$, each generator satisfies $x_i^{d/\gamma_i} \in A_d$. Replacing d by a sufficiently divisible multiple if necessary, we may assume $A^{(d)}$ is generated as an A_0-algebra by its degree-1 part $A_d = (A^{(d)})_1$.

More precisely, the inclusion map $A^{(d)} \hookrightarrow A$ is eventually surjective. Therefore, it induces an isomorphism $\mathrm{Proj}(A) \xrightarrow{\sim} \mathrm{Proj}(A^{(d)})$.

Since $A^{(d)}$ is generated by its degree-1 part, Theorem 2.3.6(2) gives

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}) \simeq \mathrm{Proj}_1(A^{(d)}).$$

Finally, $\mathrm{Proj}_1(A^{(d)})$ by definition is a closed subscheme of $\mathbb{P}(A_d^\vee)$ (defined by the relations in $A^{(d)}$). Therefore $\mathrm{Proj}(A)$ is a projective A_0-scheme. □

Example 2.3.9 (Veronese Embedding). Let k be a ring, M a k-module, and $d \geq 1$. The *d-fold Veronese embedding* is a natural transformation of k-algebraic functors

$$\rho_d \colon \mathbb{P}(M) \longrightarrow \mathbb{P}(\mathrm{Sym}_k^d(M)),$$

which sends a quotient line $M \otimes_k R \twoheadrightarrow L$ to its d-th symmetric power:

$$\mathrm{Sym}_k^d(M) \otimes_k R \twoheadrightarrow \mathrm{Sym}_R^d(L) \simeq L^{\otimes d}.$$

(Since L is an R-line, $\mathrm{Sym}_R^d(L) \simeq L^{\otimes d}$.)

Coordinate description: If $M = k^{(I)}$ is a free module, then the d-fold Veronese embedding is the map:

$$\rho_d \colon \mathbb{P}(k^{(I)}) \longrightarrow \mathbb{P}(\mathrm{Sym}^d(k^{(I)})).$$

In particular, when $M = k^{n+1}$ (i.e., $I = \{0, \dots, n\}$), $\mathrm{Sym}^d(M)$ has rank $\binom{n+d}{n}$. The embedding map is:

$$\rho_d \colon \mathbb{P}_k^n \longrightarrow \mathbb{P}_k^N, \quad \text{where } N = \binom{n+d}{n} - 1.$$

Its action on coordinates is given by:

$$[a_0 : \cdots : a_n] \mapsto [\text{all degree } d \text{ monomials } M_J(a_0, \dots, a_n)].$$

Geometric property: We claim that for $d \geq 1$, ρ_d is a *closed immersion*.

We can prove this using the functoriality of Proj. Let $S = \mathrm{Sym}_k(M)$. The map ρ_d decomposes as the composition of the following two maps:

$$\mathrm{Proj}_1(S) \xrightarrow{\sim} \mathrm{Proj}_d(S) \xrightarrow{i} \mathrm{Proj}_1(\mathrm{Sym}_k(S_d)).$$

Here $\mathrm{Proj}_1(\mathrm{Sym}_k(S_d))$ is precisely the target space $\mathbb{P}(\mathrm{Sym}^d(M))$.

(i) *First map:* $(L, \varphi) \mapsto (L^{\otimes d}, \varphi^{(d)})$ is an isomorphism. This is guaranteed by *Theorem* 2.3.6 (Proj Structure Theorem), since S is generated by $S_1 = M$, so $S^{(d)}$ is generated by S_d.

(ii) *Second map.* The morphism i is a closed immersion induced by a surjection of graded rings. Consider the canonical k-algebra map

$$\Phi \colon \mathrm{Sym}_k(S_d) \twoheadrightarrow S^{(d)} = \bigoplus_{n \geq 0} S_{nd},$$

sending the formal generators of $\mathrm{Sym}_k(S_d)$ to products of degree-d elements of S. Because S is generated by S_1, the Veronese subring $S^{(d)}$ is generated by S_d, so Φ is surjective.

By the contravariant functoriality of Proj, the surjective ring homomorphism Φ induces a closed immersion:

$$\mathrm{Proj}(\Phi) \colon \mathrm{Proj}(S^{(d)}) \hookrightarrow \mathrm{Proj}(\mathrm{Sym}_k(S_d)).$$

That is, $\mathrm{Proj}_d(S) \hookrightarrow \mathbb{P}(\mathrm{Sym}^d(M))$.

In summary, ρ_d is the composition of an isomorphism and a closed immersion, hence is a closed immersion.

We turn to Zariski descent for the Proj functor.

Proposition 2.3.10 (Zariski Descent for Proj). *Let A be an $\mathbb{N}$-graded ring, $d \in \mathbb{N}$ a natural number, and X be $\mathrm{Proj}_d(A)$ (or $\mathrm{Proj}(A)$). Then for any ring R, R-line L, and generating set $(s_i)_{i\in I}$ of L, the diagram:*

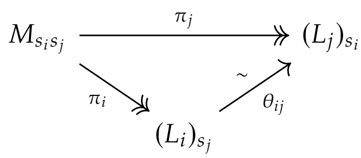
$$X(R) \longrightarrow \prod_{i\in I} X(R_{s_i}) \rightrightarrows \prod_{i,j\in I} X(R_{s_is_j})$$

is an equalizer.

Proof. We may assume $X = \mathrm{Proj}_1(B)$ (taking $B = A^{(d)}$ in the $\mathrm{Proj}_d(A)$ case). Set $M = B_1 \otimes_{B_0} R$.

Let $x_i = (L_i, \varphi_i) \in X(R_{s_i})$ be a family of local data — each L_i an R_{s_i}-line and $\varphi_i \colon B \otimes R_{s_i} \twoheadrightarrow \mathrm{Sym}(L_i)$ — inducing in degree 1 a surjection $\pi_i \colon M_{s_i} \twoheadrightarrow L_i$.

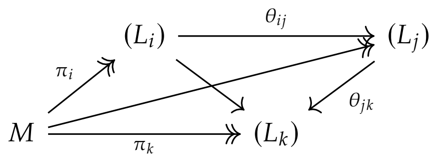
Assume these data are compatible on overlaps: there exist isomorphisms $\theta_{ij} \colon (L_i)_{s_j} \xrightarrow{\sim} (L_j)_{s_i}$ identifying (L_i, φ_i) and (L_j, φ_j) after restriction to $R_{s_is_j}$, that is, the triangle



commutes. Since π_i is surjective, any such θ_{ij} is unique.

Step 1: Descent of the line L. We construct a global R-line L with quotient map $\pi \colon M \twoheadrightarrow L$. By generalized Zariski descent it suffices to show the cocycle condition $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$ holds on $R_{s_is_js_k}$.

Consider the following diagram (all modules are localized at $R_{s_is_js_k}$):



- The upper left triangle commutes (by definition of θ_{ij}).
- The right triangle commutes (by definition of θ_{jk}).
- The outer large triangle commutes (by definition of θ_{ik}).

Since $M \twoheadrightarrow (L_i)$ is surjective, we can verify by tracing the images of elements:

$$\theta_{jk}(\theta_{ij}(\pi_i(m))) = \theta_{jk}(\pi_j(m)) = \pi_k(m) = \theta_{ik}(\pi_i(m)).$$

Since π_i is surjective, $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$.

By generalized Zariski descent for R-modules (Proposition 2.2.26), there exists a unique R-module L with isomorphisms $L_{s_i} \simeq L_i$. Moreover, since "being a line bundle (invertible module)" is a local property, and L is locally isomorphic to the line bundles L_i, L is an R-line.

Step 2: Descent of the morphism φ. Now we have the global line L and local isomorphisms. The local morphisms φ_i can be viewed as $\psi_i \colon B \otimes R_{s_i} \to \mathrm{Sym}(L)_{s_i}$. The compatibility condition ensures that ψ_i and ψ_j agree on intersections.

By Zariski descent for morphisms, this family ψ_i uniquely glues to a global algebra morphism:

$$\varphi\colon B \otimes R \longrightarrow \mathrm{Sym}(L).$$

Surjectivity is local: each φ_i is surjective and φ agrees with φ_i after localizing at s_i, so φ is surjective (in the Proj_1 case this is just ordinary surjectivity; for Proj_d via $A^{(d)}$ it amounts to eventual surjectivity).

This exhibits the unique $(L, \varphi) \in X(R)$ restricting to the given local data. □

The structure theorem rests on the following lemma about the affine open $D(f)$.

Lemma 2.3.11 (Properties of $D(f)$). *Let A be an $\mathbb{N}$-graded ring, $d \geq 1$, and $f \in A_d$. Let $D(f) \subset \mathrm{Proj}_d(A)$ be the subfunctor defined by: for a ring R, $D(f)(R) = \{(L, \varphi) \mid \varphi(f) \text{ generates } L\}$.*

(i) There is a canonical isomorphism $D(f) \simeq \mathrm{Spec}(A_{(f)})$.

(ii) For each $n \geq 1$, the map $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$ induces an isomorphism:

$$D(f) \xrightarrow{\sim} D(f^n).$$

More precisely, the diagram

$$\begin{array}{ccc} D(f) & \xrightarrow{\sim} & D(f^n) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Proj}_d(A) & \longrightarrow & \mathrm{Proj}_{nd}(A) \end{array}$$

is cartesian.

Proof. (i) We construct mutually inverse maps in two steps.

(a) The map $D(f) \to \mathrm{Spec}(A_{(f)})$.

Let $(L, \varphi) \in D(f)(R)$. By definition, $\varphi(f)$ generates L. This means $\varphi(f)$ is invertible in $\mathrm{Sym}_R(L)$ (geometrically corresponding to introducing the dual $L^\vee$). Localizing at $\varphi(f)$ gives the $\mathbb{Z}$-graded algebra $\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}$. This induces the following commutative diagram:

$$\begin{array}{ccc} A^{(d)} \otimes_{A_0} R & \xrightarrow{\varphi} & \mathrm{Sym}_R(L) \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle \varphi(f)^{-1}} \\ A_f^{(d)} \otimes_{A_0} R & \overset{\tilde{\varphi}}{\dashrightarrow} & \bigoplus_{n\in\mathbb{Z}} L^{\otimes n} \end{array}$$

Take the *degree 0 part* of the bottom row. The left side is $A_{(f)} \otimes R$; since L is trivialized by $\varphi(f)$, the right side gives $L^{\otimes 0} \simeq R$. Therefore, $\tilde{\varphi}$ restricted to degree 0 gives a homomorphism $\psi\colon A_{(f)} \to R$.

(b) The map $\mathrm{Spec}(A_{(f)}) \to D(f)$.

Conversely, given $\psi\colon A_{(f)} \to R$, take $L = R$ with generator T, so $\mathrm{Sym}_R(L) = R[T]$. Using the graded isomorphism $A_f^{(d)} \simeq A_{(f)}[u, u^{-1}]$ (where u is the degree-1 generator corresponding to f), set $\tilde{\varphi}\colon A_{(f)}[u, u^{-1}] \to R[T, T^{-1}]$ to be ψ on coefficients and $u \mapsto T$ on the generator, then restrict to $A^{(d)} \to R[T]$:

$$\begin{array}{ccc} A^{(d)} & \overset{\varphi}{\dashrightarrow} & R[T] \\ \downarrow & & \downarrow \\ A_{(f)}[u, u^{-1}] & \xrightarrow{\tilde{\varphi}} & R[T, T^{-1}] \end{array}$$

Then $\varphi(f) = T$ generates L, so $(R, \varphi) \in D(f)(R)$. The two constructions are mutually inverse by direct verification.

(ii) Localization of A at f and at f^n coincide as graded rings, $A_f = A_{f^n}$, hence so do their degree-0 parts $A_{(f)} = A_{(f^n)}$. Therefore, we have the following commutative diagram (where the vertical arrows are the isomorphisms established in part 1):

$$\begin{array}{ccc} D(f) & \longrightarrow & D(f^n) \\ \wr\downarrow & & \downarrow\wr \\ \operatorname{Spec}(A_{(f)}) & = & \operatorname{Spec}(A_{(f^n)}) \end{array}$$

This proves that $D(f) \xrightarrow{\sim} D(f^n)$ is an isomorphism.

Finally, we show that the diagram:

$$\begin{array}{ccc} D(f) & \xrightarrow{\sim} & D(f^n) \\ \downarrow & & \downarrow \\ \operatorname{Proj}_d(A) & \longrightarrow & \operatorname{Proj}_{nd}(A) \end{array}$$

is a pullback. Since we are considering algebraic functors, we only need to check the fiber product property on sets for any ring R.

The map $\operatorname{Proj}_d(A) \to \operatorname{Proj}_{nd}(A)$ sends (L, φ) to $(L^{\otimes n}, \varphi^{(n)})$, so the cartesian-square claim reduces to the equivalence

$$\varphi(f) \text{ generates } L \iff \varphi(f)^{\otimes n} \text{ generates } L^{\otimes n}.$$

This is local on $\operatorname{Spec}(R)$ and reduces to the elementary statement $u \in R^\times \iff u^n \in R^\times$ for $u \in R$.

□

We are now ready to prove the structure theorem.

Proof of Theorem 2.3.6. Morally, $\operatorname{Proj}_d(A) = \bigcup_{f \in A_d} D(f)$ — not literally, but the open subfunctors $D(f)$ do cover $\operatorname{Proj}_d(A)$, with pullback diagrams

$$\begin{array}{ccc} \operatorname{Spec}(R_{\varphi(f)}) & \longrightarrow & D(f) \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec}(R) & \xrightarrow{x} & \operatorname{Proj}_d(A) \end{array} \tag{2.3}$$

From the construction, x corresponds to $\varphi\colon A^{(d)} \otimes_{A_0} R \to \operatorname{Sym}_R(L)$. The strategy is to show $\operatorname{Proj}_d(A) \to \operatorname{Proj}_{nd}(A)$ is an open immersion (with image cut out by "$(\varphi(f))_{f \in A_d}$ generates L"), then use Zariski descent and Remark 2.3.4 to conclude.

Injectivity. We show $i\colon \operatorname{Proj}_d(A) \to \operatorname{Proj}(A)$ is a monomorphism. Suppose $x_1, x_2\colon \operatorname{Spec}(R) \to \operatorname{Proj}_d(A)$ satisfy $i \circ x_1 = i \circ x_2$. By Lemma 2.3.11, for every $f \in A_d$ and $n \geq 1$ we have the diagram

$$\begin{array}{ccccc} \operatorname{Spec}(R_{\varphi_1(f)}) & \overset{y_1}{\underset{y_2}{\rightrightarrows}} & D(f) & \xrightarrow{\sim} & D(f^n) \\ \downarrow{\scriptstyle j_f} & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec}(R) & \overset{x_1}{\underset{x_2}{\rightrightarrows}} & \operatorname{Proj}_d(A) & \longrightarrow & \operatorname{Proj}_{nd}(A) \end{array}$$

(the pullback along x_1 and along x_2 give isomorphic results). Here y_1 classifies the line $(L_1, \varphi_1 \colon A^{(d)} \otimes_{A_0} R_{\varphi_1(f)} \to L)$ on which $\varphi_1(f)$ generates L_1. Zariski descent gives an embedding

$$\mathrm{Proj}_d(A)(R) \hookrightarrow \prod_{f \in A_d} \mathrm{Proj}_d(A)(R_{\varphi_1(f)}), \quad x \mapsto (x \circ j_f)_{f \in A_d}.$$

Since $D(f) \simeq D(f^n)$ identifies y_1 and y_2, we have $x_1 \circ j_f = x_2 \circ j_f$ for every f, so $x_1 = x_2$.

Openness. We claim the pullback of i along any $x \colon \mathrm{Spec}(R) \to \mathrm{Proj}_{nd}(A)$ is the open subfunctor of $\mathrm{Spec}(R)$ where $(\varphi(f^n))_{f \in A_d}$ generates L:

$$\begin{array}{ccc} \{(\varphi(f^n))_{f \in A_d} \text{ generates } L\} & \longrightarrow & \mathrm{Spec}(R) \\ \big\downarrow & \lrcorner & \big\downarrow x \\ \mathrm{Proj}_d(A) & \hookrightarrow & \mathrm{Proj}_{nd}(A) \end{array}$$

Unpacking, x corresponds to $\varphi \colon A^{(nd)} \otimes_{A_0} R \to \mathrm{Sym}_R(L)$, and the claim is that

$$x \text{ factors through } \mathrm{Proj}_d(A) \iff L = \langle \varphi(f^n) \mid f \in A_d \rangle.$$

We again use Zariski descent to reduce to $D(f)$:

- $\Rightarrow$ If x lies in $\mathrm{Proj}_d(A)$, then x corresponds to $\psi \colon A^{(d)} \otimes_{A_0} R \to \mathrm{Sym}_R(M)$ such that $M^{\otimes n} = L$, $\psi^{(n)} = \varphi$, and $M = \langle \psi(f) \mid f \in A_d \rangle$. This is equivalent to saying $L = M^{\otimes n} = \langle \psi(f^n) \mid f \in A_d \rangle$.
- $\Leftarrow$ Suppose L is generated by $\{\varphi(f^n)\}_{f \in A_d}$; we construct the unique lift $x' \in \mathrm{Proj}_d(A)(R)$. The plan is to lift x locally on the cover $\{\mathrm{Spec}(R_{\varphi(f^n)})\}_{f \in A_d}$ and glue:

$$\begin{array}{ccccc} \exists! x' & \dashrightarrow & (x'_f)_f & & (x'_{fg})_{f,g} \\ \mathrm{Proj}_d(A)(R) & \longrightarrow & \prod_f \underbrace{D(f)(R_{\varphi(f^n)})}_{\text{local lift}} & \rightrightarrows & \prod_{f,g} D(fg)(R_{\varphi(f^n g^n)}) \\ \big\downarrow & & \big\downarrow \wr & & \big\downarrow \wr \\ \mathrm{Proj}_{nd}(A)(R) & \longrightarrow & \prod_f D(f^n)(R_{\varphi(f^n)}) & \rightrightarrows & \prod_{f,g} D(f^n g^n)(R_{\varphi(f^n g^n)}) \\ x & \longmapsto & (x_f)_f & & (x_{fg})_{f,g} \end{array}$$

  The lift is built in three steps:

  (i) *Restrict.* Restrict x to $D(f^n)(R_{\varphi(f^n)})$ to obtain $(x_f)_{f \in A_d}$; the generation hypothesis ensures the restriction lands in $D(f^n)$.

  (ii) *Lift locally.* The isomorphism $D(f^n) \simeq D(f)$ from Lemma 2.3.11(2) sends each x_f to a unique $x'_f \in D(f)(R_{\varphi(f^n)})$.

  (iii) *Glue.* The cocycle conditions for (x'_f) follow from those for (x_f) by canonicity of the vertical isomorphisms; Zariski descent (Proposition 2.3.10) then assembles (x'_f) into a global $x' \in \mathrm{Proj}_d(A)(R)$.

It remains to check that the resulting subfunctor of $\mathrm{Spec}(R)$ is open. The map $\Psi \colon R^{(A_d)} \to L$ defined by $e_f \mapsto \varphi(f^n)$ has surjection locus $\mathrm{Epi}(\Psi) \subset \mathrm{Spec}(R)$, which is open by Proposition 2.2.19. Since "L generated by $\{\varphi(f^n)\}_{f \in A_d}$" is precisely the condition that Ψ be surjective, the pullback subfunctor coincides with $\mathrm{Epi}(\Psi)$, completing the proof of openness.

Surjectivity. Assume now that A is generated as an A_0-algebra by homogeneous elements of degrees divisible by d. Every $f \in A_{nd}$ is then an A_0-linear combination of products $f_1 \cdots f_k$ with each $\deg(f_i)$ divisible by d, so by linearity it suffices to lift each such product. Given $x\colon \operatorname{Spec}(R) \to \operatorname{Proj}_{nd}(A)$ corresponding to (L, φ), we seek $x'\colon \operatorname{Spec}(R) \to \operatorname{Proj}_d(A)$ with

$$\begin{array}{ccc} & & \operatorname{Proj}_d(A) \\ & \overset{x'}{\nearrow} & \big\downarrow \\ \operatorname{Spec}(R) & \xrightarrow{x} & \operatorname{Proj}_{nd}(A) \end{array}$$

Here x corresponds to (L, φ) with L generated by the $\varphi(f)$ for $f \in A_{nd}$ a product of generators. For each generator f_i (of degree dividing d), set $m_i := d/\deg(f_i)$, so $f_i^{m_i} \in A_d$; in particular $f_1^{m_1} \in A_d$. The diagram

$$\begin{array}{ccccc} & & D(f_1^{m_1}) & \hookrightarrow & \operatorname{Proj}_d(A) \\ & & \big\downarrow \simeq & & \big\downarrow \\ D(f) & \subset & D(f_1^{m_1 n}) & \hookrightarrow & \operatorname{Proj}_{nd}(A) \end{array}$$

lifts the $D(f)$-piece of x into $\operatorname{Proj}_d(A)$. The inclusion $D(f) \subset D(f_1^{m_1 n})$ holds because $\varphi(f) = \varphi(f_1) \otimes \cdots \otimes \varphi(f_k)$ generates L iff each tensor factor $\varphi(f_i)$ generates the corresponding tensor component of L — in particular $\varphi(f_1)$, hence $\varphi(f_1^{m_1 n}) = \varphi(f_1)^{\otimes m_1 n}$, generates its component. Zariski descent assembles these local lifts into a global $x'\colon \operatorname{Spec}(R) \to \operatorname{Proj}_d(A)$.

□

2.3.1. Comparison Between Spec and Proj

With the Proj construction in place, we can systematically compare it with Spec.

***Remark* 2.3.12** (Relationship Between Spec and Proj). For an $\mathbb{N}$-graded ring A, we have the following canonical diagram over $\operatorname{Spec}(A_0)$:

$$\begin{array}{ccccc} \operatorname{Spec}(A) & \hookleftarrow & D(A_+) & \longrightarrow & \operatorname{Proj}(A) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \operatorname{Spec}(A_0) & = & \operatorname{Spec}(A_0) & = & \operatorname{Spec}(A_0) \end{array}$$

Here:

(i) $D(A_+) = \operatorname{colim}_{d \geq 1} D(A_d)$ is the *non-vanishing locus of the irrelevant ideal* $A_+ = \bigoplus_{d \geq 1} A_d$ in $\operatorname{Spec}(A)$.

(ii) The map $D(A_+) \to \operatorname{Proj}(A)$ is constructed as follows: For $f \in A_d$ with $d \geq 1$, we have $D(f) \subset \operatorname{Spec}(A)$ corresponding to the localization $\operatorname{Spec}(A_f)$. On the other hand, from Lemma 2.3.11, $D(f) \subset \operatorname{Proj}_d(A)$ corresponds to $\operatorname{Spec}(A_{(f)})$. The relationship between these two is:

$$A_f \simeq A_{(f)}[u, u^{-1}],$$

where u is the degree 1 variable corresponding to f. This gives a canonical map $\operatorname{Spec}(A_f) \to \operatorname{Spec}(A_{(f)})$, hence $D(f) \to D(f) \subset \operatorname{Proj}_d(A)$.

(iii) The map $D(f) \to \operatorname{Proj}_d(A)$ sends an A-algebra homomorphism $\varphi\colon A \to R$ to the graded map:

$$\varphi^{(d)}\colon A^{(d)} \otimes_{A_0} R \to \bigoplus_{n \in \mathbb{N}} R \cdot \varphi(f)^n \simeq \operatorname{Sym}_R(R),$$

where the right side is identified with the symmetric algebra of the trivial line R with generator $\varphi(f)$. This is precisely an element of $\operatorname{Proj}_d(A)(R)$.

(iv) $\mathbb{G}_m$-*equivariance:* Note that $D(A_+) \subset \mathrm{Spec}(A)$ carries a natural $\mathbb{G}_m$-action (coming from the grading). The map $D(A_+) \to \mathrm{Proj}(A)$ is $\mathbb{G}_m$-equivariant. When $d = 1$ (i.e., A is generated by A_1), the map $D(A_1) \to \mathrm{Proj}_1(A)$ has image precisely the *trivial quotient lines*, and induces an injection:

$$D(A_1)/\mathbb{G}_m \hookrightarrow \mathrm{Proj}_1(A).$$

***Remark* 2.3.13** (Extension to Spec). We can extend the domain of the Proj functor to include Spec as follows: For any ring R (viewed as a trivially graded ring concentrated in degree 0), we can define $\mathrm{Proj}(R[x])$ where x has degree 1. Then:

$$\mathrm{Proj}(R[x]) \simeq \mathrm{Proj}_1(R[x]) = \mathbb{P}^0_R = \mathrm{Spec}(R).$$

This shows that Proj can be viewed as a generalization of Spec.

***Remark* 2.3.14** (Comparison with Previous Definition). Earlier, we defined projective schemes using homogeneous polynomial systems. How does this relate to the current definition?

For an $\mathbb{N}$-graded k-algebra A generated by A_1, we have a canonical surjection:

$$\mathrm{Sym}_{A_0}(A_1) \twoheadrightarrow A.$$

According to the previous definition, hSol_Σ for a homogeneous system Σ is defined as the set of quotient lines $A_1 \otimes_k R \twoheadrightarrow L$ through which the composition $\mathrm{Sym}_k(A_1) \to A \to \mathrm{Sym}_R(L)$ factors. By the universal property of the symmetric algebra, this is precisely the definition of $\mathrm{Proj}_1(A)(R)$.

Therefore, the two definitions are completely consistent.

2.3.2. Vanishing Loci and Non-vanishing Loci in Proj

We now introduce vanishing and non-vanishing loci in $\mathrm{Proj}(A)$, the analogues of the corresponding affine constructions.

Definition 2.3.15 (Vanishing Loci and Non-vanishing Loci). Let A be an $\mathbb{N}$-graded ring and $F \subset A$ a set of homogeneous elements. We define:

(i) The *vanishing locus* $V(F) \subset \mathrm{Proj}(A)$ is defined as the colimit:

$$V(F) = \mathrm{colim}_{d \geq 1} V_d(F),$$

where for each $d \geq 1$:

$$V_d(F)(R) = \left\{ \varphi \colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \,\middle|\, \text{for all } f \in F^{(d)}, \varphi(f) = 0 \right\}.$$

(ii) The *non-vanishing locus* $D(F) \subset \mathrm{Proj}(A)$ is defined as the colimit:

$$D(F) = \mathrm{colim}_{d \geq 1} D_d(F),$$

where for each $d \geq 1$:

$$D_d(F)(R) = \left\{ \varphi \colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \,\middle|\, \exists n \geq 1 : \varphi((F)_{nd}) \text{ generates } L^{\otimes n} \right\}.$$

Here $(F)_{nd}$ denotes the degree nd part of the ideal generated by F in A.

***Remark* 2.3.16** (Properties of $V(F)$ and $D(F)$). These constructions enjoy properties parallel to the affine case:

- $\bigcap_i V(F_i) = V(\bigcup_i F_i)$.
- $\bigcup_i D(F_i) \subset D(\bigcup_i F_i)$ (not necessarily equality).
- $V(F)$ is determined by the ideal (F) generated by F.
- $D(F)$ is determined by the radical $\sqrt{(F)}$ of the ideal generated by F.
- When A is generated by $A_{\leq 1}$, we have simpler descriptions. For instance, for a single homogeneous element $f \in A_d$:
$$V(f)(R) = \{\varphi \mid \varphi(f) = 0 \in L^{\otimes d}\},$$
$$D(f)(R) = \{\varphi \mid \varphi(f) \text{ generates } L^{\otimes d}\}.$$

***Remark* 2.3.17** (Further Properties). (i) For a homogeneous ideal $I \subset A$, we have $\mathrm{Proj}(A/I) \simeq V(I)$ canonically. This follows from the surjection $A \twoheadrightarrow A/I$ inducing a closed immersion $\mathrm{Proj}(A/I) \hookrightarrow \mathrm{Proj}(A)$.

(ii) For $f \in A_d$ with $d \geq 1$, we have $D(f^n) \simeq D(f) \simeq \mathrm{Spec}(A_{(f)})$ for all $n \geq 1$. This was proved in Lemma 2.3.11.

(iii) For $f \in A_0$, we have $\mathrm{Proj}(A_f) \simeq D(f)$, where $A_f = A[f^{-1}]$ is the ordinary localization (which inherits the $\mathbb{N}$-grading since f has degree zero). Note that here we use A_f rather than $A_{(f)}$: the latter notation is reserved for the degree-zero part of A_f when $\deg f \geq 1$, where the two would clash.

Proposition 2.3.18 (Closedness and Openness). *Let A be an $\mathbb{N}$-graded ring and $F \subset A$ a set of homogeneous elements.*

(i) $V(F)$ *is a* closed subfunctor *of* $\mathrm{Proj}(A)$.

(ii) $D(F)$ *is an* open subfunctor *of* $\mathrm{Proj}(A)$.

Proof. We may assume A is generated by $A_{\leq 1}$ (the general case follows from passing to Veronese subrings and using cofinality).

(i) *Closedness of $V(F)$:* We can write $V(F) = \bigcap_{f \in F} V(f)$, so it suffices to show $V(f)$ is closed for a single homogeneous element $f \in A_d$.

For any R-point $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ corresponding to a quotient line $\varphi\colon A_1 \otimes_k R \twoheadrightarrow L$, we have:
$$x \in V(f) \iff \varphi(f) = 0 \in L^{\otimes d}.$$

Consider the pullback:
$$\begin{array}{ccc} x^{-1}(V(f)) & \longrightarrow & V(f) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{x} & \mathrm{Proj}(A) \end{array}$$

It suffices to show $x^{-1}(V(f)) \subset \mathrm{Spec}(R)$ is closed. By definition,
$$x^{-1}(V(f))(S) = \{\psi\colon R \to S \mid \psi^*(\varphi(f)) = 0 \in L^{\otimes d} \otimes_R S\}.$$

Since L is an R-line, hence projective, $L^{\otimes d}$ is also projective, hence reflexive. Therefore:
$$\varphi(f) = 0 \iff \text{for all } \lambda \in (L^{\otimes d})^\vee, \lambda(\varphi(f)) = 0.$$

The condition $\lambda(\varphi(f)) = 0$ defines a closed subset $V(\lambda(\varphi(f))) \subset \mathrm{Spec}(R)$ (since $\lambda(\varphi(f)) \in R$ generates an ideal). Therefore:

$$x^{-1}(V(f)) = \bigcap_{\lambda \in (L^{\otimes d})^\vee} V(\lambda(\varphi(f)))$$

is closed.

(ii) *Openness of $D(F)$:* For an R-point x corresponding to $\varphi\colon A \otimes R \twoheadrightarrow \mathrm{Sym}_R(L)$, we have:

$$x \in D(F) \iff \exists n \geq 1 : \{(\varphi(f))_{f \in F_n}\} \text{ generates } L^{\otimes n},$$

where F_n denotes the homogeneous elements in F of degrees dividing n.

This is equivalent to:

$$x^{-1}(D(F)) = \bigcup_{n \geq 1} \mathrm{Epi}(R^{(F_n)} \to L^{\otimes n}).$$

Each $\mathrm{Epi}(R^{(F_n)} \to L^{\otimes n})$ is an open subfunctor by Proposition 2.2.19. More compactly,

$$x^{-1}(D(F)) = D\Big(\bigcup_{n \geq 1} I_n\Big),$$

where $I_n \subset R$ is the radical ideal cut out by the image of (F_n) under φ. The chain $I_1 \subset I_2 \subset \cdots$ has union $\bigcup_n I_n$ which is itself an ideal, so $x^{-1}(D(F))$ is the open subfunctor it cuts out.

□

Example 2.3.19 (Non-uniqueness of Vanishing Loci)**.** Unlike the affine case, in the projective case, $V(F) = V(F')$ does *not* imply $(F) = (F')$ in general.

Example: Consider $A = \mathbb{Z}[x, y]$ with the standard grading. We have:

$$V(x) = V(x^2, xy) \subset \mathbb{P}^1.$$

Verification. The inclusion $(x^2, xy) \subset (x)$ gives $V(x) \subset V(x^2, xy)$.

For the reverse inclusion, suppose $\varphi\colon \mathbb{Z}[x, y] \to \mathrm{Sym}_R(L)$ satisfies $\varphi(x^2) = 0$ and $\varphi(xy) = 0$ in $L^{\otimes 2}$. Write $\varphi(x) = a$ and $\varphi(y) = b$ where $a, b \in L$ generate L (i.e., $L = \langle a, b\rangle$).

From $\varphi(x^2) = a^{\otimes 2} = 0$ and $\varphi(xy) = a \otimes b = 0$ in $L^{\otimes 2}$, we deduce:

- Since $a^{\otimes 2} = 0$, we have $a \cdot a = 0$ (viewing multiplication in the symmetric algebra).
- Since $a \otimes b = 0$ and $L = \langle a, b\rangle$, we must have $a \cdot L = 0$.

In particular, $a \otimes b = 0$ in $L^{\otimes 2}$ for every b generating L. Since L is invertible, the map $L \to L^{\otimes 2}$, $x \mapsto x \otimes b$, is an isomorphism (with inverse given by tensoring with $b^{-1} \in L^\vee$), so $a = 0$. Hence $\varphi(x) = a = 0$, i.e. $\varphi \in V(x)$. □

This shows that the ideals (x) and (x^2, xy) define the same vanishing locus in $\mathbb{P}^1$, even though they are different ideals.

This phenomenon leads us to introduce the concept of *saturation*.

2.3.3. Saturation of Homogeneous Ideals

Definition 2.3.20 (Saturation). Let A be an $\mathbb{N}$-graded ring and $H \subset A$ a homogeneous ideal. The *saturation* of H is:

$$H^{\mathrm{sat}} = \{x \in A \mid \text{for all } f \in A_+, \exists n \geq 1 : f^n x \in H\}.$$

We say H is *saturated* if $H = H^{\mathrm{sat}}$.

***Remark* 2.3.21** (Properties of Saturation).

(i) The saturation H^{sat} is determined by the "tail" of H: for any infinite subset $D \subset \mathbb{N}$, we have:

$$H^{\mathrm{sat}} = \left(\bigoplus_{d \in D} H_d\right)^{\mathrm{sat}}.$$

(ii) If A is generated by $A_{\leq 1}$, we can replace A_+ with A_1 in the definition of saturation.

(iii) For a radical ideal H (i.e., $\sqrt{H} = H$), the saturation H^{sat} is also radical.

(iv) A homogeneous prime ideal $\mathfrak{p} \subset A$ is saturated if and only if $A_+ \not\subset \mathfrak{p}$.

Example 2.3.22 (Examples of Saturation).

(i) For $A = \mathbb{Z}[x, y]$, $(x) = (x^2, xy)^{\mathrm{sat}}$: any $g \in A$ killed by some power of every element of $A_+ = (x, y)$ — equivalently, satisfying $y^n g, x^m g \in (x^2, xy)$ for some n, m — must lie in (x), by inspecting the y-degree.

(ii) For polynomial rings $k[x_i]$, the zero ideal (0) is saturated. Indeed, A_+-nilpotency would force every $x_i^n g = 0$ for some n; since A has no zero divisors, this forces $g = 0$.

Our goal is to show that $V(H) = V(H')$ if and only if $H^{\mathrm{sat}} = H'^{\mathrm{sat}}$.

The key tool is the theory of *I-nilpotent* and *I-local* modules, which we now develop.

Definition 2.3.23 (I-nilpotent and I-local Modules). Let R be a ring, $I \subset R$ a subset, and M an R-module.

(i) An element $x \in M$ is *I-nilpotent* if for all $f \in I$, there exists $n \geq 1$ such that $f^n x = 0$.

(ii) We denote by $\Gamma_I(M) \subset M$ the *I-nilpotent submodule*, i.e., the submodule consisting of all I-nilpotent elements.

(iii) M is *I-nilpotent* if $\Gamma_I(M) = M$.

(iv) M is *I-local* if for any homomorphism $h: P \to Q$ with I-nilpotent kernel and cokernel, the induced map:

$$h^*: \operatorname{Hom}_R(Q, M) \xrightarrow{\sim} \operatorname{Hom}_R(P, M)$$

is an isomorphism.

Proposition 2.3.24 (Categories of I-nilpotent and I-local Modules). *Let R be a ring and $I \subset R$ a subset.*

(i) $\mathrm{Mod}_R^{I\text{-nil}}$ and $\mathrm{Mod}_R^{I\text{-loc}}$ are Grothendieck abelian categories.

(ii) The inclusion $\iota: \mathrm{Mod}_R^{I\text{-nil}} \hookrightarrow \mathrm{Mod}_R$ has a right adjoint Γ_I, and the inclusion $\iota': \mathrm{Mod}_R^{I\text{-loc}} \hookrightarrow \mathrm{Mod}_R$ has a left adjoint L_I. Both Γ_I and L_I are exact.

(iii) M is I-nilpotent if and only if $\mathrm{L}_I(M) = 0$.

(iv) $\Gamma_I(M) = \ker(M \to \mathrm{L}_I(M))$, and the cokernel of $M \to \mathrm{L}_I(M)$ is I-nilpotent.

(v) For a morphism $h: M \to N$ in Mod_R, $\mathrm{L}_I(h)$ is an isomorphism if and only if $h_f: M_f \to N_f$ is an isomorphism for every $f \in I$.

Proof. (1). The class of I-nilpotent modules is closed under submodules, quotients, extensions, and arbitrary direct sums in Mod_R (each test "$f^n x = 0$" is preserved by these operations). Hence $\mathrm{Mod}_R^{I\text{-nil}}$ is a Serre subcategory closed under colimits in Mod_R, and inherits the Grothendieck abelian structure. The class of I-local modules is the right orthogonal $(\mathrm{Mod}_R^{I\text{-nil}})^{\perp}$ — by Definition 2.3.23, M is I-local iff $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(Q, M)$ is bijective for every $h: P \to Q$ with I-nilpotent kernel and cokernel. Such an orthogonal complement to a localizing subcategory is again Grothendieck abelian (this is a standard fact, see e.g. [The25, Tag 0AS6]).

(2). Right adjoint to ι: define $\Gamma_I(M) = \{x \in M \mid \forall f \in I, \exists n, f^n x = 0\}$. This is the maximal I-nilpotent submodule of M, so any R-linear map $N \to M$ from an I-nilpotent N factors uniquely through $\Gamma_I(M)$, giving the adjunction $\iota \dashv \Gamma_I$. Left adjoint to ι': L_I is the Bousfield localization at the maps with I-nilpotent kernel and cokernel, which exists because $\mathrm{Mod}_R^{I\text{-nil}}$ is a localizing subcategory of the Grothendieck abelian category Mod_R. Exactness: Γ_I is left exact as a right adjoint, and L_I is right exact as a left adjoint. For the converse exactness, observe that the kernel and cokernel of the unit $M \to \mathrm{L}_I M$ are I-nilpotent (by construction of the localization), so killing them produces an exact sequence in the localized category; this transfers exactness in both directions.

(3). If $\mathrm{L}_I(M) = 0$, then the unit $M \to 0$ exhibits $M \to 0$ as an L_I-isomorphism, hence M has I-nilpotent kernel and cokernel. The cokernel 0 is trivial, and the kernel is M, so M is I-nilpotent. Conversely, if M is I-nilpotent, then for any I-local N, $\mathrm{Hom}_R(M, N) = 0$ — every map from an I-nilpotent module to an I-local one is zero, since $M = \Gamma_I(M)$ and the unit $0 \to N$ has I-nilpotent kernel/cokernel — so $\mathrm{L}_I(M) = 0$.

(4). The unit $\eta_M : M \to \mathrm{L}_I(M)$ is an L_I-isomorphism, so its kernel and cokernel are I-nilpotent — this gives the second assertion. For the first, $\ker(\eta_M)$ is an I-nilpotent submodule of M, hence $\ker(\eta_M) \subset \Gamma_I(M)$. Conversely, the composite $\Gamma_I(M) \hookrightarrow M \xrightarrow{\eta_M} \mathrm{L}_I(M)$ factors through $\mathrm{L}_I(\Gamma_I(M)) = 0$ (by (3)), so $\Gamma_I(M) \subset \ker(\eta_M)$.

(5). The class of morphisms h such that each h_f ($f \in I$) is an isomorphism contains all L_I-isomorphisms (since L_I inverts each $f \in I$). Conversely, if every h_f is an iso, then $\ker(h)_f = \mathrm{coker}(h)_f = 0$ for all $f \in I$, so $\ker(h)$ and $\mathrm{coker}(h)$ are I-nilpotent (by Definition 2.3.23 and the fact that an element killed by some f^n becomes zero in M_f). Thus h is an L_I-isomorphism. □

***Remark* 2.3.25** (Explicit Description for Finite I). When $I = \{f_1, \ldots, f_n\}$ is finite, $\mathrm{Mod}_R^{I\text{-loc}}$ can be identified with the category of quasi-coherent modules on the open subset $D(I) \subset \mathrm{Spec}(R)$.

In this case, $\mathrm{L}_I(M)$ can be computed as the equalizer:

$$\mathrm{L}_I(M) \to \prod_{i=1}^{n} M_{f_i} \rightrightarrows \prod_{i,j} M_{f_i f_j}.$$

In particular, $\mathcal{O}(D(I)) = \mathrm{L}_I(R)$.

Example 2.3.26 (Special Cases). (i) When $I = \{f\}$ consists of a single element, "I-nilpotent" means "f-torsion", and $\mathrm{L}_I(M) = M_f$ (the localization at f).

(ii) For an $\mathbb{N}$-graded ring A and homogeneous ideal $H \subset A$, we have:

$$H^{\mathrm{sat}} = \ker(A \to A/H \to \mathrm{L}_{A_+}(A/H)).$$

(iii) In particular, $(0)^{\mathrm{sat}} = \Gamma_{A_+}(A)$.

We can now state the key proposition relating Proj to I-local modules.

Proposition 2.3.27 (Characterization of Proj via I-local Modules). *Let A, B be $\mathbb{N}$-graded rings and $\alpha : A \to B$ an eventually surjective graded map. Then the following are equivalent:*

(i) $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ *is an isomorphism.*

(ii) *For all* $d \geq 1$*, the map:*

$$\mathrm{L}_{A_d}(A^{(d)}) \to \mathrm{L}_{A_d}(B^{(d)})$$

is an isomorphism.

If α *is surjective, these are also equivalent to:*

(3) *For all* $d \geq 1$, $\ker(A \to B^{(d)})$ *is* A_d*-nilpotent.*

If A *is generated by* $A_{\leq 1}$*, then it suffices to check condition (2) or (3) for* $d = 1$ *only.*

Proof Sketch (for A *generated by* $A_{\leq 1}$*).*

$1 \Rightarrow 2$: If $\mathrm{Proj}(B) \xrightarrow{\sim} \mathrm{Proj}(A)$, then for each $f \in A_1$, the pullback diagram gives $D(\alpha(f)) \simeq D(f)$. By Lemma 2.3.11, this means $B_{\alpha(f)} \simeq A_f$ as graded rings, hence $B_{(\alpha(f))} \simeq A_{(f)}$. Taking the equalizer over all $f \in A_1$ and using Zariski descent gives $\mathrm{L}_{A_1}(A) \xrightarrow{\sim} \mathrm{L}_{A_1}(B)$.

$2 \Rightarrow 1$: Assume $\mathrm{L}_{A_1}(A) \xrightarrow{\sim} \mathrm{L}_{A_1}(B)$. This implies $A_f \xrightarrow{\sim} B_f$ as graded rings for all $f \in A_1$, hence $A_{(f)} \xrightarrow{\sim} B_{(f)}$. For any $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ corresponding to a quotient line $A_1 \otimes R \twoheadrightarrow L$, we can use Zariski descent over the open cover $\{D(f)\}_{f \in A_1}$ to lift x uniquely to $\mathrm{Proj}(B)$. This shows $\mathrm{Proj}(B) \xrightarrow{\sim} \mathrm{Proj}(A)$.

□

Corollary 2.3.28 (Functorial Projective Nullstellensatz). *For a ring* k*, the map* $F \mapsto V(F)$ *induces an order-preserving bijection:*

$$\{\textit{saturated homogeneous ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{P}^I_k\}.$$

Proof. Set $A = k[x_i \mid i \in I]$ and let $H, H' \subset A$ be homogeneous ideals. We show that $V(H) = V(H')$ iff $H^{\mathrm{sat}} = H'^{\mathrm{sat}}$.

Writing $V(H) = \mathrm{Proj}(A/H)$ and similarly for H', an inclusion $V(H') \hookrightarrow V(H)$ being an isomorphism amounts to $\mathrm{Proj}(A/H') \xrightarrow{\sim} \mathrm{Proj}(A/H)$. By Proposition 2.3.27, this happens iff the kernel of $A/H \twoheadrightarrow A/H'$ is A_+-nilpotent in A/H — equivalently, $H'/H \subset (0)^{\mathrm{sat}}$ in A/H, i.e. $H' \subset H^{\mathrm{sat}}$. By symmetry $H \subset H'^{\mathrm{sat}}$, and saturating both sides gives $H^{\mathrm{sat}} = H'^{\mathrm{sat}}$. □

We can now classify the open and closed subfunctors of $\mathrm{Proj}(A)$.

Proposition 2.3.29 (Classification of Open and Closed Subfunctors). *Let* A *be an* $\mathbb{N}$*-graded ring generated by* $A_{\leq 1}$*.*

(i) *The map* $F \mapsto V(F)$ *induces an order-preserving injection:*

$$\{\textit{saturated homogeneous ideals in } A\} \hookrightarrow \{\textit{closed subfunctors of } \mathrm{Proj}(A)\}.$$

This is a bijection if A *is finitely generated over* A_0*.*

(ii) *The map* $F \mapsto D(F)$ *induces an order-preserving bijection:*

$$\{\textit{saturated homogeneous radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Proj}(A)\}.$$

Proof. (i) *Injectivity* is Corollary 2.3.28 applied to the surjection $A \twoheadrightarrow A/H$.

Surjectivity (assuming A finitely generated over A_0). Pick a finite generating set $F \subset A_1$, so that $\{D(f)\}_{f \in F}$ is an affine open cover of $\mathrm{Proj}(A)$. For each $f \in F$, pull Z back along $D(f) \simeq \mathrm{Spec}(A_{(f)}) \hookrightarrow \mathrm{Proj}(A)$ to obtain a closed subfunctor $Z_f \subset \mathrm{Spec}(A_{(f)})$, which by the affine Nullstellensatz has the form $Z_f = V(I_f)$ for a unique ideal $I_f \subset A_{(f)}$.

Now lift each I_f back to A. Under the identification $A_f \simeq A_{(f)}[t^{\pm 1}]$ (with t in degree $\deg(f)$), the homogeneous ideal $\widetilde{I_f} := I_f[t^{\pm 1}] \subset A_f$ is the unique homogeneous ideal whose degree-zero part is I_f. Set

$$H(f) := \ker\bigl(A \twoheadrightarrow A_f/\widetilde{I_f}\bigr), \qquad H := \bigcap_{f \in F} H(f).$$

Each $H(f)$ is homogeneous and f*-saturated* ($x \in H(f) \iff f^n x \in H(f)$ for some $n \geq 0$), since it is pulled back from a localization at f.

$Z = V(H)$. The inclusion $V(H(f)) \cap D(f) = V(I_f) = Z_f$ holds by construction, so on each $D(f)$ we have $V(H) \cap D(f) = Z_f$. Since $\{D(f)\}$ covers $\mathrm{Proj}(A)$ and both $V(H)$ and Z are closed subfunctors, Zariski descent for Proj (Proposition 2.3.10) gives $V(H) = Z$.

H is saturated. Suppose $xA_+^N \subset H$. Since F generates A_+ as an ideal, A_+^N is generated by monomials in F of total degree N; in particular $xf^N \in H \subset H(f)$ for each $f \in F$. By f-saturation of $H(f)$, this forces $x \in H(f)$, and hence $x \in H$.

(ii) *Injectivity.* The key observation is that for any homogeneous subset $F \subset A_+$ and homogeneous element $a \in A_+$,

$$D(a) \subset D(F) \iff a \in \sqrt{(F)}^{\mathrm{sat}}.$$

The implication "$\Leftarrow$" is immediate, since $D(\cdot)$ is preserved by sums, powers, and saturation. For "$\Rightarrow$": pulling back to $D(a) \simeq \mathrm{Spec}(A_{(a)})$, the hypothesis becomes $\mathrm{Spec}(A_{(a)}) = D(F_a)$, where F_a is the image of F in $A_{(a)}$. The affine Nullstellensatz then gives $1 \in \sqrt{(F_a)}$, i.e. $1 = \sum_i (g_i/a^{n_i})$ in $A_{(a)}$ with each g_i a power of an element of (F). Clearing denominators yields $a^N \in (F)$ in A_a for large N, which lifts to $a^{N+M} \in (F) \subset A$ for some M — exactly the statement $a \in \sqrt{(F)}^{\mathrm{sat}}$.

Now $D(F) = D(F')$ forces $a \in \sqrt{(F')}^{\mathrm{sat}}$ for every $a \in F$, and symmetrically. Saturating gives $\sqrt{(F)}^{\mathrm{sat}} = \sqrt{(F')}^{\mathrm{sat}}$.

Surjectivity. Let $U \subset \mathrm{Proj}(A)$ be open, and pick a generating set $F \subset A_1$. For each $f \in F$ the pullback $U_f := U \cap D(f) \subset D(f) \simeq \mathrm{Spec}(A_{(f)})$ is open, hence of the form $D(J_f)$ for a unique radical ideal $J_f \subset A_{(f)}$. Set

$$H_f := \bigl\{ a \in A_+ \bigm| \deg(f) \mid \deg(a) \text{ and } a/f^{\deg(a)/\deg(f)} \in J_f \bigr\}, \qquad H := \sum_{f \in F} H_f.$$

By construction H_f generates J_f inside $A_{(f)}$, so $D(H) \cap D(f) \supset D(J_f) = U_f$ for each f, and equality follows because $D(\cdot)$ commutes with intersection over $D(f)$ for any $f' \in F$. Zariski descent over $\{D(f)\}_{f \in F}$ then gives $D(H) = U$.

□

***Remark* 2.3.30** (General $\mathbb{N}$-graded Rings). For an $\mathbb{N}$-graded ring A that is *not* generated by $A_{\leq 1}$, saturation must be controlled simultaneously at all the Veronese subrings $A^{(d)}$. Concretely, declare $H \subset A$ to be *saturated* when

$$H = \ker\Bigl(A \to \prod_{d \geq 1} \mathrm{L}_{A_d}(A/H)\Bigr),$$

i.e. a homogeneous element lies in H iff its image in A/H is annihilated by some power of A_d for every $d \geq 1$. With this definition, Proposition 2.3.29 continues to hold for general $\mathbb{N}$-graded A.

2.3.4. Quasi-projective Schemes

Just as quasi-affine schemes are open subfunctors of affine ones, quasi-projective schemes are open subfunctors of projective ones. By Proposition 2.3.29(2), every such open subfunctor of $\mathrm{Proj}(A)$ has the form $D(F)$ for a homogeneous subset $F \subset A_+$, which we may take to be finite.

Definition 2.3.31 (Quasi-projective Schemes). Let k be a ring. An algebraic k-functor X is a *quasi-projective k-scheme* if there exist an $\mathbb{N}$-graded k-algebra A finitely generated over A_0 and a finite homogeneous subset $F \subset A_+$ with $X \simeq D(F) \subset \mathrm{Proj}(A)$. Write $\mathsf{QProj}_k \subset \mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ for the full subcategory of quasi-projective k-schemes.

***Remark* 2.3.32** (Relations Among the Four Categories). The two "open subfunctor" inclusions

$$\mathsf{Aff}_k \subset \mathsf{QAff}_k, \qquad \mathrm{Proj}_k \subset \mathsf{QProj}_k$$

sit inside the larger inclusion $\mathsf{Aff}_k \subset \mathrm{Proj}_k$ (an affine scheme $\mathrm{Spec}(R)$ is identified with $\mathrm{Proj}(R[t])$ where $\deg(t) = 1$, viewed as the open locus $D(t)$). Combining these gives $\mathsf{QAff}_k \subset \mathsf{QProj}_k$. The two categories are distinct, however: e.g. $\mathbb{P}^1_k$ is projective but not quasi-affine.

2.4. Projective Closure

Recall from Example 2.2.27 that the affine space $\mathbb{A}^I$ embeds into $\mathbb{P}^{I\sqcup\{0\}}$ as the non-vanishing locus $D(x_0)$. This section compares vanishing and non-vanishing loci across this embedding.

Proposition 2.4.1. *Let I be a set not containing 0, let $F \subset k[x_0, x_i \mid i \in I]$ be a homogeneous subset, and let $F_0 \subset k[x_i \mid i \in I]$ be obtained from F by setting $x_0 = 1$. Consider the vanishing loci $V(F) \subset \mathbb{P}^{I\sqcup\{0\}}_k$ and $V(F_0) \subset \mathbb{A}^I_k$ and the nonvanishing loci $D(F) \subset \mathbb{P}^{I\sqcup\{0\}}_k$ and $D(F_0) \subset \mathbb{A}^I_k$. Then*

$$V(F) \cap \mathbb{A}^I_k = V(F_0) \quad \textit{and} \quad D(F) \cap \mathbb{A}^I_k = D(F_0).$$

Proof. We calculate the R-points for an arbitrary ring R. Recall that an R-point of $\mathbb{P}^{I\sqcup\{0\}}_k$ is a tuple $(L, s_0, (s_i)_{i\in I})$ where L is an R-line and s_0 together with $(s_i)_{i\in I}$ are sections of L generating it.

The intersection with $\mathbb{A}^I_k \simeq D(x_0)$ corresponds to the subfunctor of points where the section s_0 generates L. If s_0 generates L, the map $R \to L$ defined by $1 \mapsto s_0$ is an isomorphism. Under this trivialization $L \simeq R$, the sections correspond to elements in R:

$$s_0 \mapsto 1, \quad s_i \mapsto a_i \in R.$$

Thus, R-points of $\mathbb{A}^I_k$ are identified with tuples $(1, (a_i)_{i\in I}) \in R \times R^I$.

Now consider a homogeneous polynomial $f \in F$ of degree d. Evaluated at such a point, $\varphi(f) \in L^{\otimes d}$ corresponds under the trivialization $L^{\otimes d} \simeq R$ (induced by $s_0^{\otimes d} \mapsto 1$) to the element:

$$f(1, (a_i)_{i\in I}) \in R.$$

By definition of F_0, this value is exactly $f_0((a_i)_{i\in I})$.

We now check the conditions for $V(F)$ and $D(F)$ (via Remark 2.3.16):

- **For $V(F)$:** A point $\varphi \in \mathbb{A}^I_k(R)$ lies in $V(F)(R)$ if and only if $\varphi(f) = 0$ in $L^{\otimes d}$ for all $f \in F$. Under the trivialization, this is equivalent to $f_0((a_i)) = 0$ in R for all $f_0 \in F_0$. This is precisely the definition of $V(F_0)(R)$.

- **For** $D(F)$**:** A point $\varphi \in \mathbb{A}^I_k(R)$ lies in $D(F)(R)$ if and only if the ideal generated by $\{\varphi(f)\}_{f\in F}$ is the unit ideal in $L^{\otimes d}$ (i.e., generates the whole module). Under the trivialization, this means the ideal generated by $\{f_0((a_i))\}_{f_0\in F_0}$ is the unit ideal in R (i.e., $1 \in (f_0((a_i)))$). This is precisely the definition of $D(F_0)(R)$.

Since this holds for any ring R, the functors are isomorphic. □

We now describe the projective closure of affine vanishing loci.

Definition 2.4.2 (Closure of subfunctor)**.** Let X be an algebraic functor and $Y \subset X$ a subfunctor. The *closure* of Y in X is the smallest closed subfunctor of X containing Y.

Definition 2.4.3 (Homogenization of Polynomials)**.** Let k be a ring and I a set (not containing 0).

(i) Let $f \in k[x_i \mid i \in I]$. The *homogenization* of f is the homogeneous polynomial

$$f^{\mathrm{h}} = x_0^d f\left(\left(\frac{x_i}{x_0}\right)_{i\in I}\right) \in k[x_0, x_i \mid i \in I],$$

where d is the maximal degree of a monomial with nonzero coefficient in f, or equivalently the smallest integer for which the above expression is a polynomial.

(ii) Let $F \subset k[x_i \mid i \in I]$ be an ideal. The *homogenization of* F is the homogeneous ideal

$$F^h = \left(f^h \mid f \in F\right) \subset k[x_0, x_i \mid i \in I].$$

2.4.1. Examples of Projective Schemes

We present three classical constructions — the Segre embedding, Grassmannians, and flag schemes — illustrating how the functor-of-points language handles them.

Example 2.4.4 (Segre Embedding)**.** Let k be a ring and let M and N be k-modules. The **Segre embedding** is the map of algebraic k-functors:

$$\varsigma\colon \mathbb{P}(M) \times \mathbb{P}(N) \to \mathbb{P}(M \otimes_k N)$$

that sends a pair of quotient lines $M \otimes_k R \twoheadrightarrow K$ and $N \otimes_k R \twoheadrightarrow L$ to their tensor product $(M \otimes_k N) \otimes_k R \twoheadrightarrow K \otimes_R L$. This is well-defined since the tensor product $\otimes$ preserves lines and epimorphisms.

If $M = k^{(I)}$ and $N = k^{(J)}$, the Segre embedding is a map:

$$\varsigma\colon \mathbb{P}^I_k \times \mathbb{P}^J_k \to \mathbb{P}^{I\times J}_k.$$

If $M = k^{m+1}$ and $N = k^{n+1}$, this becomes the map:

$$\varsigma\colon \mathbb{P}^m_k \times \mathbb{P}^n_k \to \mathbb{P}^{mn+m+n}_k,$$

which is given in coordinates by the formula:

$$\big([a_0 : \ldots : a_m], [b_0 : \ldots : b_n]\big) \mapsto [\text{all products } a_i b_j].$$

More generally, we have a Segre embedding:

$$\varsigma\colon \prod_{i=1}^n \mathbb{P}(M_i) \to \mathbb{P}\left(\bigotimes_{i=1}^n M_i\right)$$

for any finite family of k-modules $M_1, \ldots, M_n$.

Claim 2.4.5. *The Segre embedding is a closed immersion.*

Proof of Claim. **Step 1: Reduction to free modules.** Choose free modules M', N' and surjections $M' \twoheadrightarrow M$ and $N' \twoheadrightarrow N$. These induce closed immersions $\mathbb{P}(M) \hookrightarrow \mathbb{P}(M')$ and $\mathbb{P}(N) \hookrightarrow \mathbb{P}(N')$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(M) \times \mathbb{P}(N) & \xrightarrow{\varsigma} & \mathbb{P}(M \otimes N) \\ \downarrow & & \downarrow \\ \mathbb{P}(M') \times \mathbb{P}(N') & \xrightarrow{\varsigma'} & \mathbb{P}(M' \otimes N') \end{array}$$

Both vertical arrows are closed immersions, being products of closed immersions. If ς' is a closed immersion (for free modules), then the composite $\mathbb{P}(M) \times \mathbb{P}(N) \to \mathbb{P}(M' \otimes N')$ is also a closed immersion, and since the right vertical arrow is separated, ς itself is a closed immersion (cancellation for closed immersions through a separated map). It thus suffices to treat the free case.

Step 2: Local verification. Assume $M = k^{(I)}$ and $N = k^{(J)}$ with bases $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$. Then $M \otimes N$ is free with basis $(e_i \otimes f_j)$. The property of being a closed immersion is local on the target (Proposition 1.3.11). The target $\mathbb{P}(M \otimes N)$ is covered by the standard affine open subfunctors $D(e_i \otimes f_j)$ for $(i, j) \in I \times J$. It suffices to show that for each pair (i, j), the restriction

$$\varsigma_{ij} \colon \varsigma^{-1}(D(e_i \otimes f_j)) \to D(e_i \otimes f_j)$$

is a closed immersion.

Step 3: Coordinate description. Fix indices $i \in I$ and $j \in J$. The open subfunctor $D(e_i \otimes f_j)$ is isomorphic to the affine space $\mathbb{A}^{(I \times J) \setminus \{(i,j)\}}$. Its coordinate ring is the polynomial ring in variables z_{kl} where $(k, l) \neq (i, j)$, corresponding to the ratio $\frac{e_k \otimes f_l}{e_i \otimes f_j}$.

The preimage is given by:

$$\varsigma^{-1}(D(e_i \otimes f_j)) = D(e_i) \times D(f_j) \simeq \mathbb{A}^{I \setminus \{i\}} \times \mathbb{A}^{J \setminus \{j\}}.$$

Its coordinate ring is $k[x_k \mid k \neq i] \otimes k[y_l \mid l \neq j]$, corresponding to ratios $x_k = \frac{e_k}{e_i}$ and $y_l = \frac{f_l}{f_j}$.

The morphism ς_{ij} corresponds to the ring homomorphism:

$$\begin{aligned} \varphi \colon k[z_{kl} \mid (k, l) \neq (i, j)] &\to k[x_k, y_l \mid k \neq i, l \neq j] \\ z_{kl} &\mapsto x_k y_l. \end{aligned}$$

(Here we adopt the convention $x_i = 1$ and $y_j = 1$).

To check that ς_{ij} is a closed immersion, we must show that φ is surjective. We check the generators of the target ring:

- For any $k \neq i$, we have $\varphi(z_{kj}) = x_k y_j = x_k \cdot 1 = x_k$. Thus $x_k \in \operatorname{im}(\varphi)$.
- For any $l \neq j$, we have $\varphi(z_{il}) = x_i y_l = 1 \cdot y_l = y_l$. Thus $y_l \in \operatorname{im}(\varphi)$.

Since all generators x_k and y_l are in the image, φ is surjective. Therefore, ς is a closed immersion. □

Corollary 2.4.6 (Products of Projective Schemes)**.** *Let X and Y be projective k-schemes. Then $X \times Y$ is a projective k-scheme.*

Proof. Since X and Y are projective, there exist closed immersions $X \hookrightarrow \mathbb{P}(M)$ and $Y \hookrightarrow \mathbb{P}(N)$ for some k-modules M and N. The product map:

$$X \times Y \hookrightarrow \mathbb{P}(M) \times \mathbb{P}(N) \xrightarrow{\varsigma} \mathbb{P}(M \otimes_k N)$$

is a closed immersion (as a composition of closed immersions), showing that $X \times Y$ is projective. □

***Remark* 2.4.7** (Veronese via Segre)**.** The Veronese embedding (Example 2.3.9) factors through the Segre embedding: for any k-module M and $d \geq 1$, there is a commutative square

$$\begin{array}{ccc} \mathbb{P}(M) & \xrightarrow{\rho_d} & \mathbb{P}(\mathrm{Sym}_k^d(M)) \\ \downarrow{\scriptstyle \Delta} & & \downarrow \\ \mathbb{P}(M)^{\times d} & \xrightarrow{\varsigma} & \mathbb{P}(M^{\otimes d}) \end{array}$$

where the vertical maps are the inclusions of the Σ_d-fixed points. This gives another proof that ρ_d is a closed immersion.

Replacing quotient lines by quotient spaces of arbitrary rank yields the Grassmannian.

Definition 2.4.8 (Grassmannian)**.** Let k be a ring, M a k-module, and $n \in \mathbb{N}$. The **rank** n **Grassmannian** of M is the algebraic k-functor $\mathrm{Gr}_n(M)$ given by:

$$\mathrm{Gr}_n(M)(R) = \{\text{quotient spaces of } M \otimes_k R \text{ of constant rank } n\}.$$

***Remark* 2.4.9** (Properties of Grassmannians)**.** (i) By definition, $\mathrm{Gr}_1(M) = \mathbb{P}(M)$.

(ii) If M is a vector space of constant rank r, then duality induces isomorphisms:

$$\mathrm{Gr}_n(M) \simeq \mathrm{Gr}_{r-n}(M^\vee)$$

Example 2.4.10 (Plücker Embedding)**.** Let M be a k-module and let $n \in \mathbb{N}$. The **Plücker embedding** is the map of algebraic k-functors:

$$\varpi\colon \mathrm{Gr}_n(M) \to \mathbb{P}(\Lambda_k^n(M))$$

that sends a quotient space of rank n to its n-th exterior power, which is a quotient line.

Claim 2.4.11. *The Plücker embedding is a closed immersion.*

Proof of Claim. **Step 1: Reduction to free modules.** For any surjective map $M' \twoheadrightarrow M$, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}_n(M) & \xrightarrow{\varpi} & \mathbb{P}(\Lambda_k^n(M)) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_n(M') & \xrightarrow{\varpi'} & \mathbb{P}(\Lambda_k^n(M')) \end{array}$$

The vertical maps are closed immersions: a surjection $M' \twoheadrightarrow M$ induces closed immersions on both Grassmannians and projective spaces. If ϖ' is a closed immersion, then the composite $\mathrm{Gr}_n(M) \to \mathbb{P}(\Lambda_k^n(M'))$ is a closed immersion. Since the right vertical map is a separated morphism, this implies ϖ is a closed immersion. Thus, we may assume M is a free module with basis $(e_i)_{i \in I}$.

Step 2: Local verification. Let $(e_J)_{J \in \binom{I}{n}}$ be the induced basis for $\Lambda_k^n(M)$. The target $\mathbb{P}(\Lambda_k^n(M))$ is covered by the affine open subfunctors $D(e_J)$. By Proposition 1.3.11 (Closed immersion is local on the target), it suffices to show that for each $J \in \binom{I}{n}$, the restriction

$$\varpi_J : \varpi^{-1}(D(e_J)) \to D(e_J)$$

is a closed immersion.

Step 3: Coordinate description. Fix a subset $J \subset I$ of size n. The open subfunctor $D(e_J) \subset \mathbb{P}(\Lambda_k^n(M))$ is the affine space with coordinate ring generated by variables X_K (representing x_K/x_J) for $K \in \binom{I}{n} \setminus \{J\}$.

The preimage $\varpi^{-1}(D(e_J))$ represents submodules $W \subset M$ that project isomorphically onto the span of $\{e_j\}_{j \in J}$. Such a submodule has a unique basis where the components along e_J form the identity matrix.

Explicitly, this open set is isomorphic to the affine space of matrices $\mathrm{Mat}_{(I\setminus J)\times J}$, with coordinates z_{ij} for $i \in I \setminus J$, $j \in J$. The subspace W is the column span of the $|I| \times n$ matrix A where:

$$A_{kj} = \begin{cases} \delta_{kj} & \text{if } k \in J, \\ z_{kj} & \text{if } k \in I \setminus J. \end{cases}$$

(Visualized as a block matrix $\begin{pmatrix} I_n \\ Z \end{pmatrix}$ after reordering rows).

The Plücker embedding sends this subspace to the point in projective space whose coordinates are the $n \times n$ minors of A. The restriction ϖ_J corresponds to the ring homomorphism:

$$\varphi \colon k[X_K \mid K \neq J] \to k[z_{ij} \mid i \notin J, j \in J],$$

where $\varphi(X_K)$ is the $n \times n$ minor of A using rows in K (normalized so the minor for J is 1).

Step 4: Surjectivity of ring map. To prove ϖ_J is a closed immersion, we show φ is surjective by finding preimages for the generators z_{ij}. Fix $i_0 \in I \setminus J$ and $j_0 \in J$. Consider the index set $K = (J \setminus \{j_0\}) \cup \{i_0\}$. The submatrix of A with rows in K consists of:

- The rows $j \in J \setminus \{j_0\}$: these are standard unit vectors (rows of I_n).
- The row i_0: this is the row $(z_{i_0,j})_{j\in J}$.

The determinant of this $n \times n$ matrix is (up to a sign ± 1) simply the entry z_{i_0,j_0}. Thus, $\varphi(X_K) = \pm z_{i_0,j_0}$. Since every generator z_{ij} is in the image (up to unit), φ is surjective.

Hence, ϖ_J is a closed immersion, and so is ϖ.

□

Definition 2.4.12 (Flag Schemes). Let M be a k-module and let $\mathbf{n} = (n_1, \ldots, n_s)$ be an increasing sequence of natural numbers.

(i) A **flag of type n** on M is a sequence of quotient modules:

$$M \twoheadrightarrow F_s \twoheadrightarrow \cdots \twoheadrightarrow F_1$$

where each F_i is a vector space of constant rank n_i.

(ii) The **type n flag scheme** of M is the algebraic k-functor $\mathrm{Flag}_{\mathbf{n}}(M)$ given by:

$$\mathrm{Flag}_{\mathbf{n}}(M)(R) = \{\text{flags of type } \mathbf{n} \text{ on } M \otimes_k R\}.$$

***Remark* 2.4.13** (Special Cases of Flag Schemes). (i) By definition, $\mathrm{Flag}_{(n)}(M) = \mathrm{Gr}_n(M)$.

(ii) If M is a vector space of constant rank r, then duality induces isomorphisms:

$$\mathrm{Flag}_{(n_1,\ldots,n_s)}(M) \simeq \mathrm{Flag}_{(r-n_s,\ldots,r-n_1)}(M^\vee).$$

Example 2.4.14 (Projectivity of Flag Schemes). Sending a flag to its components defines a monomorphism:

$$\mathrm{Flag}_{\mathbf{n}}(M) \hookrightarrow \prod_{i=1}^{s} \mathrm{Gr}_{n_i}(M).$$

Claim 2.4.15. *It is a closed immersion.*

In particular, $\mathrm{Flag}_{\mathbf{n}}(M)$ is a projective k-scheme if M is of finite type.

Proof. For any R-point $x\colon \operatorname{Spec}(R) \to \prod_{i=1}^{s} \operatorname{Gr}_{n_i}(M)$ classifying quotient spaces $F_1, \dots, F_s$ of $M \otimes_k R$, the preimage of $\operatorname{Flag}_{\mathbf{n}}(M)$ by x is exactly the joint vanishing locus of the R-linear maps:

$$\ker(M \otimes_k R \twoheadrightarrow F_{i+1}) \hookrightarrow M \otimes_k R \twoheadrightarrow F_i$$

for $1 \le i \le s-1$, which is a closed subfunctor of $\operatorname{Spec}(R)$ by Proposition 2.2.19(i). □

2.5. The Projective Nullstellensatz

Hilbert's Nullstellensatz (Theorem 1.5.1) has a homogeneous counterpart for the projective space. We record the classical statement, which complements the functorial Corollary 2.3.28.

For a ring k, define the maps:

$$\{\text{homogeneous subsets of } k[x_0, \dots, x_n]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{subsets of } \mathbb{P}^n(k)\}$$

as follows:

$$V(F) = \{x \in \mathbb{P}^n(k) \mid f(x) = 0 \text{ for all } f \in F\},$$
$$I(X) = \{f \in k[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in X\}.$$

Note that both maps are order-reversing and that $F \subset I(V(F))$ and $X \subset V(I(X))$ (in other words, this is an adjunction between posets). Note also that $I(X)$ is always a saturated homogeneous ideal in $k[x_0, \dots, x_n]$, which is radical if k is reduced. Call a subset $X \subset \mathbb{P}^n(k)$ **algebraic** if it lies in the image of V, or equivalently if $X = V(I(X))$.

Proposition 2.5.1 (Projective Nullstellensatz). *Let k be an algebraically closed field and let $n \in \mathbb{N}$. For any homogeneous subset $F \subset k[x_0, \dots, x_n]$, we have:*

$$I(V(F)) = \sqrt{(F)}^{\mathrm{sat}}.$$

Consequently, the maps V and I define a one-to-one correspondence:

$$\{\textit{saturated radical homogeneous ideals in } k[x_0, \dots, x_n]\} \underset{I}{\overset{V}{\underset{\sim}{\rightleftarrows}}} \{\textit{algebraic subsets of } \mathbb{P}^n(k)\}$$

Proof. The inclusion $\sqrt{(F)}^{\mathrm{sat}} \subset I(V(F))$ is clear: if $f \in \sqrt{(F)}^{\mathrm{sat}}$, then $f^m \in (F) + (x_0, \dots, x_n)^N$ for some $m, N \ge 1$. For any $x \in V(F)$, we have $g(x) = 0$ for all $g \in F$, and hence $f^m(x) = 0$ (since the term from $(x_0, \dots, x_n)^N$ also vanishes on projective space). Since k is a field, this implies $f(x) = 0$.

For the reverse inclusion, we use the affine Nullstellensatz. Consider the standard affine cover of $\mathbb{P}^n_k$ given by $D(x_i) \simeq \mathbb{A}^n_k$ for $i = 0, \dots, n$. For each i, let $F_i \subset k[y_1, \dots, y_n]$ be the dehomogenization of F with respect to x_i (setting $x_i = 1$).

By the affine Nullstellensatz (Theorem 1.5.1), we have:

$$I(V(F_i)) = \sqrt{(F_i)}$$

in $k[y_1, \dots, y_n]$, where $V(F_i)$ denotes the vanishing locus in k^n.

Now suppose $f \in I(V(F))$. Then for each i, the dehomogenization f_i of f with respect to x_i vanishes on $V(F_i)$, so $f_i \in \sqrt{(F_i)}$. This means $f_i^{m_i} \in (F_i)$ for some $m_i \ge 1$.

Taking $m = \max_i m_i$, we have $f_i^m \in (F_i)$ for each i. This gives us the desired m for which $f^m \in (F)^h$, which is what we wanted to show regarding the projective closure of vanishing loci. □

Looking Ahead

The projective setting is now in place: $\mathrm{Proj}(A)$ of an $\mathbb{N}$-graded ring, its open cover by $D(f) \simeq \mathrm{Spec}(A_{(f)})$, the Proj structure theorem, saturation of homogeneous ideals, projective closure, and examples ranging from Grassmannians to Segre and Veronese embeddings. We also proved the functorial projective Nullstellensatz, completing the picture of how algebra translates to projective geometry.

Two natural directions open up. First, we need to do *linear algebra* over our schemes: modules, ideals, quasi-coherent sheaves, relative constructions. The next chapter develops this machinery via the category Mod_X of quasi-coherent modules — defined for any algebraic functor X as a categorical limit of the Mod_R. Second, we eventually need to assemble affine and projective schemes into a single category that is closed under gluing: this is the category of schemes, treated in Chapter 6.

Chapter 3.

Quasi-coherent Modules

For a ring R, the category Mod_R of R-modules is the natural home for linear algebra over R. We now ask: given an algebraic functor $X \in \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$, what is the "linear algebra" attached to X?

The answer is the category Mod_X of *quasi-coherent modules on X*. The idea is compelling in its simplicity: since X assembles, via its R-points, out of affine pieces $\mathrm{Spec}(R) \to X$, the category Mod_X should assemble out of the individual Mod_R along the same diagram. This is a right Kan extension of the functor $R \mapsto \mathrm{Mod}_R$ along Spec, making Mod_X a categorical *limit*:

$$\mathrm{Mod}_X \simeq \lim_{\mathrm{Spec}(R)\to X} \mathrm{Mod}_R.$$

The universal property of this limit automatically gives us pullback functors, base change, and compatibility with all the constructions we already know for rings.

What follows is a careful unpacking of this picture. After reviewing limits in Cat, we define quasi-coherent modules and ideals, classify open and closed subfunctors via quasi-coherent ideals, construct the *relative* Spec and Proj over a base, prove Zariski descent for modules, and introduce Serre's twisting sheaves $\mathcal{O}(d)$ on projective schemes — the building blocks of projective geometry's cohomology theory.

3.1. Quasi-coherence

Before extending $R \mapsto \mathrm{Mod}_R$ along the Yoneda embedding, we need a mild amount of categorical machinery: how to take limits of category-valued functors. The resulting category Mod_X is best understood as a limit of ordinary module categories along the affine points of X.

3.1.1. Category-valued Functors

We begin with limits of category-valued functors, the categorical machinery underlying the definition of Mod_X.

Definition 3.1.1 (Category-valued Functors). Let $\mathcal{J}$ be a category. A *functor* $F : \mathcal{J} \to \mathsf{Cat}$ consists of the following data:

(i) For each object $I \in \mathcal{J}$, a category $F(I)$.

(ii) For each morphism $f : I \to J$ in $\mathcal{J}$, a functor $F(f) : F(I) \to F(J)$.

(iii) For each object $I \in \mathcal{J}$, a natural isomorphism $\eta_I : \mathrm{id}_{F(I)} \simeq F(\mathrm{id}_I)$.

(iv) For morphisms $f : I \to J$ and $g : J \to K$ in $\mathcal{J}$, a natural transformation $\mu_{f,g} : F(g) \circ F(f) \simeq F(g \circ f)$.

These data must satisfy the following compatibility conditions:

5. For any morphism $f\colon I \to J$ in $\mathcal{J}$, both of the following triangles commute:

$$\begin{array}{ccc} F(f) & \xrightarrow{\eta_I} & F(f)\circ F(\mathrm{id}_I) \\ & \searrow^{\mathrm{id}} & \downarrow \mu_{\mathrm{id}_I,f} \\ & & F(f), \end{array} \quad \text{and} \quad \begin{array}{ccc} F(f) & \xrightarrow{\eta_J} & F(\mathrm{id}_J)\circ F(f) \\ & \searrow^{\mathrm{id}} & \downarrow \mu_{f,\mathrm{id}_J} \\ & & F(f). \end{array}$$

6. For morphisms $f\colon I \to J$, $g\colon J \to K$, and $h\colon K \to L$ in $\mathcal{J}$, the following square commutes:

$$\begin{array}{ccc} F(h)\circ F(g)\circ F(f) & \xrightarrow{\mu_{f,g}} & F(h)\circ F(g\circ f) \\ \downarrow \mu_{g,h} & & \downarrow \mu_{g\circ f,h} \\ F(h\circ g)\circ F(f) & \xrightarrow{\mu_{f,h\circ g}} & F(h\circ g\circ f). \end{array}$$

Example 3.1.2 (Slice Categories). Let C be a category with pullbacks. We can consider the functor:

$$C_{/-}\colon C^{\mathrm{op}} \to \mathsf{Cat}, \quad X \mapsto C_{/X},$$

called the *self-indexing* of C. For a morphism $f\colon X \to Y$, it sends f to the pullback functor:

$$f^*\colon C_{/Y} \to C_{/X}, \quad (A \to Y) \mapsto (X \times_Y A \to X).$$

***Remark* 3.1.3** (Topos). If C is a presentable category and the above functor satisfies the property that for any small category $\mathcal{I}$ and functor $F\colon \mathcal{I} \to C$:

$$C_{/\mathrm{colim}_{i\in\mathcal{I}} F(i)} \simeq \lim_{i\in\mathcal{I}} C_{/F(i)},$$

then C is called a *topos*.

Example 3.1.4 (Module Categories). We can construct the functor $R \mapsto \mathrm{Mod}_R$ in three different ways:

(i) *Base change:* There exists a functor:

$$\mathrm{Mod}^*\colon \mathsf{CAlg} \to \mathsf{Cat}, \quad R \mapsto \mathrm{Mod}_R,$$

which sends a ring homomorphism $f\colon R \to S$ to the base change functor:

$$f^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto M \otimes_R S.$$

(ii) *Restriction of scalars:* There exists a functor:

$$\mathrm{Mod}_*\colon \mathsf{CAlg}^{\mathrm{op}} \to \mathsf{Cat}, \quad R \mapsto \mathrm{Mod}_R,$$

which sends a ring homomorphism $f\colon R \to S$ to the forgetful functor:

$$f_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R, \quad M \mapsto M,$$

where we view the S-module M as an R-module via composition with f.

(iii) *Internal Hom:* There exists a functor:

$$\mathrm{Mod}^!\colon \mathsf{CAlg} \to \mathsf{Cat}, \quad R \mapsto \mathrm{Mod}_R,$$

which sends a ring homomorphism $f\colon R \to S$ to the functor:

$$f^!\colon \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(S, M).$$

***Remark* 3.1.5.** In the above example, we can replace Mod_R with CAlg_R. Then constructions (1) and (2) work for CAlg_R, but (3) does not.

Example 3.1.6 (Properties Preserved by Base Change)**.** Any property P of modules that is preserved by base change (such as being finitely generated, finitely presented, flat, projective, free, a vector space, or a line) defines a subfunctor of Mod^*.

Example 3.1.7 (Ideal Functors)**.** For a poset P, we can automatically view it as a category. Therefore, for a category C, we can consider functors landing in Pos (the category of posets):

$$C \to \mathsf{Pos}.$$

For example, taking $C = \mathsf{CAlg}$:

- For a ring R, consider the poset Id_R of all ideals of R ordered by inclusion. For $R \to S$, we have $\mathrm{Id}_R \to \mathrm{Id}_S$ sending $I \mapsto IS$.
- For a ring R, consider the poset Rad_R of all radical ideals of R ordered by inclusion. For $R \to S$, we have $\mathrm{Rad}_R \to \mathrm{Rad}_S$ sending $I \mapsto \sqrt{IS}$.

Notation 3.1.8 (Left and Right Cones)**.** Let $\mathcal{J}$ be a category. We denote by $\mathcal{J}^{\triangleleft}$ the category $[0] \star \mathcal{J}$. In other words, it is $\mathcal{J}$ with an initial object $-\infty$ added, such that for any $I \in \mathcal{J}$, we have $\mathrm{Hom}_{\mathcal{J}^{\triangleleft}}(-\infty, I) = *$. We call this the *left cone* over $\mathcal{J}$.

Similarly, we can define the *right cone* $\mathcal{J}^{\triangleright}$.

We can now formalize the limit construction.

Definition 3.1.9 (Limits of Categories)**.** Let $F \colon \mathcal{J} \to \mathsf{Cat}$ be a functor. The *limit* $\lim F$ (or $\lim_{I \in \mathcal{J}} F(I)$) is the category defined as follows:

- *Objects:* An object of $\lim F$ is a pair (x, α) where:
  (i) For each $I \in \mathcal{J}$, we specify an object $x_I \in F(I)$.
  (ii) For each morphism $f : I \to J$ in $\mathcal{J}$, we specify an isomorphism $\alpha_f : F(f)(x_I) \xrightarrow{\sim} x_J$.

  These data must satisfy the following *cocycle condition*:
  (i) For each $I \in \mathcal{J}$, the following triangle commutes:

$$\begin{array}{ccc} x_I & \xrightarrow{\eta_I(x_I)} & F(\mathrm{id}_I)(x_I) \\ & {}_{\mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha_{\mathrm{id}_I}} \\ & & x_I. \end{array}$$

  (ii) For morphisms $f : I \to J$ and $g : J \to K$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{array}{ccc} F(g)(F(f)(x_I)) & \xrightarrow{F(g)(\alpha_f)} & F(g)(x_J) \\ \downarrow{\scriptstyle \mu_{f,g}(x_I)} & & \downarrow{\scriptstyle \alpha_g} \\ F(g \circ f)(x_I) & \xrightarrow{\alpha_{g \circ f}} & x_K. \end{array}$$

- *Morphisms:* A morphism φ from (x, α) to (y, β) in $\lim F$ consists of:
  (i) For each $I \in \mathcal{J}$, a morphism $\varphi_I : x_I \to y_I$ in $F(I)$.

(ii) For each morphism $f: I \to J$ in $\mathcal{J}$, the following diagram commutes:

$$\begin{array}{ccc} F(f)(x_I) & \xrightarrow{F(f)(\varphi_I)} & F(f)(y_I) \\ \downarrow{\alpha_f} & & \downarrow{\beta_f} \\ x_J & \xrightarrow{\varphi_J} & y_J. \end{array}$$

- *Composition:* Identity morphisms and composition of morphisms are defined pointwise.

Definition 3.1.10 (Limit Diagrams). Let $F: \mathcal{J} \to \mathsf{Cat}$ be a functor. Consider an extension $\bar{F}: \mathcal{J}^{\triangleleft} \to \mathsf{Cat}$. We say $\bar{F}$ is a *limit diagram* if the canonical map $\bar{F}(-\infty) \to \lim F$ is a categorical equivalence.

***Remark* 3.1.11** (Colimits). Similarly, we can define colimits of category-valued functors and colimit diagrams. Moreover, we can verify that given a functor $F: \mathcal{J} \to \mathsf{Cat}$ and an extension $\bar{F}: \mathcal{J}^{\triangleright} \to \mathsf{Cat}$, we have: $\bar{F}$ is a colimit diagram if and only if for any category $\mathcal{E}$:

$$\mathsf{Fun}(\bar{F}(-), \mathcal{E}): (\mathcal{J}^{\mathrm{op}})^{\triangleleft} \to \mathsf{Cat}$$

is a limit diagram. (Note: in synthetic category theory, we define pushouts in a similar way.)

Example 3.1.12 (Pullbacks of Categories). Given functors $C \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{D}$, the pullback $C \times_{\mathcal{E}} \mathcal{D}$ is the category whose objects are triples (x, y, α) with $x \in C$, $y \in \mathcal{D}$, and $\alpha: f(x) \xrightarrow{\sim} g(y)$ in $\mathcal{E}$.

Example 3.1.13 (The Loop Category as a Limit). Let C be a category. Consider the diagram in Cat:

$$C \underset{\mathrm{id}_C}{\overset{\mathrm{id}_C}{\rightrightarrows}} C.$$

The *limit* of this diagram is equivalent to the functor category $\mathsf{Fun}(S^1, C)$, where S^1 (often denoted $B\mathbb{Z}$) is the groupoid with one object and $\mathbb{Z}$ as the automorphism group.

Derivation from Definition 3.1.9: Let's check what an object in this limit looks like according to the definition.

- The index category $\mathcal{J}$ has two objects (say $0, 1$) and two non-trivial morphisms $s, t: 0 \to 1$.
- The functor F maps $0, 1 \mapsto C$ and $s, t \mapsto \mathrm{id}_C$.
- An object in $\lim F$ is a pair (x, α), which consists of:
  - Objects $x_0, x_1 \in C$.
  - Isomorphisms $\alpha_s: \mathrm{id}(x_0) \xrightarrow{\sim} x_1$ and $\alpha_t: \mathrm{id}(x_0) \xrightarrow{\sim} x_1$ in C.

Since α_s is an isomorphism, we can identify x_1 with x_0. The data then reduces to a single object $x := x_0$ and a single automorphism formed by the loop $\gamma = \alpha_t^{-1} \circ \alpha_s: x \xrightarrow{\sim} x$.

Thus, the category $\lim F$ is equivalent to the category where:

- **Objects:** Pairs (x, γ) where $x \in C$ and $\gamma \in \mathrm{Aut}_C(x)$.
- **Morphisms:** A morphism $f: (x, \gamma) \to (y, \delta)$ is a map $f: x \to y$ in C such that $\delta \circ f = f \circ \gamma$.

This is precisely the description of $\mathsf{Fun}(S^1, C)$, where a functor from S^1 selects an object and an automorphism.

Example 3.1.14 (Zariski Descent for Modules)**.** Limits of categories allow us to reformulate Theorem 1.3.1 more succinctly: it is exactly the statement that, if $(f_i)_{i\in I}$ generates the unit ideal in R, then the diagram of categories

$$\mathrm{Mod}_R \longrightarrow \textstyle\prod_{i\in I} \mathrm{Mod}_{R_{f_i}} \rightrightarrows \prod_{i,j\in I} \mathrm{Mod}_{R_{f_i f_j}} \Rrightarrow \prod_{i,j,k\in I} \mathrm{Mod}_{R_{f_i f_j f_k}}$$

obtained by restricting the functor Mod^* is a limit diagram. In the case $I = \{1, 2\}$, this is equivalent to the simpler statement that the square of categories

$$\begin{array}{ccc} \mathrm{Mod}_R & \longrightarrow & \mathrm{Mod}_{R_{f_1}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{R_{f_2}} & \longrightarrow & \mathrm{Mod}_{R_{f_1 f_2}} \end{array}$$

is cartesian. Moreover, these results hold for any subfunctor of Mod^* defined by a Zariski-local property, such as $R \mapsto \mathsf{Vect}_R$ and $R \mapsto \mathsf{Line}_R$.

3.1.2. Quasi-coherent Modules over Algebraic Functors

We are now ready to define modules over a general algebraic functor.

Idea (Modules via Kan Extension)**.** Our core idea is to extend the notion of "module" to general algebraic functors $X\colon \mathsf{CAlg} \to \mathsf{Set}$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathsf{CAlg} & \xrightarrow{\mathrm{Mod}_{(-)}} & \mathsf{Cat} \\ \big\downarrow & \nearrow & \\ \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})^{\mathrm{op}} & & \end{array}$$

Our goal is to construct the dashed arrow. A natural approach is to use *Kan extension* theory.

We can view a set-valued algebraic functor X as a functor taking values in discrete categories:

$$X\colon \mathsf{CAlg} \to \mathsf{Cat}.$$

The functor category $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Cat})$ naturally has a Cat-enriched structure (the cartesian closed structure on Cat gives a tensor action and internal Hom on $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Cat})$ via adjunction). In this framework, we can define the category of quasi-coherent modules Mod_X as the following internal Hom object:

$$\mathrm{Mod}_X \coloneqq \underline{\mathrm{Hom}}_{\mathsf{Cat}}(X, \mathrm{Mod}).$$

By the density of the Yoneda embedding, we have a canonical equivalence:

$$X \simeq \mathrm{colim}_{x\in \mathrm{El}(X)} \operatorname{Spec} R.$$

Using the property that Hom functors convert colimits to limits, we obtain:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathsf{Cat}}(X, \mathrm{Mod}) &\simeq \underline{\mathrm{Hom}}_{\mathsf{Cat}}\left(\mathrm{colim}_{x\in \mathrm{El}(X)} \operatorname{Spec} R, \mathrm{Mod}\right) \\ &\simeq \lim_{x\in \mathrm{El}(X)^{\mathrm{op}}} \underline{\mathrm{Hom}}(\operatorname{Spec} R, \mathrm{Mod}) \\ &= \lim_{x\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Mod}_R. \end{aligned}$$

Note that the indexing category $x \in \mathrm{El}(X)^{\mathrm{op}}$ is equivalent to the category of morphisms $x\colon \operatorname{Spec} R \to X$ (the opposite of the element category of X). The above limit formula precisely corresponds to the *right Kan extension* of $\mathrm{Mod}_{(-)}$ along the Yoneda embedding.

Definition 3.1.15 (Quasi-coherent Modules). Let X be an algebraic functor. A *quasi-coherent module* over X (or *quasi-coherent $\mathcal{O}_X$-module*) is an element of the limit category:

$$\mathrm{Mod}_X := \lim_{x\colon \mathrm{Spec}(R)\to X} \mathrm{Mod}_R.$$

Explicitly, a quasi-coherent module M over X consists of:

(i) For each ring R and each R-point $x\colon \mathrm{Spec}(R) \to X$, an R-module $M(x) \in \mathrm{Mod}_R$.

(ii) For each ring homomorphism $\varphi\colon R \to S$ and commutative triangle:

$$\begin{array}{ccc} \mathrm{Spec}(S) & \xrightarrow{\;y\;} & X \\ & {\scriptstyle \mathrm{Spec}(\varphi)}\searrow \quad \nearrow{\scriptstyle x} & \\ & \mathrm{Spec}(R) & \end{array}$$

an isomorphism $\alpha_\varphi\colon M(x) \otimes_R S \xrightarrow{\sim} M(y)$ in Mod_S.

These data must satisfy:

- *Transitivity:* For composable ring homomorphisms $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ with compatible points x, y, z, we have:
$$\alpha_{\psi\circ\varphi} = \alpha_\psi \circ (\alpha_\varphi \otimes_S T).$$
- *Identity:* For $\varphi = \mathrm{id}_R$, we have $\alpha_{\mathrm{id}} = \mathrm{id}$.

***Remark* 3.1.16** (Functoriality). The assignment $X \mapsto \mathrm{Mod}_X$ is contravariant functorial. For a morphism $f\colon Y \to X$ of algebraic functors, we obtain a pullback functor:

$$f^*\colon \mathrm{Mod}_X \to \mathrm{Mod}_Y, \quad M \mapsto f^*M,$$

where $(f^*M)(y) = M(f \circ y)$ for any $y\colon \mathrm{Spec}(R) \to Y$.

Similarly, we can define quasi-coherent algebras, ideals, and radical ideals.

Definition 3.1.17 (Quasi-coherent Algebras and Ideals). Let X be an algebraic functor.

(i) A *quasi-coherent $\mathcal{O}_X$-algebra* is an element of:

$$\mathsf{CAlg}_X := \lim_{x\colon \mathrm{Spec}(R)\to X} \mathsf{CAlg}_R.$$

(ii) A *quasi-coherent ideal* is an element of:

$$\mathsf{Id}_X := \lim_{x\colon \mathrm{Spec}(R)\to X} \mathsf{Id}_R.$$

(iii) A *quasi-coherent radical ideal* is an element of:

$$\mathsf{Rad}_X := \lim_{x\colon \mathrm{Spec}(R)\to X} \mathsf{Rad}_R.$$

***Warning* 3.1.18** (Incompatibility of forgetful functors). It should be noted that the forgetful functors

$$\mathsf{Rad}_R \hookrightarrow \mathsf{Id}_R \to \mathrm{Mod}_{\mathrm{Spec}(R)}$$

are not compatible with base change. Therefore, they cannot automatically extend to general algebraic functors X (although they will when X is a scheme). Specifically:

- A quasi-coherent radical ideal on X does not automatically induce a quasi-coherent ideal structure.
- A quasi-coherent ideal on X does not automatically induce a quasi-coherent module structure.

This can be seen by observing base change. For a ring homomorphism $\varphi\colon R \to S$ and a radical ideal I in R, we have the following **non-**commutative diagram:

$$\begin{array}{ccc} \mathsf{Rad}_R & \xrightarrow{\text{forget}} & \mathsf{Id}_R \\ {\scriptstyle I\mapsto\sqrt{IS}}\downarrow & & \downarrow{\scriptstyle I\mapsto IS} \\ \mathsf{Rad}_S & \xrightarrow{\text{forget}} & \mathsf{Id}_S \end{array}$$

(In general $\sqrt{IS} \neq IS$, so the distinction matters.)

Notation 3.1.19 ($\mathcal{O}_X$)**.** Let X be an algebraic functor. Denote by $\mathcal{O}_X \in \mathsf{CAlg}_X$ the initial object of the category of quasi-coherent algebras on X, i.e., for $x\colon \operatorname{Spec}(R) \to X$, we have $\mathcal{O}_X(x) = R$. It is not hard to see that its underlying quasi-coherent module is the unit of Mod_X under the tensor product. Furthermore, for any morphism $f : Y \to X$, we have $f^*(\mathcal{O}_X) = \mathcal{O}_Y$.

Definition 3.1.20 (Global Sections)**.** Let X be an algebraic functor and M be a quasi-coherent module on X. Then a *global section* of M refers to a morphism $\mathcal{O}_X \to M$ in Mod_X. The set formed by the global sections of M can be characterized as follows:

$$M(X) = \Gamma(X, M) = \lim_{(R,x)\in\mathrm{El}(X)^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Mod}_R}(R, M(x)) = \lim_{(R,x)\in\mathrm{El}(X)^{\mathrm{op}}} M(x).$$

***Remark* 3.1.21** (Functions as Global Sections)**.** One can find that a global section of $\mathcal{O}_X$ is exactly a function on X in the sense of Definition 1.2.37: $\Gamma(X, \mathcal{O}_X) \simeq \mathcal{O}(X)$. Hence for any $M \in \mathrm{Mod}_X$, the abelian group $M(X)$ has a canonical structure of module over the ring $\mathcal{O}(X)$.

***Remark* 3.1.22** (Properties Compatible with Base Change)**.** Since quasi-coherent objects are essentially collections of local data compatible with base change, any property or construction in the category of R-modules (or algebras, ideals) that is compatible with base change can be extended to the category of quasi-coherent objects.

The following lists some important properties and constructions:

- **Properties of modules:** Properties such as finite type, finite presentation, flat, projective, free, etc., can all be defined. In particular, vector bundles (Vect_X) and line bundles (Line_X) form full subcategories of Mod_X.
- **Properties of module morphisms:** Zero morphism, surjection, universal injection, etc.
- **Sum of morphisms:** The additive structure on $\mathrm{Mod}_X(M, N)$ makes it an abelian group.
- **Colimits of modules:** Can be computed pointwise in Mod_X, i.e., $(\mathrm{colim}_i\, M_i)(x) = \mathrm{colim}_i(M_i(x))$.
- **Tensor product:** Mod_X has a symmetric monoidal structure, given by the pointwise tensor product $(M \otimes N)(x) = M(x) \otimes N(x)$.
- **Power operations on modules:** Symmetric powers Sym^d, exterior powers Λ^d, and divided powers Γ^d can all be defined.
- **Operations on ideals:** The sum $I + J$ and product IJ of quasi-coherent ideals are both defined by pointwise operations.
- **Radical of an ideal:** The radical $\sqrt{I}$ of a quasi-coherent ideal is defined as $(\sqrt{I})(x) = \sqrt{I(x)}$.
- **Properties of algebras:** Finite type, finite presentation, finite, flat, free, etc. For $\mathbb{N}$-graded algebras, properties such as "generated by A_1" can also be defined.

- **Properties of algebra morphisms:** Surjection, localization, having nilpotent kernel, eventually surjective, etc.
- **Other constructions:** Including the forgetful functor from algebras to modules, the symmetric algebra construction $\mathrm{Sym}(M)$, etc.

Note: Injectivity is usually not defined in Mod_X, because injectivity is not necessarily preserved under base change (unless the module is flat). Similarly, limits of modules usually exist (if X is accessible), but cannot necessarily be computed pointwise.

3.2. Classification of Open and Closed Subfunctors

We have seen for affine and projective schemes that open and closed subfunctors correspond bijectively to radical and arbitrary ideals respectively (with saturation in the projective case). We now extend this correspondence to general algebraic functors.

Proposition 3.2.1 (Classification Theorem). *Let X be an algebraic functor.*

(i) There exists an isomorphism of posets:

$$V\colon \mathsf{Id}_X^{\mathrm{op}} \xrightarrow{\sim} \{\textit{closed subfunctors of } X\}.$$

(ii) There exists an isomorphism of posets:

$$D\colon \mathsf{Rad}_X \xrightarrow{\sim} \{\textit{open subfunctors of } X\}.$$

Proof. Strategy. The affine case is already in hand: closed subfunctors of $\mathrm{Spec}(R)$ correspond to ideals via V, and open subfunctors to radical ideals via D (the affine Nullstellensatz, Corollary 1.2.11). We reduce the general case to the affine one using the canonical colimit presentation $X \simeq \mathrm{colim}_{x\colon \mathrm{Spec}(R)\to X}\mathrm{Spec}(R)$ and the fact that subfunctors of X are uniquely determined by their pullbacks to each affine chart.

Let $\mathrm{Sub}(X)$ denote the poset of all subfunctors of X. Since $X \simeq \mathrm{colim}_{x\colon \mathrm{Spec}(R)\to X}\mathrm{Spec}(R)$, we have an isomorphism:

$$\begin{aligned}\mathrm{Sub}(X) &\xrightarrow{\sim} \lim_{x\colon \mathrm{Spec}(R)\to X}\mathrm{Sub}(\mathrm{Spec}(R))\\ Y &\mapsto (x^{-1}(Y)\subset \mathrm{Spec}(R))_x.\end{aligned}$$

The inverse is given by $(Y_x \subset \mathrm{Spec}(R))_x \mapsto (Y(R) = \{x\colon \mathrm{Spec}(R)\to X \mid x^{-1}(Y) = Y_x = \mathrm{Spec}(R)\})$.

(i) Consider the diagram:

$$\begin{array}{ccc}\mathrm{Sub}(X) & \xrightarrow{\sim} & \lim_x \mathrm{Sub}(\mathrm{Spec}(R))\\ \uparrow & & \uparrow\\ \mathsf{Closed}(X) & \xrightarrow{\sim} & \lim_x \mathsf{Closed}(\mathrm{Spec}(R))\end{array}$$

By the definition of closed subfunctors and $X \simeq \mathrm{colim}_x\mathrm{Spec}(R)$, the bottom row is an isomorphism. Therefore, we reduce to showing:

$$\lim_x \mathsf{Closed}(\mathrm{Spec}(R)) \simeq \lim_x \mathsf{Id}_R^{\mathrm{op}}.$$

For this, we must show that the bijection $V\colon \mathsf{Id}_R^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Closed}(\mathrm{Spec}(R))$ is functorial. For $\varphi\colon R \to S$, consider the diagram:

$$\begin{array}{ccc} \mathsf{Id}_R & \xrightarrow{V_R} & \mathsf{Closed}(\mathrm{Spec}(R)) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \mathrm{Spec}(\varphi)^{-1}} \\ \mathsf{Id}_S & \xrightarrow{V_S} & \mathsf{Closed}(\mathrm{Spec}(S)). \end{array}$$

We verify commutativity by showing $V(IS) = \mathrm{Spec}(\varphi)^{-1}(V(I))$. The right side is:

$$V(I) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) \simeq V(S \otimes_R I) \simeq \mathrm{Spec}(S \otimes_R R/I).$$

Using $S/IS \simeq S \otimes_R R/I$, we get:

$$\mathrm{Spec}(\varphi)^{-1}(V(I)) \simeq \mathrm{Spec}(S/IS) = V(IS) = V(\varphi^*(I)).$$

This shows V is a natural transformation from $\mathsf{Id}_{(-)}^{\mathrm{op}}$ to $\mathsf{Closed}(\mathrm{Spec}(-))$.

(ii) Similarly, using $X \simeq \operatorname{colim} \mathrm{Spec}(R)$ and the definition of open subfunctors, we reduce to proving $D\colon \mathsf{Rad}_R \xrightarrow{\sim} \mathsf{Open}(\mathrm{Spec}(R))$ is functorial.

For $\varphi\colon R \to S$ (let $f = \mathrm{Spec}(\varphi)$), we must show the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Rad}_R & \xrightarrow{D_R} & \mathsf{Open}(\mathrm{Spec}(R)) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle f^{-1}} \\ \mathsf{Rad}_S & \xrightarrow{D_S} & \mathsf{Open}(\mathrm{Spec}(S)) \end{array}$$

Note: In the category Rad, for $J \in \mathsf{Rad}_R$, the pullback is defined as $\varphi^*(J) := \sqrt{JS}$.

We verify $f^{-1}(D(J)) = D(\sqrt{JS})$:

- *Left side:* By the definition of open subfunctor pullback:

  $$f^{-1}(D(J)) = D(J) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S).$$

  For affine schemes, the pullback of $D(J)$ is the open subfunctor defined by the extended ideal JS, so $f^{-1}(D(J)) = D(JS)$.
- *Right side:* By definition, this is $D(\sqrt{JS})$.

Since $D(K)$ depends only on the radical of K (i.e., $D(K) = D(\sqrt{K})$), we have $D(JS) = D(\sqrt{JS})$ identically. This means the diagram commutes, inducing an isomorphism at the limit level:

$$\mathsf{Rad}_X = \lim_x \mathsf{Rad}_R \xrightarrow{\sim} \lim_x \mathsf{Open}(\mathrm{Spec}(R)) \simeq \mathsf{Open}(X).$$

□

Definition 3.2.2 (Open Complement)**.** Let $Z \subset X$ be a closed subfunctor determined by a quasi-coherent ideal I. The *open complement* $X - Z$ is the open subfunctor corresponding to the quasi-coherent radical ideal $\sqrt{I}$.

Example 3.2.3 (Open Complements)**.**

- Let A be a ring and $F \subset A$ a subset. Then in $\mathrm{Spec}(A)$, $D(F)$ is the open complement of $V(F)$. For example, for a set I, take $A = \mathbb{Z}[x_i \mid i \in I]$ and $F = \{x_i\}_{i \in I}$. Then $D(F) = \mathbb{A}^I - 0$ is the open complement of $V(F) = 0$.

- Let A be an $\mathbb{N}$-graded ring and $F \subset A$ a subset of homogeneous elements. Then in $\mathrm{Proj}(A)$, $D(F)$ is the open complement of $V(F)$. For example, for a set I, in $\mathbb{P}^{I \sqcup \{0\}}$, the affine space $D(x_0) \simeq \mathbb{A}^I$ is the open complement of $V(x_0) \simeq \mathbb{P}^I$.

***Warning* 3.2.4** (No Closed Complement). Different closed subfunctors can have the same open complement, because different ideals can have the same radical. Therefore, there is no notion of "closed complement" of an open subfunctor.

Example 3.2.5 (Scheme-theoretic Image). For any ring homomorphism $\varphi\colon A \to B$, we can factor it as a surjection followed by an injection:

$$A \twoheadrightarrow A/\ker(\varphi) \hookrightarrow B.$$

Therefore, $V(\ker(\varphi)) \subset \mathrm{Spec}(A)$ is the smallest closed subfunctor through which $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ factors.

This construction generalizes to arbitrary algebraic functors: Let $f : Y \to X$ be a morphism of algebraic functors. The *scheme-theoretic image* of f is the smallest closed subfunctor Z through which f factors. By Proposition 3.2.1 (the bijection between closed subfunctors and quasi-coherent ideals), there exists a unique quasi-coherent ideal $I \in \mathsf{Id}_X$ such that $Z = V(I)$.

Explicit characterization: I is the *largest* quasi-coherent ideal such that the algebra homomorphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ factors through the quotient $\mathcal{O}_X/I$. This is because f factoring through the closed subfunctor $V(J)$ is equivalent to the algebra morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ factoring through the quotient $\mathcal{O}_X/J$ (which can be understood intuitively by examining the geometric shape of $V(J(x)) \subset \mathrm{Spec}(\mathcal{O}_X(x))$ pointwise). Since $V(\cdot)$ is an order-reversing bijection, "smallest closed subfunctor" naturally corresponds to "largest quasi-coherent ideal".

It is important to note that although when X is a scheme, I coincides with the kernel of the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ in the module category Mod_X, for general algebraic functors X, the kernel in the module category may not form a quasi-coherent ideal (i.e., it may not satisfy the naturality requirements in Id_X). Therefore, I should be understood through the above universal property (largest quasi-coherent ideal), rather than simply taking the kernel at the module level.

***Remark* 3.2.6** (Loci of Maps Between Quasi-coherent Modules). Recall the loci of linear maps $u\colon M \to N$ from §Section 2.3: we set

$$V(u)(R) := \{\varphi\colon R \to A \mid \varphi^*(u) = 0\}.$$

We can now generalize this to general algebraic functors X. For a morphism $f : M \to N$ of quasi-coherent modules over X, we define $V(f)$ as:

$$V(f)(R) := \{x\colon \mathrm{Spec}(R) \to X \mid f(x) = 0\}.$$

Note: we have $f(x)\colon M(x) \to N(x)$, so $f(x) = 0$ means the zero homomorphism. We can similarly define $\mathrm{Epi}(f)$, $\mathrm{Mono}(f)$, and $\mathrm{Iso}(f)$.

We can establish the following analogous proposition: If $N \in \mathsf{Vect}_X$ is a vector bundle, then:

- $V(f)$ is a closed subfunctor.
- $\mathrm{Epi}(f)$ is an open subfunctor.

If M is also a vector bundle, then:

- $\mathrm{Mono}(f)$ is an open subfunctor.
- $\mathrm{Iso}(f)$ is an open subfunctor.

3.2.1. Relative Spec and Relative Proj

We now globalize Spec and Proj, working relative to an arbitrary algebraic functor as base.

***Observation* 3.2.7** (Base Change Compatibility). Note that the functor:

$$\mathrm{Spec}\colon \mathsf{CAlg}_R^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{Spec}(R)}$$

is compatible with base change in the following sense: For an R-algebra A and a ring homomorphism $f: R \to S$, we have a pullback diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A \otimes_R S) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

Therefore, we can extend the Spec functor to quasi-coherent algebras.

Definition 3.2.8 (Relative Spec). Let X be an algebraic functor and $A \in \mathsf{CAlg}_X$ a quasi-coherent algebra over X. We define $\mathrm{Spec}(A) \in \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ by:

$$\mathrm{Spec}(A)(R) \coloneqq \{(x, a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Spec}(A(x)) \to \mathrm{Spec}(R)\}.$$

This construction is functorial, giving a functor:

$$\mathrm{Spec}\colon \mathsf{CAlg}_X^{\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}.$$

Similarly, for an $\mathbb{N}$-graded R-algebra A and a ring homomorphism $f: R \to S$, we have a pullback diagram:

$$\begin{array}{ccc} \mathrm{Proj}(A \otimes_R S) & \longrightarrow & \mathrm{Proj}(A) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

Therefore, we can also extend the Proj functor to quasi-coherent algebras.

Definition 3.2.9 (Relative Proj). Let X be an algebraic functor and $A \in \mathsf{CAlg}_X$ a quasi-coherent $\mathbb{N}$-graded algebra over X. We define $\mathrm{Proj}(A) \in \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ by:

$$\mathrm{Proj}(A)(R) \coloneqq \{(x, a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Proj}(A(x)) \to \mathrm{Spec}(R)\}.$$

This construction is functorial, giving a functor:

$$\mathrm{Proj}\colon \mathsf{CAlg}_X^{\mathbb{N},\mathrm{op}} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}.$$

We can also define relative affine and projective spaces.

Definition 3.2.10 (Relative Affine and Projective Spaces). Let X be an algebraic functor and M a quasi-coherent module over X.

(i) The *affine space over* X associated to M is $\mathbb{A}(M) \coloneqq \mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbb{A}(M)(R) = \{(x, a) \mid x \in X(R) \text{ and } a\colon M(x) \to R \text{ is an } R\text{-linear map}\}.$$

(ii) The *punctured affine space over* X is $\mathbb{A}(M) - 0 := D(M) \subset \mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$(\mathbb{A}(M) - 0)(R) = \{(x, a) \mid x \in X(R) \text{ and } a\colon M(x) \twoheadleftarrow R \text{ is surjective}\}.$$

(iii) The *projective space over* X associated to M is $\mathbb{P}(M) := \mathrm{Proj}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbb{P}(M)(R) = \{(x, a) \mid x \in X(R) \text{ and } a\colon M(x) \twoheadrightarrow L \text{ is a quotient } R\text{-line}\}.$$

***Remark* 3.2.11** (Functorial Characterization). Let X and Y be algebraic functors.

(i) *Relative Spec:* For any quasi-coherent algebra $A \in \mathsf{CAlg}_X$, a morphism $Y \to \mathrm{Spec}(A)$ consists of:

- A morphism of algebraic functors $f\colon Y \to X$;
- An algebra homomorphism in CAlg_Y: $\varphi\colon f^*(A) \to \mathcal{O}_Y$.

(ii) *Relative Proj:* Let A be a quasi-coherent $\mathbb{N}$-graded algebra over X generated by A_1. A morphism $Y \to \mathrm{Proj}(A)$ consists of:

- A morphism of algebraic functors $f\colon Y \to X$;
- A quotient line bundle on Y, i.e., a surjection $q\colon f^*(A_1) \twoheadrightarrow L$;
- A compatibility condition: the induced map $\mathrm{Sym}(f^*(A_1)) \to \mathrm{Sym}(L)$ must factor through $f^*(A)$. That is, there exists a unique dashed arrow making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Sym}(f^*(A_1)) & \xrightarrow{\mathrm{Sym}(q)} & \mathrm{Sym}(L) \\ \big\downarrow & \overset{\exists!}{\nearrow} & \\ f^*(A) & & \end{array}$$

(This is equivalent to saying that homogeneous relations in A must vanish when mapped to $L^{\otimes n}$.)

In fact, relative Spec and relative Proj allow us to extend the notions of affine schemes, quasi-affine schemes, projective schemes, and quasi-projective schemes to the relative setting. For an algebraic functor V, let $\mathsf{Rad}_V^{\mathrm{ft}}$ denote the category of finitely generated radical ideals over V.

Definition 3.2.12 (Classification of Relative Morphisms). Let $f\colon Y \to X$ be a morphism of algebraic functors.

(i) *Affine morphism:* If there exists $A \in \mathsf{CAlg}_X$ such that $Y \simeq \mathrm{Spec}(A)$, then f is called an *affine morphism*, or we say Y is *affine over* X. We denote by $\mathsf{Aff}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory spanned by affine schemes over X.

(ii) *Quasi-affine morphism:* If there exist $V \in \mathsf{Aff}_X$ and $I \in \mathsf{Rad}_V^{\mathrm{ft}}$ such that $Y \simeq D(I)$, then f is called a *quasi-affine morphism*, or we say Y is *quasi-affine over* X. We denote by $\mathsf{QAff}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory of quasi-affine schemes over X.

(Note: The condition $I \in \mathsf{Rad}_V^{\mathrm{ft}}$ means I is a radical ideal generated by finitely many sections. This is equivalent to saying $I \subset \mathcal{O}_V$ is a finite subset.)

(iii) *Projective morphism:* If there exists $A \in \mathsf{CAlg}_X^{\mathbb{N}}$ such that A_1 is *finitely generated* (i.e., A_1 is a finitely generated A_0-module and A as an algebra is generated by A_1) and $Y \simeq \mathrm{Proj}(A)$, then f is called a *projective morphism*, or we say Y is *projective over* X. We denote by $\mathsf{Proj}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory of projective schemes over X.

(iv) *Quasi-projective morphism:* If there exist $V \in \mathsf{Proj}_X$ and $I \in \mathsf{Rad}_V^{\mathrm{ft}}$ such that $Y \simeq D(I)$, then f is called a *quasi-projective morphism*, or we say Y is *quasi-projective over* X. We denote by $\mathsf{QProj}_X \subset \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ the full subcategory of quasi-projective schemes over X.

We characterize affine morphisms via their fibers.

***Observation* 3.2.13** (Fibers of Affine Morphisms). Let $f: Y \to X$ be an affine morphism. Then there exists $A \in \mathsf{CAlg}_X$ such that $\mathrm{Spec}(A) \simeq Y$. For any R-point $x\colon \mathrm{Spec}(R) \to X$, consider the pullback:

$$\begin{array}{ccc} Y_x & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{x} & X. \end{array}$$

For any ring S, by the universal property of fiber products, we have $Y_x(S) = Y(S) \times_{X(S)} \mathrm{Hom}_{\mathsf{CAlg}}(R, S)$. Concretely, elements of $Y_x(S)$ have the form $((r, a), r')$ where:

- $(r, a) \in Y(S) = \mathrm{Spec}(A)(S)$;
- $r' \in \mathrm{Hom}_{\mathsf{CAlg}}(R, S)$,

satisfying $x \circ r' = r$ and a is a section of $\mathrm{Spec}(A(r)) \to \mathrm{Spec}(S)$.

By the characterization of quasi-coherent modules (more precisely, quasi-coherent algebras), we have $(r')^*(A(x)) \simeq A(r)$. Therefore, a can be interpreted as an S-algebra homomorphism:

$$\varphi\colon (r')^*(A(x)) \simeq A(x) \otimes_R S \to S.$$

Using the fact that tensor product is the pushout in the category of rings, the above map corresponds to an R-algebra homomorphism:

$$\psi\colon A(x) \to S.$$

Therefore, we obtain an isomorphism of sets:

$$Y_x(S) \simeq \mathrm{Hom}_{\mathsf{CAlg}_R}(A(x), S) \simeq \mathrm{Spec}(A(x))(S),$$

yielding an isomorphism:

$$Y_x \simeq \mathrm{Spec}(A(x)).$$

Conversely, if for any ring R and any morphism $x\colon \mathrm{Spec}(R) \to X$, the pullback Y_x is an affine scheme, we define $A(x) := \mathcal{O}(Y_x)$. Since Y_x satisfies pullback properties with respect to x and the $\mathcal{O}$ functor preserves base change, the assignment $x \mapsto A(x)$ forms a quasi-coherent X-algebra A. Recalling the definition of relative Spec, we have:

$$\mathrm{Spec}(A)(R) = \{(x, \sigma) \mid x \in X(R), \sigma \in \mathrm{Hom}_{\mathsf{CAlg}}(A(x), R)\}.$$

Since Y_x is affine, we have $\mathrm{Hom}_{\mathsf{CAlg}}(A(x), R) \simeq Y_x(R)$ (the fiber of $Y(R)$ projecting to x). Thus we establish $\mathrm{Spec}(A) \simeq Y$.

Proposition 3.2.14 (Affine Morphisms). *Let X be an algebraic functor. We have a categorical equivalence:*

$$\mathrm{Spec}\colon \mathsf{CAlg}_X^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}_X.$$

The inverse sends $(Y \to X)$ to $f_*(\mathcal{O}_Y)$.

Proof. Everything reduces to showing:

$$\mathsf{Aff}_X \xrightarrow{\sim} \lim_x \mathsf{Aff}_R.$$

Note that we have a pullback:

$$\begin{array}{ccc} \mathsf{Aff}_X & \longrightarrow & \lim_x \mathsf{Aff}_R \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X} & \longrightarrow & \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/\mathrm{colim}_x \mathrm{Spec}(R)}. \end{array}$$

Since $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is a topos, all colimits in it are van Kampen. This means the bottom row is an isomorphism, hence the top row is also an isomorphism. □

3.3. Modules over Quasi-projective Schemes

We now describe Mod_X when X is quasi-projective. The motivating example is $X = \mathbb{P}^n_k$, where we will identify line bundles $O(d)$ with $\Gamma(\mathbb{P}^n_k, O(d)) = k[x_0, \ldots, x_n]_d$.

3.3.1. Generalized Zariski Descent

Recall the generalized Zariski descent for an R-line L with generating sections $(s_i)_{i\in I}$:

$$\mathrm{Mod}_R \longrightarrow \prod_{i\in I} \mathrm{Mod}_{R_{s_i}} \rightrightarrows \prod_{i,j\in I} \mathrm{Mod}_{R_{s_i s_j}} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i,j,k\in I} \mathrm{Mod}_{R_{s_i s_j s_k}}$$

is a limit diagram.

This raises the following question: For $X = \mathbb{P}^n_k$, is there a family of affine open subschemes $\{D(x_i)\}_{i\in I}$ such that for any $\mathrm{Spec}(R) \to X$, we have a pullback diagram:

$$\begin{array}{ccc} \mathrm{Mod}_{D(x_i)} & \longrightarrow & D(x_i) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X? \end{array}$$

This motivates the following definition:

Definition 3.3.1 (Zariski-Local Epimorphism). A morphism $Y \to X$ of algebraic functors is called *Zariski-local epimorphism* if for any affine scheme $\mathrm{Spec}(R)$ and morphism $\mathrm{Spec}(R) \to X$, there exist elements $f_1, \ldots, f_n \in R$ satisfying $(f_1, \ldots, f_n) = R$ (i.e., they generate the unit ideal) such that for each i, the following dashed lift exists:

$$\begin{array}{ccccc} & & & \nearrow & Y \\ & & & & \downarrow \\ \mathrm{Spec}(R_{f_i}) & \hookrightarrow & \mathrm{Spec}(R) & \longrightarrow & X. \end{array}$$

Example 3.3.2 (Examples of Zariski-Local Epimorphism). (i) Every epimorphism is a Zariski-local epimorphism. In particular, morphisms with sections, such as $\mathbb{A}^n \times X \to X$, are also Zariski-local epimorphism.

(ii) Let $(F_i)_{i\in I}$ be a family of subsets of $O(X)$ such that $(\bigcup_{i\in I} F_i)$ generates $O(X)$. Then $\coprod_{i\in I} D(F_i) \to X$ is a Zariski-local epimorphism.

(iii) Let L be a line bundle over X. We say L is *generated by a family of global sections* $(s_i)_{i\in I}$ if for any $x \in \mathrm{El}(X)$, the elements $(s_i(x))_{i\in I}$ generate $L(x)$. In this case, $\coprod_{i\in I} D(s_i) \to X$ is a Zariski-local epimorphism.

(iv) The morphism $\mathbb{A}^I - 0 \to \mathbb{P}^I$ is a Zariski-local epimorphism. In fact, for a general $\mathbb{N}$-graded algebra A, we have $D(A_1) \to \mathrm{Proj}_1(A)$ is a Zariski-local epimorphism.

Proposition 3.3.3 (Zariski Descent for Quasi-coherent Modules). *Let $(Y_i \to X)_{i\in I}$ be a family of morphisms such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the following diagram is a limit diagram:*

$$\mathrm{Mod}_X \longrightarrow \prod_{i\in I} \mathrm{Mod}_{Y_i} \rightrightarrows \prod_{i,j\in I} \mathrm{Mod}_{Y_i\times_X Y_j} \Rrightarrow \prod_{i,j,k\in I} \mathrm{Mod}_{Y_i\times_X Y_j \times_X Y_k}$$

In particular, $\mathrm{Mod}_X \to \prod_{i\in I} \mathrm{Mod}_{Y_i}$ *is a conservative functor.*

Proof. This is an instance of descent for category-valued sheaves on the Zariski site. The functor $X \mapsto \mathrm{Mod}_X$ defined in this chapter is the right Kan extension of $R \mapsto \mathrm{Mod}_R$ along the Yoneda embedding $\mathsf{Aff} \hookrightarrow \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ (Definition 3.1.15); the Comparison Proposition 5.6.2, applied to the Zariski topology, shows that this right Kan extension takes Zariski-covers to limit diagrams. See Remark 5.6.9 for the higher-categorical formulation. □

Corollary 3.3.4 (Zariski Descent for Functions). *Let $(Y_i \to X)_{i\in I}$ be a family of morphisms such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the following diagram of rings is an equalizer*

$$\mathcal{O}(X) \longrightarrow \prod_{i\in I} \mathcal{O}(Y_i) \rightrightarrows \prod_{i,j\in I} \mathcal{O}(Y_i \times_X Y_j).$$

Corollary 3.3.5 (Zariski Descent for $D(I)$). *Let A be a ring and $I \subset A$ a subset such that $\coprod_{f\in I} D(f) \to D(I)$ is a Zariski-local epimorphism. Then:*

$$\mathrm{Mod}_{D(I)} \longrightarrow \prod_{f\in I} \mathrm{Mod}_{D(f)} \rightrightarrows \prod_{f,g\in I} \mathrm{Mod}_{D(fg)} \Rrightarrow \prod_{f,g,h\in I} \mathrm{Mod}_{D(fgh)}$$

is a limit diagram.

Proof. Set $X = D(I)$ and $Y_f = D(f)$. The claim reduces to the identification $D(fg) \simeq D(f) \times_{D(I)} D(g)$, which follows from

$$D(f) \times_{D(I)} D(g) \simeq D(f) \times_{\mathrm{Spec}(A)} D(g) \simeq D(fg).$$

□

We can now describe quasi-coherent modules on $\mathrm{Proj}(A)$.

Example 3.3.6 (Modules over Proj). Let A be an $\mathbb{N}$-graded ring.

(i) Let $I \subset A_+$ be a subset of homogeneous elements satisfying $A_+ \subset \sqrt{(I)}$. Then $\coprod_{f\in I} \mathrm{Spec}(A_{(f)}) \to \mathrm{Proj}(A)$ is a Zariski-local epimorphism. Therefore, Proposition 3.3.3 shows that we have a limit diagram:

$$\mathrm{Mod}_{\mathrm{Proj}(A)} \longrightarrow \prod_{f\in I} \mathrm{Mod}_{A_{(f)}} \rightrightarrows \prod_{f,g\in I} \mathrm{Mod}_{A_{(fg)}} \Rrightarrow \prod_{f,g,h\in I} \mathrm{Mod}_{A_{(fgh)}}.$$

(ii) Consider $D(A_1) \subset \mathrm{Spec}(A)$. The map $D(A_1) \to \mathrm{Proj}_1(A)$ is a Zariski-local epimorphism inducing an injection $D(A_1)/\mathbb{G}_m \hookrightarrow \mathrm{Proj}_1(A)$. The injectivity means that the iterated fiber product

$$D(A_1) \times_{\mathrm{Proj}_1(A)} \cdots \times_{\mathrm{Proj}_1(A)} D(A_1)$$

(n factors) coincides with the iterated fiber product over $D(A_1)/\mathbb{G}_m$. Since the $\mathbb{G}_m$-action on $D(A_1)$ is free, the latter is just $D(A_1) \times \mathbb{G}_m^{n-1}$. We thus obtain the limit diagram:

$$\mathrm{Mod}_{\mathrm{Proj}_1(A)} \longrightarrow \mathrm{Mod}_{D(A_1)} \rightrightarrows \mathrm{Mod}_{D(A_1)\times\mathbb{G}_m} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathrm{Mod}_{D(A_1)\times\mathbb{G}_m\times\mathbb{G}_m}.$$

Example 3.3.7 (Modules over $\mathbb{P}^1$)**.** Let k be a ring. Consider the projective line $\mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$. Let $u = y/x$. We have $k[x,y]_{(x)} = k[u]$. Let $v = x/y$. Then $k[x,y]_{(y)} = k[v]$ and $k[x,y]_{(xy)} = k[u,v] = k[t^{\pm 1}]$. Therefore, the limit diagram from Example 3.3.6 can be rewritten as a pullback diagram:

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbb{P}^1_k} & \longrightarrow & \mathrm{Mod}_{k[u]} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Mod}_{k[v]} & \longrightarrow & \mathrm{Mod}_{k[t^{\pm 1}]} \end{array}$$

Thus we can write $\mathrm{Mod}_{\mathbb{P}^1_k}$ as:

$$\mathrm{Mod}_{\mathbb{P}^1_k} = \left\{ (M,N,\alpha) \,\middle|\, \substack{M\in\mathrm{Mod}_{k[u]},\, N\in\mathrm{Mod}_{k[v]},\\ \alpha\colon M[u^{\pm1}]\xrightarrow{\sim}N[v^{\pm1}] \text{ is a } k[t^{\pm1}]\text{-linear isomorphism}} \right\}.$$

Example 3.3.8 (Modules over the Affine Line with Two Origins)**.** Let X be the affine line with two origins. From previous exercises, we know the explicit characterization of X is:

$$X(R) := \{(f,e) \mid f \in R, e \in R/(f), e^2 = e\}.$$

Let $X_0 \subset X$ and $X_1 \subset X$ be the loci where $e = 0$ and $e = 1$, respectively. Then we can verify:

- X_0 and X_1 are both open subfunctors of X, and $X_0 \sqcup X_1 \to X$ is a Zariski-local epimorphism.
- The projection $X \xrightarrow{\sim} \mathbb{A}^1_k$ given by $(f,e) \mapsto f$ induces isomorphisms:

$$X_0 \xrightarrow{\sim} \mathbb{A}^1_k, \quad X_1 \xrightarrow{\sim} \mathbb{A}^1_k, \quad X_0 \cap X_1 \xrightarrow{\sim} \mathbb{G}_{m,k}.$$

Therefore, combining with Proposition 3.3.3, we obtain a pullback diagram:

$$\begin{array}{ccc} \mathrm{Mod}_X & \longrightarrow & \mathrm{Mod}_{k[x]} \\ \downarrow & \lrcorner & \downarrow {\scriptstyle x\mapsto x} \\ \mathrm{Mod}_{k[x]} & \xrightarrow{x\mapsto x} & \mathrm{Mod}_{k[x^{\pm 1}]} \end{array}$$

That is:

$$\mathrm{Mod}_X = \left\{ (M,N,\alpha) \,\middle|\, \substack{M\in\mathrm{Mod}_{k[x]},\, N\in\mathrm{Mod}_{k[x]},\\ \alpha\colon M[x^{-1}]\xrightarrow{\sim}N[x^{-1}] \text{ is a } k[x^{\pm1}]\text{-linear isomorphism}} \right\}.$$

Recall the notion of I-local modules from §Section 2.3: for a subset $I \subset A$ there is a localization functor

$$L_I \colon \mathrm{Mod}_A \to \mathrm{Mod}_A^{I\text{-loc}}.$$

Moreover, for a homomorphism $h \colon M \to N$, we have $L_I(h)$ is an isomorphism if and only if for all $f \in I$, the map $h_f \colon M_f \to N_f$ is an isomorphism.

Theorem 3.3.9 (Modules over $D(I)$). *Let A be a ring and $I \subset A$ a finite subset. Then the base change functor $\mathrm{Mod}_A \to \mathrm{Mod}_{D(I)}$ induced by $D(I) \hookrightarrow \mathrm{Spec}(A)$ gives a categorical equivalence:*

$$\mathrm{Mod}_A^{I\text{-loc}} \xrightarrow{\sim} \mathrm{Mod}_{D(I)}.$$

Moreover, we have:

$$L_I(M) = \lim\left(\prod_{f\in I} M_f \rightrightarrows \prod_{f,g\in I} M_{fg} \right).$$

***Remark* 3.3.10** (Infinite Sets). Note that when I is an infinite set, since $\prod_I$ is no longer a finite product, we no longer have full faithfulness.

Proof. First, we construct the functor $\mathrm{Mod}_A^{I\text{-loc}} \to \mathrm{Mod}_{D(I)}$. Consider the diagram:

$$\begin{array}{ccc} \mathrm{Mod}_A & \xrightarrow{i^*} & \mathrm{Mod}_{D(I)} \\ {\scriptstyle L_I}\downarrow & \nearrow & \\ \mathrm{Mod}_A^{I\text{-loc}} & & \end{array}$$

We need to characterize the dashed arrow. By Corollary 3.3.5, we can write $\mathrm{Mod}_{D(I)}$ as a limit diagram. We know that $h \in \mathrm{Mod}_A^{I\text{-loc}}$ is an isomorphism if and only if for all $f \in I$, we have h_f is an isomorphism. Thus the dashed functor, if it exists, is a conservative functor. Its existence is guaranteed by $\mathrm{Mod}_A^{I\text{-loc}} \hookrightarrow \mathrm{Mod}_A$.

In fact, the functor $\mathrm{Mod}_A^{I\text{-loc}} \to \mathrm{Mod}_{D(I)}$ has a right adjoint, given by:

$$(M(f), \alpha_{fg}) \mapsto \lim\left(\prod_{f\in I} M(f) \rightrightarrows \prod_{f,g\in I} M(f)_g \right).$$

By the basic properties of adjoint functors, everything reduces to showing that the right adjoint is fully faithful (left adjoint conservative + right adjoint fully faithful $\Rightarrow$ categorical equivalence). This further reduces to showing that the counit is a pointwise isomorphism. We must show that for all $(M(f), \alpha_{fg})$:

$$i^*\left(\lim\left(\prod_{f\in I} M(f) \rightrightarrows \prod_{f,g\in I} M(f)_g \right)\right) \simeq (M(f), \alpha_{fg}).$$

By conservativity, everything reduces to showing that for all $h \in I$:

$$\left(\lim\left(\prod_{f\in I} M(f) \rightrightarrows \prod_{f,g\in I} M(f)_g \right)\right)_h \simeq M(h).$$

This follows from the exactness of $(-)_h$, finiteness of $\prod_I$, and Zariski descent. □

Corollary 3.3.11 (Affine Completion of Quasi-affine Schemes). *Let X be a quasi-affine scheme. Then the canonical morphism $X \to \mathrm{Spec}(\mathcal{O}(X))$ is an open immersion cut out by some finitely generated radical ideal.*

Proof. Since X is a quasi-affine scheme, there exist a ring A and a finite subset $I \subset A$ such that $X \simeq D(I)$. By Theorem 3.3.9, we have a categorical equivalence:

$$\mathrm{Mod}_A^{I\text{-loc}} \simeq \mathrm{Mod}_{D(I)} \simeq \mathrm{Mod}_X.$$

This actually says that $\mathcal{O}_X \in \mathrm{Mod}_X$ corresponds to $L_I(A)$. In other words, the global sections ring $\mathcal{O}(X) \in \mathsf{CAlg}$ corresponds to $\mathrm{End}_A(L_I(A))$. Moreover, we can see that the ring homomorphism $\lambda\colon A \to L_I A$ induces the canonical morphism:

$$X \to \mathrm{Spec}(\mathcal{O}(X)).$$

Next, consider the pullback:

$$\begin{array}{ccc} D(\lambda(I)) & \hookrightarrow & \operatorname{Spec}(\mathcal{O}(X)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \operatorname{Spec}(\lambda)} \\ D(I) & \hookrightarrow & \operatorname{Spec}(A). \end{array}$$

Everything reduces to showing $D(\lambda(I)) \to D(I)$ is an isomorphism. For this, we only need to show that for any $\varphi\colon A \to R$, if φ lies in $D(I)(R)$, then R can be viewed as an I-local A-module via φ. This follows from $D(I)(R) = \{\varphi\colon A \to R \mid (\varphi(I)) = R\}$ and Zariski descent. ◻

Construction 3.3.12 (Quasi-coherent Module from $\mathbb{Z}$-graded Module). Let A be an $\mathbb{N}$-graded ring. We will define a functor:

$$\{\mathbb{Z}\text{-graded } A\text{-modules}\} \to \operatorname{Mod}_{\operatorname{Proj}(A)}, \quad M \mapsto \widetilde{M}.$$

For a $\mathbb{Z}$-graded A-module M, let $x\colon \operatorname{Spec}(R) \to \operatorname{Proj}(A)$ be an R-point of $\operatorname{Proj}(A)$. Specifically, we place it in $\operatorname{Proj}_d(A)$ corresponding to a quotient $\mathbb{N}$-graded algebra $\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \operatorname{Sym}_R(L)$ (of course, when A is generated by $A_{\le 1}$, by the structure theorem we can take $d = 1$).

Let $\widetilde{\varphi}$ be the induced map between $\mathbb{Z}$-graded rings $A^{(d)} \to \bigoplus_{n\in\mathbb{Z}} L^{\otimes n}$. Define:

$$\widetilde{M}(x) = \widetilde{\varphi}^*(M^{(d)})_0 \in \operatorname{Mod}_R.$$

Note that for $d \ge 1$ and $f \in A_d$, the pullback functor $\operatorname{Mod}_{\operatorname{Proj}(A)} \to \operatorname{Mod}_{A_{(f)}}$ sends $\widetilde{M}$ to $M_{(f)}$. Example 3.3.6 shows that $M \mapsto \widetilde{M}$ factors through L_{A_+}.

Theorem 3.3.13 (Quasi-coherent Modules on Quasi-projective Schemes). *Let A be an $\mathbb{N}$-graded ring generated by $A_{\le 1}$, and let $I \subset A_1$ be a finite subset such that $D(I) \subset \operatorname{Proj}(A)$. Then the functor $M \mapsto \widetilde{M}$ induces an equivalence of categories*

$$\{I\text{-local } \mathbb{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \operatorname{Mod}_{D(I)}.$$

In particular, if A is finitely generated as an A_0-algebra, then

$$\{A_1\text{-local } \mathbb{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \operatorname{Mod}_{\operatorname{Proj}(A)}.$$

Proof. Since A is generated by $A_{\le 1}$, the open subschemes $\operatorname{Spec}(A_{(f)}) \simeq D(f)$ for $f \in I$ form an affine open cover of $D(I) \subset \operatorname{Proj}(A)$. By Example 3.3.6(i) (an instance of Proposition 3.3.3), we have a limit description

$$\operatorname{Mod}_{D(I)} \xrightarrow{\sim} \lim\left(\prod_{f\in I}\operatorname{Mod}_{A_{(f)}} \rightrightarrows \prod_{f,g\in I}\operatorname{Mod}_{A_{(fg)}} \rightrightarrows \prod_{f,g,h\in I}\operatorname{Mod}_{A_{(fgh)}}\right). \tag{3.1}$$

On the other side, write $\operatorname{Mod}_A^{\mathbb{Z}\text{-gr}}$ for $\mathbb{Z}$-graded A-modules. For each $f \in A_1$, the degree-zero functor $N \mapsto N_0$ on graded A_f-modules gives an equivalence $\operatorname{Mod}_{A_f}^{\mathbb{Z}\text{-gr}} \xrightarrow{\sim} \operatorname{Mod}_{A_{(f)}}$, with inverse $P \mapsto \bigoplus_{n\in\mathbb{Z}} P \otimes_{A_{(f)}} (A_f)_n$ (using the trivialization of the line A_f-module $(A_f)_1 \simeq A_{(f)} \cdot f$). Composing with the localization $\operatorname{Mod}_A^{\mathbb{Z}\text{-gr}} \to \operatorname{Mod}_{A_f}^{\mathbb{Z}\text{-gr}}$, $M \mapsto M_f$, yields a functor

$$\Phi_f : \operatorname{Mod}_A^{\mathbb{Z}\text{-gr}} \to \operatorname{Mod}_{A_{(f)}}, \quad M \mapsto (M_f)_0 = M_{(f)}.$$

Bringing these together, Construction 3.3.12 sends a graded A-module M to the descent datum $\big((M_{(f)})_f, \text{evident gluings}\big)$ on the cover $\{D(f)\}_{f\in I}$, which under (3.1) corresponds to $\widetilde{M}|_{D(I)}$. Thus the diagram

$$\begin{array}{ccc} \mathrm{Mod}_A^{\mathbb{Z}\text{-gr}} & \xrightarrow{\widetilde{(-)}} & \mathrm{Mod}_{D(I)} \\ & \searrow^{(\Phi_f)_f} & \downarrow \wr \\ & & \lim_{f,g,h\in I} \mathrm{Mod}_{A_{(\dots)}} \end{array}$$

commutes.

It remains to identify the kernel and essential image of $\widetilde{(-)}$. By Theorem 3.3.9, the right-hand limit is equivalent to I-local A-modules where I is regarded as a subset of A; localizing degree-by-degree on the graded level gives the same equivalence at the graded level. Concretely, $\widetilde{M} \simeq \widetilde{N}$ iff $M_f \simeq N_f$ as graded A_f-modules for every $f \in I$, which by Proposition 2.3.24(v) is exactly $L_I M \simeq L_I N$. Restricting to I-local graded modules makes $\widetilde{(-)}$ fully faithful.

For essential surjectivity: any descent datum $(N_f, \alpha_{fg})_{f\in I}$ with $N_f \in \mathrm{Mod}_{A_{(f)}}$ assembles into a graded A-module via

$$M = \bigoplus_{n\in\mathbb{Z}} \lim \Big(\prod_f N_f \otimes_{A_{(f)}} (A_f)_n \rightrightarrows \prod_{f,g} N_f \otimes_{A_{(f)}} (A_{fg})_n \Big),$$

whose localizations at each $f \in I$ recover $\bigoplus_n N_f \otimes_{A_{(f)}} (A_f)_n$ — i.e., the graded module corresponding to N_f. By construction M is I-local, and $\widetilde{M}$ recovers the original descent datum.

The "in particular" statement: when A is finitely generated over A_0 and generated by $A_{\leq 1}$, the degree-1 part A_1 is itself finitely generated. Choosing I to be a finite generating set of A_1, we have $D(I) = D(A_1) = \mathrm{Proj}(A)$ inside $\mathrm{Proj}(A)$, and "I-local" coincides with "A_1-local". □

3.3.2. Serre Twisting

Let A be a $\mathbb{Z}$-graded ring, M a $\mathbb{Z}$-graded A-module, and $d \in \mathbb{Z}$. Recall the shift notation $M(d) := \bigoplus_{n\in\mathbb{Z}} M_{n+d}$ (so $(M(d))_n = M_{n+d}$).

Definition 3.3.14 (Serre Twisting). Let A be an $\mathbb{N}$-graded ring and $d \in \mathbb{Z}$. We denote by $\mathcal{O}(d)$ the quasi-coherent module corresponding to the $\mathbb{Z}$-graded A-module $A(d)$. For a given $M \in \mathrm{Mod}_{\mathrm{Proj}(A)}$, the *$d$-th Serre twist* of M is:

$$M(d) := M \otimes \mathcal{O}(d).$$

***Remark* 3.3.15** (Explicit Description). If A is generated by $A_{\leq 1}$, an R-point $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ corresponds to a quotient line $A_1 \otimes_{A_0} R \twoheadrightarrow L$ such that $A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ factors through A. Under this correspondence $\mathcal{O}(d)(x) = L^{\otimes d}$.

***Remark* 3.3.16** (Notation Variation). In some references, $\mathcal{O}(-d)$ may be used to denote what we call $\mathcal{O}(d)$.

Definition 3.3.17 (Tautological Line Bundle). Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$. We call $\mathcal{O}(1)$ the *tautological line bundle* on $\mathrm{Proj}(A)$.

We collect the basic properties of Serre twisting.

Proposition 3.3.18 (Properties of Serre Twisting). *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$.*

(i) For any $d \in \mathbb{Z}$, $\mathcal{O}(d)$ is a line bundle on $\mathrm{Proj}(A)$.

(ii) For a $\mathbb{Z}$-graded A-module M and $d \in \mathbb{Z}$, there exists an isomorphism $\widetilde{M}(d) \xrightarrow{\sim} \widetilde{M(d)}$. In particular, for all $d, e \in \mathbb{Z}$, there exist isomorphisms:

$$\mathcal{O}(d) \otimes \mathcal{O}(e) \simeq \mathcal{O}(d+e) \quad \text{and} \quad \mathcal{O}(1)^{\otimes d} \xrightarrow{\sim} \mathcal{O}(d).$$

Therefore, $\bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$ can be viewed as a $\mathbb{Z}$-graded quasi-coherent algebra on $\operatorname{Proj}(A)$.

(iii) There exist canonical maps of $\mathbb{Z}$-graded rings:

$$L_{A_1}A \to \bigoplus_{d\in\mathbb{Z}} \Gamma(\operatorname{Proj}(A), \mathcal{O}(d)),$$

and for any $\mathbb{Z}$-graded A-module M, there exist canonical maps of $\mathbb{Z}$-graded A-modules:

$$L_{A_1}M \to \bigoplus_{d\in\mathbb{Z}} \Gamma(\operatorname{Proj}(A), \widetilde{M}(d)).$$

When A is finitely generated as an A_0-algebra, these are isomorphisms.

Proof. (i) For $x \colon \operatorname{Spec}(R) \to \operatorname{Proj}(A)$, we know it corresponds to $\varphi \colon A \otimes_{A_0} R \twoheadrightarrow \operatorname{Sym}_R(L)$. Therefore, we can consider the commutative diagram:

$$\begin{array}{ccc} A & \overset{\widetilde{\varphi}}{\dashrightarrow} & \bigoplus_{n\in\mathbb{Z}} L^{\otimes n} \\ \downarrow & & \uparrow \\ A \otimes_{A_0} R & \xrightarrow{\varphi} & \operatorname{Sym}_R(L). \end{array}$$

This gives the dashed $\widetilde{\varphi}$. By definition:

$$\mathcal{O}(d)(x) = \widetilde{A(d)}(x) = \widetilde{\varphi}^*(A(d))_0 = \left(\bigoplus_{n\in\mathbb{Z}} L^{\otimes n}(d)\right)_0 = L^{\otimes d}.$$

(ii) We have:

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}(d))(x) = \widetilde{M}(x) \otimes_R \mathcal{O}(d)(x) = \widetilde{\varphi}^*(M) \otimes_R L^{\otimes d} = \widetilde{\varphi}^*(M(d))_0.$$

(iii) Since $\operatorname{Proj}(A)$ is a quasi-projective scheme, by Theorem 3.3.13, there exists an isomorphism:

$$\operatorname{Hom}_A^{A_1\text{-loc},\mathbb{Z}\text{-graded}}(L_{A_1}A, L_{A_1}A(d)) \xrightarrow[M\mapsto\widetilde{M}]{\sim} \operatorname{Hom}_{\operatorname{Mod}_{\operatorname{Proj}(A)}}(\mathcal{O}, \mathcal{O}(d)) = \Gamma(\operatorname{Proj}(A), \mathcal{O}(d)).$$

The left side is nothing but $\operatorname{Hom}_{\operatorname{Mod}_A^{\mathbb{Z}}}(A, L_{A_1}A(d)) \simeq (L_{A_1}A)_d$.

□

Looking Ahead

We have now constructed the ambient linear algebra of algebraic functors: Mod_X and its subposet $\mathsf{Id}_X \supset \mathsf{Rad}_X$, the relative $\underline{\operatorname{Spec}}$ and $\underline{\operatorname{Proj}}$ over a base, Zariski descent for modules, and Serre's twisting sheaves $\mathcal{O}(d)$ on projective schemes. Together with the content of Chapters 1 and 2, the sheaf-theoretic / module-theoretic vocabulary is complete.

What is missing is *topology*. To talk about open covers, generic points, connected components, or dimension, we need to attach to each algebraic functor X a genuine topological space $|X|$ compatible with its subfunctor structure. This is the subject of the next chapter, where we develop point-free topology (locales) and prove that the poset $\mathsf{Open}(X)$ of open subfunctors is always a spatial locale, realised by the space $|X|$. From Chapter 5 onward the topological and categorical pictures will be available in parallel.

Chapter 4.

Locale

In this chapter we introduce *pointless topology*, otherwise known as the theory of *locales*.

When studying a topological space, we often care more about the lattice of open sets than the underlying points. Locales formalize this — open-set data, abstracted away from any concrete point set. The goal is to construct the diagram

$$\begin{array}{ccc} & & \mathsf{Loc} \\ & & \downarrow \\ \mathsf{Top} & \xrightarrow{X \mapsto \mathrm{Open}(X)} & \mathsf{Pos}^{\mathrm{op}}. \end{array}$$

Not every space is recoverable from its open-set lattice; those that are constitute the class of *sober spaces*, and we have

$$\mathsf{Top}^{\mathrm{Haus}} \subset \mathsf{Top}^{\mathrm{sober}} \hookrightarrow \mathsf{Loc}.$$

In fact, M. Hochster proved the following theorem in 1969:

Theorem 4.0.1 (Hochster [Hoc69]). *Let X be a topological space. The following are equivalent:*

(i) There exists a ring A such that $X \simeq \mathrm{Prim}(A) = |\mathrm{Spec}(A)|$.

(ii) X is quasi-compact and has a basis $\mathcal{B}$ of quasi-compact open sets that is closed under finite intersections, and X is sober.

(iii) X can be written as a limit of finitely many T_0 spaces.

The proof — that every space satisfying (ii) arises as a prime spectrum — is delicate; we refer to the original paper.

Definition 4.0.2 (Spectral Space). A *spectral space* is a topological space satisfying any of the equivalent conditions in Theorem 4.0.1.

For algebraic geometry, sober spaces will be sufficient throughout.

The main goal of this chapter is to show that for any algebraic functor X, the poset $\mathsf{Open}(X)$ of open subfunctors is a locale, and it lies in the essential image of:

$$\mathsf{Top} \xrightarrow{\mathrm{Open}} \mathsf{Loc}.$$

That is, there exists $|X|$ such that $\mathsf{Open}(X) \simeq \mathrm{Open}(|X|)$. We call $|X|$ the *geometric realization* or *underlying topological space* of X.

4.1. Pointless Topology

The starting point of "pointless topology" is the observation that most topological spaces (including all Hausdorff spaces) are determined by their posets of open subsets. Since posets are ubiquitous in mathematics, this observation lets topology appear in unexpected places. Our goal here is to develop the formalism that will later express $\mathsf{Open}(X)$ of an algebraic functor X as the open-set poset of a topological space $|X|$.

Notation 4.1.1. Let P be a poset. Given a family $(x_i)_{i\in I}$ in P, we write $\bigvee_{i\in I} x_i$ for its supremum (least upper bound) and $\bigwedge_{i\in I} x_i$ for its infimum (greatest lower bound), when they exist.

***Remark* 4.1.2** (Posets as Categories).

(i) Viewing a poset as a category, suprema are colimits and infima are limits.

(ii) The supremum of the empty family is the smallest element (initial object); dually, the infimum of the empty family is the largest element.

(iii) A poset admits all suprema iff it admits all infima: the infimum of a family equals the supremum of all elements below it.

Definition 4.1.3 (Locale). A *locale* is a poset O satisfying:

(i) *(Completeness)* O admits all suprema (hence all infima).

(ii) *(Distributivity)* For any $u \in O$ and family $(v_i)_{i\in I}$,

$$u \wedge \Big(\bigvee_{i\in I} v_i\Big) = \bigvee_{i\in I} (u \wedge v_i).$$

A *morphism of locales* $f : O' \to O$ is a morphism of posets $f^* : O \to O'$ (in the opposite direction) that preserves colimits (arbitrary suprema) and finite limits (finite infima). We denote the category of locales by Loc.

Example 4.1.4 (Totally Ordered Sets). Any totally ordered set with suprema is a locale. For instance, $\mathbb{N} \cup \{+\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$ are locales.

Example 4.1.5 (The Locale of a Topological Space). Let T be a topological space. The poset $\mathrm{Open}(T)$ of open subsets of T is a locale:

- Suprema: $\bigvee_i U_i = \bigcup_i U_i$; infima: $\bigwedge_i U_i = \mathrm{interior}\big(\bigcap_i U_i\big)$.
- Distributivity follows from the set-theoretic identity $U \cap (\bigcup_i V_i) = \bigcup_i (U \cap V_i)$ combined with the fact that finite intersections of opens are open.

A continuous map $f : T \to S$ induces a locale morphism $\mathrm{Open}(T) \to \mathrm{Open}(S)$ via $f^{-1} : \mathrm{Open}(S) \to \mathrm{Open}(T)$ (which preserves colimits and finite limits). We thus obtain a functor

$$\mathrm{Open} : \mathsf{Top} \to \mathsf{Loc}.$$

***Remark* 4.1.6** (Frames). Locales are also known as *frames* or *complete Heyting algebras*. These names carry different conventions for morphisms and hence describe different categories.

Proposition 4.1.7 (Limits and Colimits of Locales). *The category* Loc *admits limits and colimits, and the inclusion* $\mathsf{Loc} \hookrightarrow \mathsf{Pos}^{\mathrm{op}}$ *preserves colimits. The initial object is* $0 = \mathrm{Open}(\emptyset) = \{\emptyset\}$ *and the terminal object is* $1 = \mathrm{Open}(\{*\}) = \{\emptyset \to *\}$.

Definition 4.1.8 (Points of a Locale). Let O be a locale. The *point space* $\mathrm{Pt}(O)$ is the topological space defined by:

- Points: morphisms of locales $p\colon 1 \to O$.
- Open subsets: for each $u \in O$, the subset $\mathrm{Pt}(u) := \{p \mid p^*(u) = *\}$.

A locale morphism $f\colon O' \to O$ induces a continuous map $\mathrm{Pt}(O') \to \mathrm{Pt}(O)$ (since $\mathrm{Pt}(u)$ pulls back to $\mathrm{Pt}(f^*(u))$), giving a functor

$$\mathrm{Pt}\colon \mathsf{Loc} \to \mathsf{Top}.$$

***Remark* 4.1.9** (Points as Completely Prime Filters). Unwinding the definition: a poset map $O \to \{\emptyset \to *\}$ is specified by the preimage $F \subset O$ of $*$. Such an F is a *filter* when (i) upward-closed and (ii) nonempty + closed under binary meets; a filter is *completely prime* when (iii) $\bigvee_i x_i \in F$ implies $x_i \in F$ for some i. The points of a locale O correspond bijectively to completely prime filters on O.

Proposition 4.1.10 ($\mathrm{Open} \dashv \mathrm{Pt}$). *The functor* $\mathrm{Open}\colon \mathsf{Top} \to \mathsf{Loc}$ *is left adjoint to* $\mathrm{Pt}\colon \mathsf{Loc} \to \mathsf{Top}$. *The adjunction is* idempotent: *it restricts to an equivalence between the essential images of both functors.*

Definition 4.1.11 (Sober Spaces and Spatial Locales).
(i) A topological space T is *sober* if it lies in the essential image of Pt, equivalently if the unit $T \to \mathrm{Pt}(\mathrm{Open}(T))$ is a homeomorphism.
(ii) A locale O is *spatial* if it lies in the essential image of Open, equivalently if the counit $\mathrm{Open}(\mathrm{Pt}(O)) \to O$ is an isomorphism.

We write $\mathsf{Top}^{\mathrm{sob}} \subset \mathsf{Top}$ and $\mathsf{Loc}^{\mathrm{spa}} \subset \mathsf{Loc}$ for the respective full subcategories.

***Remark* 4.1.12** (Soberification and Spatialization). By Proposition 4.1.10, the adjunction $\mathrm{Open} \dashv \mathrm{Pt}$ restricts to an equivalence $\mathsf{Top}^{\mathrm{sob}} \simeq \mathsf{Loc}^{\mathrm{spa}}$. The inclusion $\mathsf{Top}^{\mathrm{sob}} \hookrightarrow \mathsf{Top}$ has a left adjoint $\mathrm{Pt} \circ \mathrm{Open}$ called *soberification*; the inclusion $\mathsf{Loc}^{\mathrm{spa}} \hookrightarrow \mathsf{Loc}$ has a right adjoint $\mathrm{Open} \circ \mathrm{Pt}$ called *spatialization*. Consequently, limits of sober spaces are sober and colimits of spatial locales are spatial.

Proposition 4.1.13 (Characterisation of Sober Spaces). *Let T be a topological space. The points of the locale* $\mathrm{Open}(T)$ *are in bijection with the irreducible closed subsets of T, and the unit $T \to \mathrm{Pt}(\mathrm{Open}(T))$ sends $t \mapsto \overline{\{t\}}$. Hence T is sober iff the map*

$$\{\text{points of } T\} \to \{\text{irreducible closed subsets of } T\}, \qquad t \mapsto \overline{\{t\}},$$

is a bijection — i.e., every irreducible closed subset has a unique generic point.

***Remark* 4.1.14** (Sober Examples and Locality).
(i) Every Hausdorff space is sober, since its irreducible subsets are singletons.
(ii) Sobriety is a local property: if T admits a cover by sober open subsets, then T is sober.

4.2. Locales of Radical Ideals

Let R be a ring. Recall (Proposition 3.2.1) that open subfunctors of $\mathrm{Spec}(R)$ correspond to radical ideals in R via $D\colon \mathsf{Rad}_R \xrightarrow{\sim} \mathsf{Open}(\mathrm{Spec}(R))$. We verify that Rad_R is a locale and compute its points.

Proposition 4.2.1 (Rad_R is a Locale). *For any ring R, the poset* Rad_R *of radical ideals is a locale. Explicitly:*

- *The supremum of a family $(K_i)_{i\in I}$ is $\sqrt{\sum_i K_i}$.*
- *The infimum (distributing over sums) satisfies*

$$K \cap \sqrt{\textstyle\sum_i L_i} = \sqrt{\textstyle\sum_i (K \cap L_i)}.$$

The distributivity rests on the identities $K \cap L = \sqrt{KL}$ for radical ideals and the distributivity of the product of ideals over sums. (The full lattice Id_R of arbitrary ideals is not *a locale, since intersection does not distribute over sums.)*

***Remark* 4.2.2** (Base Change). For a ring map $R \to S$, the base-change map $\mathsf{Id}_R \to \mathsf{Id}_S$, $I \mapsto IS$, preserves products and sums of ideals. Consequently the induced $\mathsf{Rad}_R \to \mathsf{Rad}_S$, $I \mapsto \sqrt{IS}$, preserves finite infima and suprema, hence is a morphism of locales $\mathsf{Rad}_S \to \mathsf{Rad}_R$ (note the direction).

Definition 4.2.3 (Prime Spectrum). Let R be a ring. The *prime spectrum* $\mathrm{Prim}(R)$ is the topological space with

- underlying set: the set of prime ideals of R;
- open subsets: $\mathrm{Prim}(I) := \{\mathfrak{p} \mid I \not\subset \mathfrak{p}\}$ for $I \subset R$.

Since prime ideals are radical, $\mathrm{Prim}(I) = \mathrm{Prim}(\sqrt{I})$ depends only on the radical.

Proposition 4.2.4 (The Locale of a Ring). *Let R be a ring.*

(i) There is a homeomorphism

$$\mathrm{Prim}(R) \xrightarrow{\sim} \mathrm{Pt}(\mathsf{Rad}_R), \qquad \mathfrak{p} \mapsto \{I \in \mathsf{Rad}_R \mid I \not\subset \mathfrak{p}\},$$

under which $\mathrm{Pt}(I)$ corresponds to $\mathrm{Prim}(I)$.

(ii) The locale Rad_R is spatial: the map in (1) induces $\mathrm{Open}(\mathrm{Prim}(R)) \xrightarrow{\sim} \mathsf{Rad}_R$.

(iii) $\mathrm{Prim}(R)$ is a spectral space*: it is sober, and its quasi-compact open subsets form a basis closed under finite intersections.*

***Remark* 4.2.5** (Converse to Hochster). Conversely, every spectral space is homeomorphic to $\mathrm{Prim}(R)$ for some ring R (the R is far from unique; see Theorem 4.0.1).

Now let A be an $\mathbb{N}$-graded ring, and write hRad_A for the poset of *saturated radical homogeneous* ideals of A. By the projective Nullstellensatz, $\mathsf{hRad}_A \simeq \mathsf{Open}(\mathrm{Proj}(A))$. This poset is again a locale: distributivity inherits from Rad_A combined with the fact that saturation preserves finite intersections.

Definition 4.2.6 (Homogeneous Prime Spectrum). Let A be an $\mathbb{N}$-graded ring. The *homogeneous prime spectrum* $\mathrm{hPrim}(A)$ is the topological space with

- underlying set: the saturated homogeneous prime ideals of A (equivalently, the homogeneous primes not containing A_+);
- open subsets: $\mathrm{hPrim}(I) := \{\mathfrak{p} \mid I \not\subset \mathfrak{p}\}$ for $I \subset A$ homogeneous.

$\mathrm{hPrim}(I)$ depends only on $\sqrt{(I)}^{\mathrm{sat}}$.

Proposition 4.2.7 (The Locale of an $\mathbb{N}$-Graded Ring). *Let A be an $\mathbb{N}$-graded ring.*

(i) There is a homeomorphism $\mathrm{hPrim}(A) \xrightarrow{\sim} \mathrm{Pt}(\mathsf{hRad}_A)$, under which $\mathrm{Pt}(I)$ corresponds to $\mathrm{hPrim}(I)$.

(ii) hRad_A is spatial: $\mathrm{Open}(\mathrm{hPrim}(A)) \xrightarrow{\sim} \mathsf{hRad}_A$.

(iii) $\mathrm{hPrim}(A)$ is sober with a basis of quasi-compact open subsets closed under binary intersections. If A is finitely generated over A_0, $\mathrm{hPrim}(A)$ is spectral.

4.3. Geometric Realization of Algebraic Functors

We now turn to a general algebraic functor X and the poset $\mathsf{Open}(X)$ of its open subfunctors. The classification theorem (Proposition 3.2.1) gives equivalences

$$\begin{array}{ccc}
\mathsf{Rad}_X & \xrightarrow{\sim} & \mathsf{Open}(X) \\
{\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\
\lim\limits_{\mathrm{El}(X)^{\mathrm{op}}} \mathsf{Rad}_R & \longrightarrow & \lim\limits_{\mathrm{El}(X)^{\mathrm{op}}} \mathsf{Open}(\mathrm{Spec}(R)).
\end{array}$$

The limit is taken in $\mathsf{Pos} \subset \mathsf{Cat}$. Each Rad_R is a locale, and the embedding $\mathsf{Loc} \hookrightarrow \mathsf{Pos}^{\mathrm{op}}$ preserves colimits — so this Pos-limit is a colimit in Loc. Concretely, the left Kan extension of Open along $\mathsf{Aff} \hookrightarrow \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ stays inside Loc:

$$\begin{array}{ccc}
\mathsf{Aff} & \xrightarrow{\ \mathsf{Open}\ } & \\
\downarrow & & \searrow \\
\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) & \xrightarrow[\mathsf{Open}]{} & \mathsf{Loc}
\end{array}$$

Since left Kan extension preserves colimits, the resulting functor $\mathsf{Open}\colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Loc}$ preserves colimits.

Thus, we immediately obtain:

Corollary 4.3.1 (Open Subfunctors Form Spatial Locales). *For any algebraic functor X, $\mathsf{Open}(X)$ is a spatial locale.*

Proof. Each $\mathsf{Rad}_R \simeq \mathrm{Open}(\mathrm{Prim}(R))$ is spatial (Theorem 4.0.1), and the colimit-preserving extension above sends X to a colimit of spatial locales. The inclusion $\mathsf{Loc}^{\mathrm{spa}} \hookrightarrow \mathsf{Loc}$ is a left adjoint (its right adjoint is the spatialization $\mathrm{Open} \circ \mathrm{Pt}$), hence preserves colimits — so this colimit lies in $\mathsf{Loc}^{\mathrm{spa}}$, i.e. $\mathsf{Open}(X)$ is spatial. □

***Remark* 4.3.2** (Properties of the Open Set Locale). For an algebraic functor X, $\mathsf{Open}(X)$ as a locale has the following properties:

- *Terminal object:* $D(1) = X$ (the entire functor).
- *Initial object:* $D(0) = \varnothing_X$, where $\varnothing_X(0) = X(0)$ (a singleton over the zero ring) and $\varnothing_X(R) = \varnothing$ for $R \neq 0$.
- *Meet operation:* For open subfunctors $U, V \in \mathsf{Open}(X)$, the locale meet is the functorial intersection: $U \wedge V = U \cap V$.

Our goal is to find a topological space $|X|$ with $\mathrm{Open}(|X|) \simeq \mathsf{Open}(X)$ — i.e., to fill in the dashed functor $|-|$ below, compatibly with the locale-theoretic data:

$$\begin{array}{ccc}
\mathsf{Aff} & \xrightarrow{\ \mathrm{Prim}\ } & \\
{\scriptstyle\mathrm{Spec}}\big\downarrow & & \searrow \\
\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) & \overset{|-|}{\dashrightarrow} & \mathsf{Top} \\
& \searrow{\scriptstyle\mathsf{Open}} & \downarrow{\scriptstyle\mathrm{Open}} \\
& & \mathsf{Loc}^{\mathrm{spa}}
\end{array}$$

Since Top is cocomplete and $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$ is generated by representables, the universal property identifies $|-|\colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Top}$ as the *left Kan extension* of Prim along Spec.

One checks that this construction satisfies

$$\mathsf{Open}(X) \simeq \mathrm{Open}(|X|),$$

and the soberization of $|X|$ recovers the point space of $\mathsf{Open}(X)$:

$$|X|^{\mathrm{sob}} \simeq \mathrm{Pt}(\mathsf{Open}(X)).$$

We now describe $|X|$ explicitly. Recall first that a category C is *connected* if any two objects $X, Y \in C$ can be joined by a zigzag

$$X = i_0 \leftarrow i_1 \rightarrow i_2 \leftarrow \cdots \rightarrow i_{2n} = Y.$$

Proposition 4.3.3 (Explicit Description of Geometric Realization). *Let* $\mathsf{Field}_X \subset \mathrm{El}(X)^{\mathrm{op}}$ *be the full subcategory spanned by pairs* (k, x)*, where* k *is a field and* $x\colon \mathrm{Spec}(k) \to X$ *is a* k*-point of* X*. Then the map* $(k, x) \mapsto (|x|\colon |\mathrm{Spec}(k)| \to |X|)$ *defines a bijection:*

$$\{\textit{connected components of } \mathsf{Field}_X\} \xrightarrow{\sim} |X|.$$

Proof. First, consider the case where X is an affine scheme. Let $X \simeq \mathrm{Spec}(R)$. We know that:

$$\mathsf{Field}_R \simeq \coprod_{\mathfrak{p} \in \mathrm{Prim}(R)} \mathsf{Field}_{\kappa(\mathfrak{p})}.$$

Hence we get a bijection

$$\{\text{connected components of } \mathsf{Field}_R\} \xrightarrow{\sim} \mathrm{Prim}(R),$$
$$(R \to k) \mapsto \ker(R \to k),$$

with inverse $\mathfrak{p} \mapsto (R \to \kappa(\mathfrak{p}))$.

For a general algebraic functor $X \simeq \mathrm{colim}_{\mathrm{El}(X)} \mathrm{Spec}(R)$, we have $|X| \simeq \mathrm{colim}_{\mathrm{El}(X)} \mathrm{Prim}(R)$, so it remains to exhibit a bijection

$$\mathrm{colim}_{(R,x) \in \mathrm{El}(X)} \{\text{connected components of } \mathsf{Field}_R\} \xrightarrow{\sim} \{\text{connected components of } \mathsf{Field}_X\}.$$

Surjectivity: For any object (k, x) in Field_X, i.e., $x\colon \mathrm{Spec}(k) \to X$, since X is a colimit of affine schemes, this map must factor as $\mathrm{Spec}(k) \to \mathrm{Spec}(R) \xrightarrow{x'} X$. This means the connected component comes from some Field_R, so the map is surjective.

Injectivity: We prove that elements identified in the left-side colimit also lie in the same connected component on the right side. This reduces to checking compatibility of morphisms. Consider a morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ in $\mathrm{El}(X)$. This induces a functor $\mathsf{Field}_S \to \mathsf{Field}_R$ (covariant, since Field_X is a subcategory of $\mathrm{El}(X)^{\mathrm{op}}$).

If two points (k_1, x_1) and (k_2, x_2) in Field_X belong to the same connected component, this means there exists a series of morphisms between them. The most crucial step is handling the case of spans: Suppose in CAlg we have a diagram $k_1 \leftarrow k_0 \rightarrow k_2$ (corresponding to $x_1 \rightarrow x_0 \leftarrow x_2$ in Field_X). Consider the pushout (tensor product):

$$\begin{array}{ccc} k_0 & \longrightarrow & k_1 \\ \downarrow & \ulcorner & \downarrow \\ k_2 & \longrightarrow & k_1 \otimes_{k_0} k_2 \end{array}$$

Although $A = k_1 \otimes_{k_0} k_2$ may not be a field, since k_i are all fields, this tensor product is non-zero. Therefore, there exists a prime ideal $\mathfrak{p}$ of A and a residue field $K = \kappa(\mathfrak{p})$ such that we have a composite map $A \to K$. This induces the following commutative diagram:

$$\begin{array}{ccc} k_0 & \longrightarrow & k_1 \\ \downarrow & & \downarrow \\ k_2 & \longrightarrow & K \end{array}$$

In the category Field_X, this means there exists an object (K, x_K) and morphisms $(K, x_K) \to (k_1, x_1)$ and $(K, x_K) \to (k_2, x_2)$. This shows that x_1 and x_2 are in the same connected component via x_K.

This construction shows that: any gluing between $\mathrm{Prim}(R)$ given by diagram relations in $\mathrm{El}(X)$ corresponds to connectivity of objects in Field_X. Specifically, the equivalence relation in $\mathrm{colim}_{\mathrm{El}(X)}$ is precisely generated by such zigzag paths. Since tensor products of fields always have non-trivial residue fields, Field_X has enough morphisms to realize these equivalence relations. Therefore, the map is injective. □

***Remark* 4.3.4** (Two Meanings of "Point"). In algebraic geometry, "point" has two different meanings:

- For any ring R, an "R-point" is an element of $X(R)$. We know it gives the element category $\mathrm{El}(X)$.
- A point of X generally refers to a point of $|X|$, i.e., an equivalence class of field-valued points of X, which gives the topological space $|X|$.

Therefore, we obtain the following relationship:

$$\mathrm{El}(X)^{\mathrm{op}} \hookleftarrow \mathsf{Field}_X \twoheadrightarrow |X|.$$

The functor $|-|$ does not preserve limits in general — for example, the pullback

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{C} \times \mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{R}) \end{array}$$

But after applying $|-|$, we get the diagram:

$$\begin{array}{ccc} S^0 & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

is plainly not a pullback. We now characterize when realization preserves pullbacks.

Lemma 4.3.5 (Pullback Preservation). *Consider a diagram in* $\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})$*:*

$$Y \to X \leftarrow Z.$$

Then there is a canonical map:

$$|Y \times_X Z| \to |Y| \times_{|X|} |Z|.$$

This map is surjective, and is a bijection when $Y \to X$ *is a monomorphism.*

Proof. Surjectivity: For $y\colon \operatorname{Spec}(k) \to Y$ and $z\colon \operatorname{Spec}(k') \to Z$ such that they map to the same point in $|X|$, i.e., given $(y,z) \in |Y| \times_{|X|} |Z|$, we need to find $\operatorname{Spec}(k'') \to Y \times_X Z$ corresponding to it. We know that y and z mapping to the same point in $|X|$ is equivalent to there existing a field k'' such that the following diagram commutes:

$$\begin{array}{ccccc} & & \operatorname{Spec}(k) & \longrightarrow & Y & & \\ & \nearrow & & & & \searrow & \\ \operatorname{Spec}(k'') & & & & & & X \\ & \searrow & & & & \nearrow & \\ & & \operatorname{Spec}(k') & \longrightarrow & Z & & \end{array}$$

This is equivalent to saying there is a canonical morphism:

$$\operatorname{Spec}(k'') \to Y \times_X Z.$$

Therefore, the map is surjective.

Bijection when $Y \hookrightarrow X$ is a monomorphism. Pullbacks of monomorphisms are monomorphisms, so $Y \times_X Z \hookrightarrow Z$ is a monomorphism. The commutative diagram

$$\begin{array}{ccc} |Y \times_X Z| & \hookrightarrow & |Z| \\ & \searrow \quad \nearrow & \\ & |Y| \times_{|X|} |Z| & \end{array}$$

reduces the claim to showing $|Y| \times_{|X|} |Z| \to |Z|$ is a monomorphism, equivalently that $|Y| \hookrightarrow |X|$ is injective — and this follows from $Y \hookrightarrow X$ being a monomorphism (then $|Y| \xrightarrow{\sim} |Y| \times_{|X|} |Y|$). ☐

Proposition 4.3.6 (Geometric Realization of Immersions)**.** *Let X be an algebraic functor, and $Y \hookrightarrow X$ an open immersion/closed immersion/immersion. Then it induces an open immersion/closed immersion/immersion $|Y| \hookrightarrow |X|$.*

Proof. We treat the closed case (open and locally-closed are analogous). For a closed immersion $Y \to X$ and any R-point $x\colon \operatorname{Spec}(R) \to X$, the pullback

$$\begin{CD} Y_R @>>> \operatorname{Spec}(R) \\ @VVV @VVV \\ Y @>>> X \end{CD}$$

satisfies $Y_R \simeq \operatorname{Spec}(R/I)$ for some $I \subset R$, hence $|Y_R| \simeq \operatorname{Prim}(R/I) \hookrightarrow \operatorname{Prim}(R)$. By Lemma 4.3.5 (applied to the monomorphism $Y \hookrightarrow X$), the geometric realization gives a pullback square in Set:

$$\begin{CD} |Y_R| @>>> \operatorname{Prim}(R) \\ @VVV @VVV \\ |Y| @>>> |X| \end{CD}$$

It remains to verify that $|Y| \hookrightarrow |X|$ is closed. A closed $Z \subset |Y|$ pulls back to a closed $Z_R \subset |Y_R| \subset \operatorname{Prim}(R)$ for each R. Since $|X|$ has the colimit topology, $Z \subset |X|$ is closed iff each $Z_R \subset \operatorname{Prim}(R)$ is closed, which we have just verified. ☐

***Remark* 4.3.7** (Closed Subfunctors vs Closed Subspaces). Proposition 4.3.6 yields a map of posets

$$\mathsf{Closed}(X) \to \mathrm{Closed}(|X|), \quad Z \mapsto |Z|.$$

This map is rarely a bijection. It is, however, surjective when $X = \mathrm{Spec}(A)$ or $X = \mathrm{Proj}(A)$: any closed subset of $|X|$ is the realization of the vanishing locus of the radical ideal cutting out its open complement.

In general, if $Z \subset X$ is a closed subfunctor with open complement U, then $|Z| = |X| \setminus |U|$.

We can transport topological properties from spaces to algebraic functors via geometric realization.

Definition 4.3.8 (Topological Properties of Algebraic Functors). Let P be a property of topological spaces (such as Hausdorff, connected, locally connected, irreducible, discrete). An algebraic functor X is said to have property P if its underlying space $|X|$ does.

Example 4.3.9 (Properties of Affine Schemes). $\mathrm{Spec}(R)$ is connected iff $\mathrm{Prim}(R)$ is connected iff R has no nontrivial idempotents. It is irreducible iff $\mathrm{Nil}(R)$ is prime, and quasi-compact iff $1 \in R$ can be written as a finite sum (always), hence *always*.

Definition 4.3.10 (Topological Properties of Morphisms). A morphism of algebraic functors $f : Y \to X$ has a topological property (such as surjective, open, closed, homeomorphism onto image) if the realisation $|f|: |Y| \to |X|$ has that property.

Proposition 4.3.11 (Universal Properties). *Let $f : Y \to X$ be a morphism of algebraic functors.*

(i) f is a monomorphism iff $|f|: |Y| \to |X|$ is injective on field-valued points in a compatible way.

(ii) f is an epimorphism iff $|f|: |Y| \to |X|$ is surjective on field-valued points.

(iii) f is an isomorphism iff it is simultaneously a mono- and epimorphism.

In particular, the functor $|-|: \mathrm{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Top}$ detects these universal properties.

Example 4.3.12 (Non-universal Properties). Not every property transports perfectly between X and $|X|$. For instance, X can be reduced while $|X|$ has the same underlying space as a non-reduced scheme, and vice versa; the functor $|-|$ collapses schemes to their underlying reduced spaces topologically.

Corollary 4.3.13 (Zariski Codescent). *The functor $|-|: \mathrm{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Top}$ preserves colimits along Zariski-covering diagrams. Concretely, if $(U_i \subset X)_{i \in I}$ is a Zariski open cover, then*

$$|X| = \bigcup_{i \in I} |U_i|,$$

with the topology glued from the $|U_i|$.

4.3.1. Open Coverings

Proposition 4.3.14 (Characterisation of Open Covers). *Let X be an algebraic functor and $(U_i \hookrightarrow X)_{i \in I}$ a family of open subfunctors. The following are equivalent:*

(i) $\bigvee_{i \in I} U_i = X$ in $\mathsf{Open}(X)$;

(ii) $|X| = \bigcup_{i \in I} |U_i|$;

(iii) for every R-point $x : \mathrm{Spec}(R) \to X$, the pullback $(U_i \times_X \mathrm{Spec}(R))_{i \in I}$ is a Zariski cover of $\mathrm{Spec}(R)$;

(iv) for every field-valued point $\mathrm{Spec}(k) \to X$, it factors through some U_i.

Definition 4.3.15 (Open Cover). A family of open subfunctors $(U_i \hookrightarrow X)_{i \in I}$ satisfying the equivalent conditions of Proposition 4.3.14 is called an *open cover* of X.

Looking Ahead

We have constructed the underlying space functor $|-|\colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{Top}$, shown that it is a left Kan extension of Prim, and proved that $\mathsf{Open}(X) \simeq \mathrm{Open}(|X|)$ as locales — so every algebraic functor has a genuine topological space attached to it. Morphisms, immersions, open covers, and topological properties (connectedness, irreducibility) all transfer cleanly between the functorial and the topological pictures.

Locales and topological spaces remain, however, *set-valued* gadgets. To talk about covering data at the level of sheaves — where one really does local-to-global analysis with sections and descent — we need the full generality of Grothendieck topologies. The next chapter introduces sieves as the categorical analogue of open covers, defines Grothendieck topologies and sheaves on arbitrary sites, constructs the sheafification functor, and sets up the language in which "scheme" will eventually mean "Zariski sheaf on Aff with an affine open cover".

Chapter 5.

Sheaves

This chapter aims to establish the theory of *Grothendieck topologies* and *sheaves* on them.

For any category C, the presheaf category $\mathsf{PShv}(C)$ is too coarse to capture the "local-to-global" properties of geometric objects. The right notion is to single out those presheaves with a *descent* property — these are τ*-sheaves*, where τ is a Grothendieck topology on C encoding the local data.

In algebraic geometry, we are primarily concerned with the following scenarios:

(i) *The case of topological spaces:* Let $C = \mathrm{Open}(T)$ be the category of open sets (poset) of a topological space T. Endow it with the *canonical Grothendieck topology*. The resulting sheaves are precisely the classical sheaves on T. This concept was originally introduced by Leray.

(ii) *The case of affine schemes:* Let $C = \mathsf{Aff}_k \simeq \mathsf{CAlg}_k^{\mathrm{op}}$. The main topologies we use include:

- *Zariski topology* (i.e., satisfying Zariski descent);
- *Étale topology*;
- *Nisnevich topology*;
- *fppf topology* (fidèlement plat de présentation finie — faithfully flat of finite presentation);
- *fpqc topology* (fidèlement plat quasi-compact — faithfully flat quasi-compact).

(iii) *Extension and generalization:* We can naturally extend the above topologies from Aff_k to the category of k-schemes Sch_k, and even further generalize them to the functor category $\mathsf{Fun}(\mathsf{CAlg}_k, \mathsf{Set})$ (i.e., the presheaf category).

5.1. Introduction: Descent and Sheaves

We sketch the main ideas before proceeding to the formal definitions.

We know that colimits in categories generally represent "gluing". For example, in the category of topological spaces Top, we use colimits to glue spaces together.

Consider a category C, its presheaf category $\mathsf{PShv}(C) = \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{Set})$, and a presheaf $F\colon C^{\mathrm{op}} \to \mathsf{Set}$. We ask whether F preserves the gluing information of C: given a diagram $X\colon \mathcal{I} \to C$, how does $F(\mathrm{colim}_{i\in\mathcal{I}} X_i)$ compare with $\lim_{i\in\mathcal{I}} F(X_i)$? (Note that colimits in C become limits when viewed in C^{op}.)

More precisely, we want the canonical map

$$\mathrm{Hom}_{\mathsf{PShv}(C)}(\mathrm{colim}_{i\in\mathcal{I}}\, y(X_i), F) \to \mathrm{Hom}_{\mathsf{PShv}(C)}(y(\mathrm{colim}_{i\in\mathcal{I}}\, X_i), F)$$

to be an isomorphism. From F's perspective, this asks whether the family $(X_i)_i$ glues into $\mathrm{colim}_i\, X_i$. This isn't true in general, but it is the defining condition for those F we are interested in: we say such an F *satisfies descent with respect to* X.

In the category of topological spaces, this local-to-global property is entirely controlled by the concept of open covers. Specifically, for a topological space X, we do not need to concern ourselves with the complex gluing data inside it; we only need to focus on a family of open embeddings $\{U_i \to X\}_{i\in I}$ whose union equals X. This family of maps is called an *open cover* of X. A section s of a sheaf on X is then determined by its restrictions $s|_{U_i}$, provided these agree on the overlaps $U_i \cap U_j$.

To do local-to-global analysis on a general category C, we generalize the geometric notion of open cover. In C open subsets get replaced by arbitrary morphisms $U \to X$, and a cover is then a family of morphisms with target X.

To treat such families of morphisms rigorously — and in particular to allow refinements and base changes — we introduce *sieves*, which play the role in category theory that open sets play in topology. This is the starting point of Grothendieck topologies.

5.2. Grothendieck Topology

A Grothendieck topology on a category C is a coherent rule for saying "a family of morphisms $(U_i \to X)$ covers X". Using sieves — subfunctors of representable presheaves — we make this rule functorial, descriptive, and compatible with base change.

5.2.1. Sieves and Descent

Definition 5.2.1 (Sieves and Pullbacks). Let C be a category and $X \in C$ an object.

(i) *Sieve:* A *sieve* on X is a full subcategory $\mathcal{R} \subset C_{/X}$ of the slice category satisfying: for any morphism $h\colon V' \to V$ in $C_{/X}$ (i.e., a morphism making the diagram commute), if $V \in \mathcal{R}$, then $V' \in \mathcal{R}$. (In other words, sieves are families of morphisms closed under precomposition.)

(ii) *Pullback:* Let $f : X' \to X$ be a morphism in C, and $\mathcal{R}$ a sieve on X. The *pullback* $f^*\mathcal{R}$ of $\mathcal{R}$ along f is a sieve on X' defined by:

$$f^*\mathcal{R} := \{(g\colon Y \to X') \in C_{/X'} \mid (f \circ g) \in \mathcal{R}\}.$$

***Remark* 5.2.2** (Right Fibration Composition and Subfunctor Viewpoint). Note that *right fibrations are closed under composition*.

We can view a sieve $\mathcal{R}$ on X as the composite of the following maps:

$$\mathcal{R} \hookrightarrow C_{/X} \to C.$$

Where:

(i) The inclusion $\mathcal{R} \hookrightarrow C_{/X}$ is a right fibration. (This follows directly from the definition of a sieve: a sieve is a full subcategory closed under precomposition, which is precisely the right fibration property for full subcategory inclusions.)

(ii) The forgetful functor (domain projection) $C_{/X} \to C$ is also a right fibration (this is a standard property of slice categories).

Therefore, the composite functor $\mathcal{R} \to C$ is still a *right fibration*.

By straightening, there is an equivalence between right fibrations over C and its presheaf category:

$$\mathrm{St}^R\colon \mathsf{RFib}(C) \xrightarrow{\sim} \mathsf{PShv}(C) = \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{An}).$$

Under this correspondence:

- The right fibration $C_{/X} \to C$ corresponds to the representable functor $y(X)$.
- The right fibration $\mathcal{R} \to C$ corresponds to a *subfunctor* of $y(X)$.

  Furthermore, if we consider the straightening of the inclusion $\mathcal{R} \hookrightarrow C_{/X}$ itself, we get a functor taking values in (-1)-truncated objects:

  $$\chi_{\mathcal{R}} : (C_{/X})^{\mathrm{op}} \to \mathsf{An}_{\leq -1} \simeq [1].$$

  Here we view $[1]$ as the full subcategory consisting of $\{\varnothing, [0]\}$ (i.e., Boolean values in the homotopy sense: false and true). In this viewpoint:

  – Objects belonging to the sieve $\mathcal{R}$ are mapped to $[0]$ (true/one-point space);

  – Objects not belonging to the sieve $\mathcal{R}$ are mapped to $\varnothing$ (false/empty space).

  Functoriality ensures that: if $V \to X$ is mapped to $[0]$ (i.e., in the sieve), then any factorization $V' \to V \to X$ must also be mapped to $[0]$ (i.e., also in the sieve).

This is the higher categorical interpretation of why we usually define sieves as subfunctors of $y(X)$ (i.e., $\mathcal{R}(V) \subset \mathrm{Hom}(V, X)$).

It is not hard to see that pullbacks of sieves along morphisms are precisely pullbacks in the presheaf category $\mathsf{PShv}(C)$. The viewpoint of $\mathcal{R} \subset y(X)$ as a subfunctor is sometimes more useful.

We now describe sieves generated by morphisms.

Definition 5.2.3 (Generated Sieve). Let C be a category and $X \in C$ an object. The sieve *generated* by a family of morphisms $\{f_i : Y_i \to X\}_{i \in I}$ is defined as the image of the natural transformation:

$$\coprod_{i \in I} y(Y_i) \to y(X).$$

Concretely, this sieve $\mathcal{R} \subset C_{/X}$ contains precisely those morphisms that can be factored through some f_i. That is:

$$\mathcal{R} = \{(g : V \to X) \in C_{/X} \mid \exists i \in I, \exists h : V \to Y_i \text{ such that } g = f_i \circ h\}.$$

We now introduce the descent condition for a sieve.

Definition 5.2.4 (Descent Along a Sieve). Let C be a category, $X \in C$ an object, and $\mathcal{R} \subset C_{/X}$ a sieve on X. Let $F : C^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf.

(i) *Descent data:* The *descent data* of F with respect to the sieve $\mathcal{R}$ is defined as the following limit:

$$\mathrm{Desc}(\mathcal{R}, F) := \lim_{(V \to X) \in \mathcal{R}^{\mathrm{op}}} F(V) \simeq \mathrm{Nat}(\mathcal{R}, F).$$

(Note: If we view the sieve $\mathcal{R}$ as a subfunctor of $y(X)$, we can understand this limit as the set of natural transformations from $\mathcal{R}$ to F.)

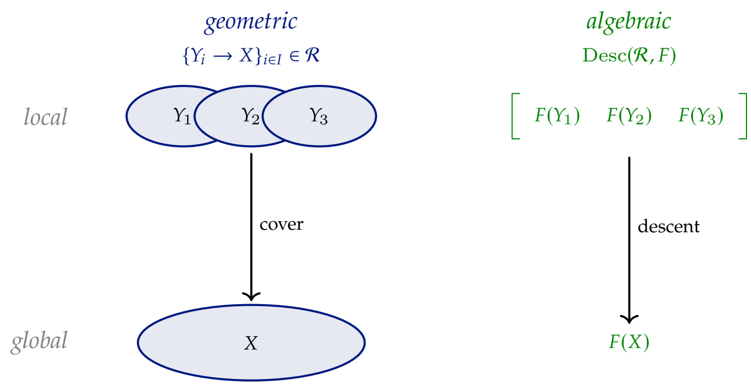
(ii) *Descent condition:* We say F *satisfies descent along* $\mathcal{R}$ if the canonical map:

$$F(X) \xrightarrow{\sim} \mathrm{Desc}(\mathcal{R}, F)$$

is an isomorphism.

(iii) *Universal descent:* We say F *universally descends* along $\mathcal{R}$ if for any morphism $f : Y \to X$, F satisfies descent along the pullback sieve $f^*\mathcal{R}$.

***Remark* 5.2.5** (Intuitive Picture). Intuitively, we can explain "descent" through the following correspondence diagram:



- On the right is the *geometric* picture: a family of "pieces" $\{Y_i\}$ (belonging to the sieve $\mathcal{R}$) covers X.
- On the left is the *algebraic* picture: $\mathrm{Desc}(\mathcal{R}, F)$ contains all local sections on the pieces (satisfying compatibility).

Descent asks: given a compatible family of local data (top row), can it be glued uniquely into a global section (bottom row)? When the canonical map is an isomorphism, the local data fully determines the global.

We list several examples of sieves.

Example 5.2.6 (Empty Sieve). For any $X \in \mathcal{C}$, the empty subfunctor $\varnothing \subset y(X)$ is a sieve. A presheaf F satisfies descent along $\varnothing$ iff $F(X)$ is a singleton.

Example 5.2.7 (Sieve Generated by Two Subobjects). Let $X \in \mathcal{C}$ and $U, V \subset X$ be subobjects. Let $\mathcal{R}$ be the sieve on X generated by U and V. If $U \cap V$ exists, then $\mathcal{R}$ is actually the pushout in $\mathsf{PShv}(\mathcal{C})$:

$$\begin{array}{ccc} y(U \cap V) & \longrightarrow & y(U) \\ \downarrow & & \downarrow \\ y(V) & \longrightarrow & \mathcal{R} \end{array}$$

Therefore, a presheaf F on $\mathcal{C}$ satisfying descent along $\mathcal{R}$ is equivalent to the following diagram being a pullback:

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

In a 1-category, the descent condition can be reinterpreted as an equalizer diagram.

Lemma 5.2.8 (Sieve as Coequalizer). *Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. Let $\mathcal{R}$ be the sieve generated by a family $(Y_i \to X)_{i \in I}$. Then the canonical maps $y(Y_i) \to \mathcal{R}$ induce an isomorphism in the presheaf category:*

$$\operatorname{colim}\left(\coprod_{i,j \in I} y(Y_i) \times_{y(X)} y(Y_j) \rightrightarrows \coprod_{i \in I} y(Y_i) \right) \xrightarrow{\sim} \mathcal{R}.$$

Proof. By Definition 5.2.3, the sieve $\mathcal{R}$ is the image of:

$$\coprod_{i\in I} y(Y_i) \to y(X).$$

The *coimage* of a morphism $f : A \to B$ is the coequalizer

$$\mathrm{coim}(f) := \mathrm{colim}\,(A \times_B A \rightrightarrows A)\,.$$

In $\mathsf{PShv}(C)$ (which inherits image factorizations pointwise from Set), every morphism has $\mathrm{im}(f) \simeq \mathrm{coim}(f)$. Substituting $A = \coprod_i y(Y_i)$ and $B = y(X)$ yields the claim. ◻

***Remark* 5.2.9** (Higher Categorical Perspective: Image and Čech Nerve)**.** In higher categories (∞-categories), we directly use the definition of image from Kerodon Tag 04X1.

Let $p\colon \coprod_{i\in I} y(Y_i) \to y(X)$ be the canonical morphism. The sieve $\mathcal{R}$ generated by $\{Y_i \to X\}$ is the image $\mathrm{im}(p)$. In this case, $\mathrm{im}(p)$ is identified with the *geometric realization* of the *Čech nerve* $\check{C}(p)_\bullet$ of p:

$$\mathcal{R} \simeq \mathrm{im}(p) \simeq |\check{C}(p)_\bullet| := \mathrm{colim}_{\Delta^{\mathrm{op}}}\left(\cdots \rightrightarrows U \times_{y(X)} U \times_{y(X)} U \rightrightarrows U \times_{y(X)} U \to U\right).$$

Where $U = \coprod_{i\in I} y(Y_i)$. Since $\mathsf{Fun}(C, \mathsf{An})$ is a topos, colimits in it are universal, i.e., coproducts commute with fiber products. Therefore, we recover the form of Lemma 5.2.8.

Alternatively, one can verify cofinality using Quillen's Theorem A.

Applying Lemma 5.2.8 to descent yields the following.

Proposition 5.2.10 (Equivalent Forms of Descent)**.** *Let C be a category, $\mathcal{R}$ a sieve on $X \in C$, and $(Y_i \to X)_{i\in I}$ a family of morphisms generating $\mathcal{R}$. Then for any presheaf $F \in \mathsf{PShv}(C)$, the following conditions are equivalent:*

(i) F satisfies descent along $\mathcal{R}$.

(ii) The canonical morphism:

$$F(X) \to \lim_{Y\in \mathrm{El}(\mathcal{R})} F(Y)$$

is an isomorphism.

(iii) There is an equalizer diagram:

$$F(X) \longrightarrow \textstyle\prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} F(Y_i \times_X Y_j)$$

Proof. (i) ⇔ (ii): By definition, F satisfies descent along $\mathcal{R}$ if $F(X) = \mathrm{Hom}(y(X), F) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{R}, F)$. Since $\mathcal{R} = \mathrm{colim}_{Y\in\mathrm{El}(\mathcal{R})}\, y(Y)$ (the canonical Yoneda density colimit), we have

$$\mathrm{Hom}(\mathcal{R}, F) = \lim_{Y\in\mathrm{El}(\mathcal{R})^{\mathrm{op}}} \mathrm{Hom}(y(Y), F) = \lim_{Y\in\mathrm{El}(\mathcal{R})^{\mathrm{op}}} F(Y),$$

the second equality by Yoneda. This is exactly the limit in (ii) (the $(-)^{\mathrm{op}}$ on the indexing category is the standard convention for limits over slice categories).

(ii) ⇔ (iii): By Lemma 5.2.8, $\mathcal{R}$ is the coequalizer

$$\mathrm{colim}\left(\coprod_{i,j\in I} y(Y_i) \times_{y(X)} y(Y_j) \rightrightarrows \coprod_{i\in I} y(Y_i)\right).$$

Applying $\mathrm{Hom}(-, F)$ converts this to the equalizer

$$F(X) \to \prod_{i \in I} F(Y_i) \rightrightarrows \prod_{i,j \in I} \mathrm{Hom}(y(Y_i) \times_{y(X)} y(Y_j), F),$$

which by Yoneda computes to $\prod_{i,j} F(Y_i \times_X Y_j)$ when the displayed pullback is representable (otherwise the same equalizer is taken with the presheaf-level pullback). □

Example 5.2.11 (Monogenic Sieve). Let C be a category and $\mathcal{R}$ a sieve on $X \in C$. We say $\mathcal{R}$ is *monogenic* if it is generated by a single morphism $Y \to X$. Assuming the pullback $Y \times_X Y$ exists in C, a presheaf F satisfies descent along $\mathcal{R}$ if and only if

$$F(X) \longrightarrow F(Y) \rightrightarrows F(Y \times_X Y)$$

is an equalizer diagram.

***Remark* 5.2.12** (Descent for Categories). Descent extends to category-valued presheaves $F : C^{\mathrm{op}} \to \mathsf{Cat}$. Since Cat is a $(2, 1)$-category, Remark 5.2.9 tells us that the relevant data is:

$$F(X) \longrightarrow \prod_{i \in I} F(Y_i) \rightrightarrows \prod_{i,j \in I} F(Y_i \times_X Y_j) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i,j,k \in I} F(Y_i \times_X Y_j \times_X Y_k)$$

5.2.2. Grothendieck Topologies

To capture the structure of open covers axiomatically, we now distinguish a class of sieves with three closure properties.

Definition 5.2.13 (Grothendieck Topology). Let C be a category. A *Grothendieck topology* on C (or just a *topology*, for short) is a choice, for each $X \in C$, of a family of sieves $\tau(X)$ on X — the *τ-covering sieves* — satisfying:

(i) *(Pullback stability)* If $\mathcal{R}$ is a τ-covering sieve on X and $f : Y \to X$ is any morphism, then the pullback sieve $f^*\mathcal{R}$ is also a τ-covering sieve on Y.

(ii) *(Local character)* Let $\mathcal{S}$ be a sieve on X and $\mathcal{R}$ a τ-covering sieve on X. If for every $f \in \mathcal{R}$, we have $f^*\mathcal{S}$ is a τ-covering sieve, then $\mathcal{S}$ is a τ-covering sieve.

(iii) *(Maximality)* For any $X \in C$, $y(X)$ is a τ-covering sieve.

We call a category C equipped with a Grothendieck topology a *site*.

***Remark* 5.2.14** (Additional Properties). Two further closure properties follow from the three axioms above:

(4) *(Refinement)* If $\mathcal{R}$ is a τ-covering sieve on X and $\mathcal{S}$ is a sieve containing $\mathcal{R}$, then $\mathcal{S}$ is also a τ-covering sieve.

(5) *(Finite intersections)* Any finite intersection of τ-covering sieves on X is still a τ-covering sieve.

We can now define sheaves on a site.

Definition 5.2.15 (Sheaves). Let (C, τ) be a site. A presheaf F is called a *τ-sheaf* if for any τ-covering sieve $\mathcal{R}$, F satisfies descent along $\mathcal{R}$. We denote by:

$$\mathsf{Shv}_\tau(C) \subset \mathsf{PShv}(C)$$

the full subcategory of τ-sheaves.

***Remark* 5.2.16** (Poset of Topologies). Given two topologies τ, ρ on C, we say τ is *coarser* than ρ (or ρ *finer* than τ), written $\tau \leq \rho$, if every τ-covering sieve is also ρ-covering. The topologies on C form a poset under $\leq$.

In particular, C admits a coarsest and a finest topology.

Example 5.2.17 (Extreme Topologies). For any category C, consider the following topologies:

(i) The finest topology on C is the *discrete topology*: all sieves are covering sieves. The terminal object $* \in \mathsf{PShv}(C)$ is a sheaf only in the discrete topology.

(ii) The coarsest topology on C is the *indiscrete topology* (or *trivial topology*): only maximal sieves are covering sieves. In this case, all presheaves are sheaves.

Example 5.2.18 (Canonical Topology). Let O be a locale, viewed as a category. Define a topology by declaring a sieve on $u \in O$ to be a covering sieve iff its supremum is u. The pullback-stability and local-character axioms follow from the distributive law (note that pullback in O is meet, $\wedge$); maximality is automatic. This is the *canonical topology* on O.

For a general topological space T, we denote by $\mathsf{Shv}(T)$ the sheaf category $\mathsf{Shv}_{\mathrm{can}}(\mathrm{Open}(T))$, i.e., the category of sheaves on $\mathrm{Open}(T)$ with respect to the canonical topology. We also simply call this the *sheaf category* on T. The resulting sheaves are precisely the classical sheaves on topological spaces.

Example 5.2.19 (κ-Weiss Topology). For a topological space $T \in \mathsf{Top}$ and a cardinal κ, we can consider the *κ-Weiss topology* on the category of open sets $\mathrm{Open}(T)$.

In this topology, a sieve $\mathcal{R}$ on an open set U is a covering sieve if and only if every subset $S \subset U$ of cardinality less than κ (i.e., $|S| < \kappa$) is *contained* in some member of $\mathcal{R}$. Formally:

$$\forall S \subset U,\ \text{if } |S| < \kappa,\ \text{then } \exists V \in \mathcal{R} \text{ such that } S \subseteq V.$$

(Note: When $\kappa = \aleph_0$, this is the standard *Weiss topology*, which requires that covers can capture all finite point configurations.) Additionally:

- When $\kappa = 0$, we get the discrete topology.
- When $\kappa = 1$, we get the topology consisting of non-empty sieves. The corresponding sheaf category is the category of constant sheaves.
- When $\kappa = 2$, we get the canonical topology.
- When $\kappa \geq |T|$, we get the trivial topology.

***Remark* 5.2.20** (Motivation for Grothendieck Topologies). Grothendieck topologies are not a direct generalization of point-set topology — Example 5.2.18 shows the relation is more subtle. Historically, Grothendieck introduced the notion entirely from the sheaf-theoretic side: classical sheaves on a topological space depend only on the lattice of open sets and their covers, not on the underlying points.

Grothendieck's insight was: since sheaves only care about "covers", we can abandon the underlying space and directly axiomatize the concept of "covering" on any category.

The next proposition is the cornerstone of the section.

Proposition 5.2.21 (Adjunction Between Topologies and Sheaf Categories). *Let C be a category. Then the map between posets:*

$$\{\text{topologies on } C\}^{\mathrm{op}} \to \{\text{subcategories of } \mathsf{PShv}(C)\}$$
$$\tau \mapsto \mathsf{Shv}_\tau(C)$$

has a left adjoint $\mathcal{E} \mapsto \tau_{\mathcal{E}}$. That is, for any subcategory $\mathcal{E}$ of the presheaf category, there exists a finest topology *$\tau_{\mathcal{E}}$ making $\mathcal{E}$ into sheaves.*

Concretely, a sieve $\mathcal{R}$ on X is $\tau_{\mathcal{E}}$-covering iff every $F \in \mathcal{E}$ universally descends along $\mathcal{R}$ — equivalently, iff $F(X) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{R}, F)$ holds for all such F.

Proof. It suffices to check that $\tau_{\mathcal{E}}$ is a Grothendieck topology. Conditions (1) and (3) of Definition 5.2.13 are immediate; the content lies in condition (2).

Let $\mathcal{S}, \mathcal{R} \subset y(X)$ be two sieves on X. Suppose $\mathcal{R}$ is a $\tau_{\mathcal{E}}$-covering sieve, and for every $(f : Y \to X) \in \mathcal{R}$, the pullback $f^*\mathcal{S}$ is a $\tau_{\mathcal{E}}$-covering sieve. We prove that $\mathcal{S}$ is also a $\tau_{\mathcal{E}}$-covering sieve.

This means for any $F \in \mathcal{E}$, we prove the isomorphism:

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{PShv}(C)}(\mathcal{S}, F).$$

We establish this isomorphism through the intermediate term $\mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$ in three steps:

(i) *Using $\mathcal{R}$ as covering sieve:* Since $\mathcal{R}$ is a covering sieve and $F \in \mathcal{E}$, we have an isomorphism:

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{PShv}(C)}(\mathcal{R}, F).$$

(ii) *Using condition (2):* Since $\mathcal{R}$ is a presheaf, we can write $\mathcal{R} = \mathrm{colim}_{Y \in \mathrm{El}(\mathcal{R})}\, y(Y)$. Since $\mathrm{Hom}(-, F)$ takes colimits to limits, the restriction map from $\mathcal{R}$ to $\mathcal{R} \cap \mathcal{S}$

$$\mathrm{Hom}(\mathcal{R}, F) \to \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F).$$

Note that $\mathcal{R} \cap \mathcal{S} = \mathcal{R} \times_{y(X)} \mathcal{S} \simeq \mathrm{colim}_{Y \in \mathcal{R}}(y(Y) \times_{y(X)} \mathcal{S})$. The above map can be expanded as a limit:

$$\lim_{Y \in \mathrm{El}(\mathcal{R})^{\mathrm{op}}} \Big(F(Y) \to \mathrm{Hom}_{\mathsf{PShv}(C)}(y(Y) \times_{y(X)} \mathcal{S}, F) \Big).$$

For each $Y \in \mathcal{R}$ (i.e., morphism $f : Y \to X$ in $\mathcal{R}$), the term $y(Y) \times_{y(X)} \mathcal{S}$ is precisely the pullback sieve $f^*\mathcal{S}$. By assumption, $f^*\mathcal{S}$ is a covering sieve, so $F(Y) \xrightarrow{\sim} \mathrm{Hom}(f^*\mathcal{S}, F)$ is an isomorphism. Since limits preserve isomorphisms, we get $\mathrm{Hom}(\mathcal{R}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$.

(iii) *Establishing $\mathrm{Hom}(\mathcal{S}, F) \simeq \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$:* Finally, we use the colimit decomposition of $\mathcal{S}$: $\mathcal{S} \simeq \mathrm{colim}_{Z \in \mathrm{El}(\mathcal{S})}\, y(Z)$. Similarly, the map $\mathrm{Hom}(\mathcal{S}, F) \to \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$ can be written as:

$$\lim_{Z \in \mathrm{El}(\mathcal{S})^{\mathrm{op}}} \Big(F(Z) \to \mathrm{Hom}_{\mathsf{PShv}(C)}(\mathcal{R} \times_{y(X)} y(Z), F) \Big).$$

For $Z \in \mathcal{S}$ (i.e., $g : Z \to X$ in $\mathcal{S}$), the term $\mathcal{R} \times_{y(X)} y(Z)$ is precisely the pullback sieve $g^*\mathcal{R}$. Since $\mathcal{R}$ is a covering sieve, $g^*\mathcal{R}$ is also a covering sieve on Z. This means $F(Z) \to \mathrm{Hom}(g^*\mathcal{R}, F)$ is an isomorphism. Again, since limits preserve isomorphisms, we get $\mathrm{Hom}(\mathcal{S}, F) \simeq \mathrm{Hom}(\mathcal{R} \cap \mathcal{S}, F)$.

Combining the three steps gives $F(X) \simeq \mathrm{Hom}(\mathcal{S}, F)$. □

Corollary 5.2.22 (Supremum of Topologies). *Let C be a category and $(\tau_i)_{i\in I}$ a family of topologies on C. Let $\tau = \bigvee_{i\in I} \tau_i$ be their supremum. Then:*

$$\mathsf{Shv}_\tau(C) = \bigcap_{i\in I} \mathsf{Shv}_{\tau_i}(C).$$

Proof. Simply take $\mathcal{E} = \{F \in \mathsf{PShv}(C) \mid F \text{ descends along } \tau_i \text{ for each } i \in I\}$. □

There is a simpler way to specify topologies, with two caveats:

- It requires the existence of corresponding pullbacks in C.
- The relationship with topologies is not injective, meaning there may be multiple pretopologies determining the same topology.

Definition 5.2.23 (Pretopology). Let C be a category. A *pretopology* π on C consists of the following data: for each object X in C, we specify a family of objects in $C_{/X}$, called *π-coverings*, satisfying:

(i) *(Pullback stability)* For a π-covering $(U_i \to X)_{i\in I}$ and any morphism $f : Y \to X$, the pullbacks $U_i \times_X Y$ exist and $(U_i \times_X Y \to Y)_{i\in I}$ is a π-covering.

(ii) *(Transitivity)* For a π-covering $(U_i \to X)_{i\in I}$ and π-coverings $(V_{ij} \to U_i)_{j\in J_i}$, the family $(V_{ij} \to X)_{i\in I, j\in J_i}$ is a π-covering.

(iii) *(Identity)* $(\mathrm{id}_X : X \to X)$ is a π-covering.

The *topology generated by the pretopology* π is the coarsest topology containing all sieves generated by π-coverings.

Remark **5.2.24** (From Topology to Pretopology). Let τ be a topology on C. The τ-coverings produce a pretopology on C only when C has pullbacks.

Proposition 5.2.25 (Pretopology and Descent). *Let C be a category, π a pretopology on C, and τ the topology generated by π. Then:*

(i) A sieve is a τ-covering sieve if and only if it contains at least one π-covering.

(ii) A presheaf F is a τ-sheaf if and only if for each π-covering $(U_i \to X)_{i\in I}$, there is an equalizer diagram:

$$F(X) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_X U_j)$$

Proof. This follows immediately from the fact that τ is the coarsest topology containing all covering sieves generated by π-coverings. □

5.2.3. Examples

We now turn to concrete examples.

Example 5.2.26 (Standard Topology on Top). On the category of topological spaces Top, open covers form a pretopology. Let τ be the corresponding topology. Then:

(i) A sieve on X is a τ-covering sieve if and only if it contains an open cover of X. Therefore, $(Y_i \to X)_{i\in I}$ is a τ-covering if and only if it locally admits sections.

(ii) A presheaf $F : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ is a τ-sheaf if and only if for each topological space T, the restriction $F|_{\mathrm{Open}(T)}$ is a sheaf on T.

We now describe the most commonly used topologies in algebraic geometry. (We omit the Nisnevich, étale, and smooth topologies, which require étale and smooth morphisms not yet introduced.)

Example 5.2.27 (Zariski Topology). On $\mathsf{Aff} \simeq \mathsf{CAlg}^{\mathrm{op}}$, declare a family $\{\mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R)\}_{i\in I}$ to be a covering whenever $(f_i)_{i\in I}$ generates the unit ideal of R. These families form a pretopology, and the topology it generates is the *Zariski topology*.

We have the following characterizations:

(i) *Covering sieve criterion:* A sieve $\mathcal{S}$ on $\mathrm{Spec}(R)$ is a Zariski-covering sieve iff the set

$$\{f \in R \mid (\mathrm{Spec}(R_f) \to \mathrm{Spec}(R)) \in \mathcal{S}\}$$

generates the unit ideal of R.

(ii) *Sheaf condition:* An algebraic functor $X\colon \mathsf{CAlg} \to \mathsf{Set}$ is a Zariski sheaf if and only if for any ring R and elements $(f_i)_{i\in I}$ generating the unit ideal, there is an equalizer diagram:

$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j}).$$

Example 5.2.28 (fpqc and fppf Topologies). The Zariski topology is a special case of a more general construction. Fix a class E of ring maps containing all isomorphisms and closed under composition and pushouts. The pretopology on $\mathsf{CAlg}^{\mathrm{op}}$ whose coverings are families $(R \to R_i)_{i\in I}$ satisfying:

- The index set I is finite;
- The induced map $\coprod_{i\in I} \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ is surjective;
- Each map $R \to R_i$ belongs to the class E.

Depending on the choice of E, we obtain different topologies:

(i) If E is the class of localizations $R \to R_f$, this recovers the Zariski topology.

(ii) If E is the class of flat morphisms, the corresponding topology is called the *fpqc topology*.

(iii) If E is the class of finitely presented flat morphisms, the corresponding topology is called the *fppf topology*.

Since localizations $R \to R_f$ are flat and finitely presented, the topologies are ordered

$$\text{Zariski} \le \text{fppf} \le \text{fpqc}.$$

In fact, we have

$$\text{Zariski} \le \text{Nisnevich} \le \text{Étale} \le \text{Smooth} \le \text{fppf} \le \text{fpqc}$$

5.3. Sheafification

Every presheaf has a universal sheaf approximation, its *sheafification*. We construct this functor $(-)^\sharp\colon \mathsf{PShv}(C) \to \mathsf{Shv}_\tau(C)$ via a small-object-argument style plus construction and verify its universal property — the left adjoint to the inclusion of sheaves.

Theorem 5.3.1 (Sheafification). *Let C be a small category and τ be a topology on C. Then the inclusion functor $\mathsf{Shv}_\tau(C) \hookrightarrow \mathsf{PShv}(C)$ admits a left adjoint:*

$$a_\tau\colon \mathsf{PShv}(C) \to \mathsf{Shv}_\tau(C),$$

called τ-sheafification. *Moreover:*

(i) the functor a_τ is left exact (i.e., it preserves finite limits);

(ii) a sieve $\mathcal{R} \subset y(X)$ is a τ-covering sieve if and only if $a_\tau(\mathcal{R}) \to a_\tau(y(X))$ is an isomorphism.

Proof. Rather than reproduce the full construction, we sketch the idea and refer to [The25, Tag 00ZG] for details.

The *plus construction* $F \mapsto F^+$ sends a presheaf F to the presheaf

$$F^+(X) := \operatorname{colim}_{\mathcal{R} \in \tau(X)} \operatorname{Hom}_{\mathsf{PShv}}(\mathcal{R}, F),$$

where the colimit is over all τ-covering sieves on X. Informally: $F^+(X)$ consists of "sections of F compatible along some cover of X". The plus construction is *not* a sheaf in general; applying it twice is required for F^{++} to satisfy descent (this iteration is the "a_τ" — it is a small-object argument in disguise). Left exactness follows because filtered colimits commute with finite limits, and (ii) is essentially the definition of a covering sieve. □

An immediate corollary:

Corollary 5.3.2. *Let $(\mathcal{C}, \tau)$ be a small site. Then $\mathsf{Shv}_\tau(\mathcal{C})$ admits limits and colimits:*

- *Limits are computed as in $\mathsf{PShv}(\mathcal{C})$.*
- *Colimits are computed as in $\mathsf{PShv}(\mathcal{C})$ and then sheafified.*

Combining Theorem 5.3.1 with Proposition 5.2.21 yields an adjunction between topologies and categories of sheaves.

Corollary 5.3.3. *Let $\mathcal{C}$ be a small category. Then the map between posets*

$$\{\textit{topologies on } \mathcal{C}\}^{\mathrm{op}} \to \{\textit{full subcategories of } \mathsf{PShv}(\mathcal{C}) \textit{ admitting left exact left adjoint}\}$$
$$\tau \mapsto \mathsf{Shv}_\tau(\mathcal{C})$$

is an isomorphism.

Proof. Injectivity follows from Theorem 5.3.1(ii). For surjectivity, let $\mathcal{E}$ be a full subcategory of $\mathsf{PShv}(\mathcal{C})$ such that the inclusion admits a left exact left adjoint $L\colon \mathsf{PShv}(\mathcal{C}) \to \mathcal{E}$. We must show that $\mathcal{E}$ is the category of sheaves for some topology τ. Let $\tau = \{\mathcal{R} \subset y(X) \mid L(\mathcal{R}) \xrightarrow{\sim} L(y(X))\}$. We verify that τ is a topology and $\mathcal{E} = \mathsf{Shv}_\tau(\mathcal{C})$.

(i) Since L is left exact, it preserves finite limits. Therefore, for any morphism $f : Y \to X$ and sieve $\mathcal{R} \subset y(X)$, we have $L(f^*\mathcal{R}) \simeq L(y(Y) \times_{y(X)} \mathcal{R}) \simeq L(y(Y)) \times_{L(y(X))} L(\mathcal{R})$. This implies that if $\mathcal{R}$ is a τ-covering sieve, then $f^*\mathcal{R}$ is also a τ-covering sieve. Thus, condition (i) in Definition 5.2.13 is satisfied.

(ii) Now, let $\mathcal{S}$ be a sieve on X and $\mathcal{R}$ be a τ-covering sieve on X. If for every $(f : Y \to X) \in \mathcal{R}$, we have $f^*\mathcal{S}$ is a τ-covering sieve, then we have $L(f^*\mathcal{S}) \simeq L(y(Y)) \times_{L(y(X))} L(\mathcal{S}) \simeq L(y(Y))$, thus we have $L(\mathcal{S}) \simeq L(y(X))$.

(iii) Direct from the definition of τ.

Thus τ is a topology and $\mathcal{E} \subseteq \mathsf{Shv}_\tau(\mathcal{C})$, fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \underset{\hookrightarrow}{\overset{L}{\longleftarrow}} & \\ \big\downarrow & & \mathsf{PShv}(\mathcal{C}) \\ \mathsf{Shv}(\mathcal{C}) & \underset{\hookrightarrow}{\overset{a_\tau}{\longleftarrow}} & \end{array}$$

Thus, it suffices to show that for all $f\colon G \to F$ in $\mathsf{PShv}(\mathcal{C})$, if $L(f)$ is an isomorphism, then $a_\tau(f)$ is an isomorphism (which implies that for every $F \in \mathsf{Shv}_\tau(\mathcal{C})$, $F \to L(F)$ is an a_τ-isomorphism).

- If $f : \mathcal{R} \hookrightarrow y(X)$ is the inclusion of a sieve, then $L(f)$ is an isomorphism iff $\mathcal{R}$ is a τ-covering sieve, and hence $a_\tau(f)$ is an isomorphism.
- Suppose f is a monomorphism. Writing F as the canonical Yoneda colimit $F \simeq \operatorname{colim}_{c \in \mathrm{El}(F)} y(c)$, each $y(c) \to F$ produces the pullback square

$$\begin{array}{ccc} G & \overset{f}{\hookrightarrow} & F \\ \uparrow & \lrcorner & \uparrow \\ \mathcal{R} & \hookrightarrow & y(c) \end{array}$$

  in which $\mathcal{R} \subset y(c)$ is a sieve (since f is mono). If $L(f)$ is an isomorphism, then $\mathcal{R}$ is a τ-covering sieve, so by Theorem 5.3.1(ii), $a_\tau(\mathcal{R}) \xrightarrow{\sim} a_\tau(y(c))$. Since f can be written as the colimit of such $\mathcal{R} \hookrightarrow y(c)$ and a_τ is a left adjoint functor (preserving colimits), we have $a_\tau(f)$ is an isomorphism.
- Now, for an arbitrary $f : G \to F$, we can consider the epimorphism-monomorphism factorization of f, i.e.,

$$G \xrightarrow{f_{\mathrm{epi}}} \mathrm{im}(f) \xhookrightarrow{f_{\mathrm{mono}}} F.$$

  If $L(f)$ is an isomorphism, then $L(f_{\mathrm{mono}})$ is an isomorphism (as $L(f)$ factors through it, and L preserves monos, so $L(f_{\mathrm{mono}})$ is both mono and split epi). Since f_{mono} is a monomorphism, by the previous case, $a_\tau(f_{\mathrm{mono}})$ is an isomorphism.

  It remains to show $a_\tau(f_{\mathrm{epi}})$ is an isomorphism. Consider the kernel pair of f_{epi}:

$$\begin{array}{ccc} G \times_{\mathrm{im}(f)} G & \xrightarrow{\pi_2} & G \\ \downarrow{\scriptstyle \pi_1} & \lrcorner & \downarrow{\scriptstyle f_{\mathrm{epi}}} \\ G & \xrightarrow{f_{\mathrm{epi}}} & \mathrm{im}(f) \end{array}$$

  The diagonal $\Delta\colon G \to G \times_{\mathrm{im}(f)} G$ is a monomorphism. Since $L(f_{\mathrm{epi}})$ is an isomorphism (as $L(f)$ and $L(f_{\mathrm{mono}})$ are isos), applying the left exact functor L shows that $L(\Delta)$ is an isomorphism. Since Δ is a monomorphism, by the previous case, $a_\tau(\Delta)$ is an isomorphism.

  Since a_τ is left exact, $a_\tau(\Delta)$ is the diagonal of the kernel pair of $a_\tau(f_{\mathrm{epi}})$. The fact that the diagonal is an isomorphism implies $a_\tau(f_{\mathrm{epi}})$ is a monomorphism. Additionally, a_τ is a left adjoint and thus preserves epimorphisms; hence $a_\tau(f_{\mathrm{epi}})$ is an epimorphism. In a topos (like $\mathsf{Shv}_\tau(\mathcal{C})$), a morphism that is both a monomorphism and an epimorphism is an isomorphism.

  Therefore, $a_\tau(f) = a_\tau(f_{\mathrm{mono}}) \circ a_\tau(f_{\mathrm{epi}})$ is an isomorphism.

This confirms that the essential image of L coincides with $\mathsf{Shv}_\tau(\mathcal{C})$. □

***Remark* 5.3.4** (Non-topological localizations). The isomorphism established in Corollary 5.3.3 relies essentially on the fact that objects in a 1-category are 0-truncated, which implies that for any morphism f, the diagonal Δ_f is a monomorphism.

In the context of ∞-topoi, the situation is different. As pointed out by Marc (see MO 273085), there are many natural examples of ∞-topoi that arise as left exact localizations but have *a priori* nothing to do with Grothendieck topologies. Examples include:

- The ∞-category of n-excisive functors (in the sense of Goodwillie calculus) with values in an ∞-topos.
- The ∞-category of coalgebras for a left exact comonad in an ∞-topos.

While these ∞-topoi might happen to be equivalent to sheaf ∞-categories "by accident", their defining localizations involve inverting morphisms that are not monomorphisms (e.g., hypercovers), a feature that is invisible in the 1-categorical setting.

We can now describe sheaf-theoretic properties of morphisms in terms of local properties at the presheaf level.

Definition 5.3.5 (Local epimorphism, monomorphism, isomorphism). Let (C, τ) be a site and let $f : F \to G$ be a morphism in $\mathrm{PShv}(C)$. Let $\Delta_f : F \to F \times_G F$ denote the diagonal morphism.

(i) f is called a *τ-local epimorphism* if for every $X \in C$ and every section $s \in G(X)$, the sieve on X consisting of all morphisms $u : Y \to X$ such that $u^*(s)$ lies in the image of the map $f_Y : F(Y) \to G(Y)$ is a τ-covering sieve.

(ii) f is called a *τ-local monomorphism* if the diagonal Δ_f is a τ-local epimorphism.

(iii) f is called a *τ-local isomorphism* if it is both a τ-local epimorphism and a τ-local monomorphism.

Proposition 5.3.6 (Stability of τ-local Morphisms under Base Change). *Let $f : F \to G$ be a morphism of presheaves, and let $h : H \to G$ be any morphism. Consider the base change $f' : F \times_G H \to H$ fitting into the cartesian diagram:*

$$\begin{array}{ccc} F \times_G H & \longrightarrow & F \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ H & \xrightarrow{h} & G \end{array}$$

If f is a τ-local epimorphism (resp. τ-local monomorphism, τ-local isomorphism), then f' is a τ-local epimorphism (resp. τ-local monomorphism, τ-local isomorphism).

Proof. (i) **Stability of τ-local epimorphisms.** Let $f : F \to G$ be a τ-local epimorphism. We must show that $f' : F \times_G H \to H$ is a τ-local epimorphism. Let $X \in C$ and let $t \in H(X)$. We consider the sieve S' on X generated by f' over t:

$$S' = \{u : Y \to X \mid u^*(t) \in \mathrm{im}(f'_Y)\}.$$

We must show S' is a τ-covering sieve. Let $s = h_X(t) \in G(X)$. Since f is a τ-local epimorphism, the sieve

$$S = \{u : Y \to X \mid u^*(s) \in \mathrm{im}(f_Y)\}$$

is a τ-covering sieve on X. We claim that $S \subseteq S'$.

Indeed, let $u : Y \to X$ be in S. Then there exists $a \in F(Y)$ such that $f_Y(a) = u^*(s)$. Since $s = h_X(t)$, we have:

$$f_Y(a) = u^*(h_X(t)) = h_Y(u^*(t)).$$

By the universal property of the fiber product, the pair $(a, u^*(t))$ determines a unique section $b \in (F \times_G H)(Y)$ such that $f'_Y(b) = u^*(t)$. Consequently, $u^*(t) \in \mathrm{im}(f'_Y)$, so $u \in S'$.

Since S is a τ-covering sieve and $S \subseteq S'$, S' is also a τ-covering sieve. Thus f' is a τ-local epimorphism.

(ii) **Stability of τ-local monomorphisms.** Let $f : F \to G$ be a τ-local monomorphism. By definition, this means the diagonal $\Delta_f : F \to F \times_G F$ is a τ-local epimorphism. We must show that $\Delta_{f'} : F \times_G H \to (F \times_G H) \times_H (F \times_G H)$ is a τ-local epimorphism.

There is a canonical isomorphism of presheaves:

$$(F \times_G H) \times_H (F \times_G H) \cong (F \times_G F) \times_G H.$$

Under this identification, $\Delta_{f'}$ corresponds to the morphism $\Delta_f \times_G \mathrm{id}_H$. This fits into the following diagram, which is a pullback square in the category of presheaves:

$$\begin{array}{ccc} F \times_G H & \longrightarrow & F \\ {\scriptstyle \Delta_{f'}} \downarrow & \lrcorner & \downarrow {\scriptstyle \Delta_f} \\ (F \times_G F) \times_G H & \longrightarrow & F \times_G F \end{array}$$

Since Δ_f is a τ-local epimorphism, and we established in part (i) that τ-local epimorphisms are stable under base change, it follows that $\Delta_{f'}$ is a τ-local epimorphism. Therefore, f' is a τ-local monomorphism.

(iii) **Stability of τ-local isomorphisms.** If f is a τ-local isomorphism, it is both a τ-local epimorphism and a τ-local monomorphism. By parts (i) and (ii), the base change f' retains both properties. Hence, f' is a τ-local isomorphism.

□

Proposition 5.3.7. *Let (C, τ) be a small site and let f be a morphism in $\mathsf{PShv}(C)$. Then $a_\tau(f)$ is an epimorphism (resp. monomorphism, isomorphism) in $\mathsf{Shv}_\tau(C)$ if and only if f is a τ-local epimorphism (resp. monomorphism, isomorphism).*

Proof. We treat the three cases in turn; throughout, write $i\colon \mathsf{Shv}_\tau(C) \hookrightarrow \mathsf{PShv}(C)$ for the inclusion, with adjunction $a_\tau \dashv i$.

Epimorphisms. $a_\tau(f)$ is an epimorphism in $\mathsf{Shv}_\tau(C)$ iff for every sheaf G, the map $\mathrm{Hom}_{\mathsf{Shv}}(a_\tau(F'), G) \to \mathrm{Hom}_{\mathsf{Shv}}(a_\tau(F), G)$ is injective. By $a_\tau \dashv i$, this is equivalent to: $\mathrm{Hom}_{\mathsf{PShv}}(F', i(G)) \to \mathrm{Hom}_{\mathsf{PShv}}(F, i(G))$ is injective for every sheaf G. Concretely, given two maps $g_1, g_2\colon F' \to i(G)$ agreeing after restricting along f, we must have $g_1 = g_2$. By Definition 5.3.5(i), f is a τ-local epimorphism precisely when, for every $X \in C$ and every $s \in F'(X)$, the sieve of $u\colon Y \to X$ with $u^*(s) \in \mathrm{im}(f_Y)$ is τ-covering. Combining: g_1 and g_2 agree on this covering sieve at s, so by the sheaf condition for G they agree at s. Conversely, if f is not a τ-local epimorphism, there exists X and $s \in F'(X)$ for which the lifting sieve $\mathcal{R}$ is not τ-covering; then for the sheaf $G = a_\tau(y(X)/\mathcal{R})$ one constructs two maps $F' \to G$ that agree along f but differ at s.

Monomorphisms. A morphism h is a monomorphism iff its diagonal Δ_h is an isomorphism. Sheafification preserves finite limits (Theorem 5.3.1), so $\Delta_{a_\tau(f)} \simeq a_\tau(\Delta_f)$. Hence $a_\tau(f)$ is a monomorphism iff $a_\tau(\Delta_f)$ is an isomorphism, equivalently a sheaf-theoretic epimorphism (since Δ_f is automatically a monomorphism). By the epimorphism case, this is equivalent to Δ_f being a τ-local epimorphism, i.e., to f being a τ-local monomorphism (Definition 5.3.5(ii)).

Isomorphisms. An iso is both a mono and an epi; combining the two cases above yields the claim. □

5.4. Pushforward and Pullback of Sheaves

Let $f: X \to Y$ be a continuous map of topological spaces. The induced map of locales corresponds to the order-preserving map $f^{-1}\colon \operatorname{Open}(Y) \to \operatorname{Open}(X)$ sending U to $f^{-1}(U)$, which preserves open coverings. It follows that the pushforward functor

$$f_*^{\mathrm{pre}}\colon \mathsf{PShv}(\operatorname{Open}(X)) \to \mathsf{PShv}(\operatorname{Open}(Y)), \quad f_*^{\mathrm{pre}}(F)(V) = F(f^{-1}(V)),$$

preserves sheaves, so it lifts to a functor between categories of sheaves:

$$\begin{array}{ccc} \mathsf{PShv}(\operatorname{Open}(X)) & \xrightarrow{f_*^{\mathrm{pre}}} & \mathsf{PShv}(\operatorname{Open}(Y)) \\ \uparrow & & \downarrow a \\ \mathsf{Shv}(X) & \overset{f_*}{\dashrightarrow} & \mathsf{Shv}(Y). \end{array}$$

Proposition 5.4.1 (Pullbacks of Sheaves). *Let $f: X \to Y$ be a continuous map between topological spaces. Then the pushforward functor $f_*\colon \mathsf{Shv}(X) \to \mathsf{Shv}(Y)$ admits a left exact left adjoint $f^*\colon \mathsf{Shv}(Y) \to \mathsf{Shv}(X)$.*

Proof. First, we define the pullback functor in the context of presheaves. Let G be a presheaf on Y. We define its presheaf pullback $f^*_{\mathrm{pre}}G$ via the left Kan extension along the functor $f^{-1}\colon \operatorname{Open}(Y)^{\mathrm{op}} \to \operatorname{Open}(X)^{\mathrm{op}}$ induced by f:

$$\begin{array}{ccc} \operatorname{Open}(Y)^{\mathrm{op}} & \xrightarrow{G} & \mathsf{Set} \\ {\scriptstyle f^{-1}}\downarrow & \nearrow_{f^*_{\mathrm{pre}}G = \operatorname{Lan}_{f^{-1}} G} & \\ \operatorname{Open}(X)^{\mathrm{op}} & & \end{array}$$

By the pointwise formula for left Kan extensions, we obtain an explicit description:

$$(f^*_{\mathrm{pre}}G)(U) = \operatorname{colim}_{U \subset f^{-1}(V)} G(V).$$

The indexing category $\{V \in \operatorname{Open}(Y) \mid U \subset f^{-1}(V)\}$ is filtered (since the intersection of two open sets containing $f(U)$ is again an open set containing $f(U)$). Since filtered colimits in Set commute with finite limits, the functor $f^*_{\mathrm{pre}}\colon \mathsf{PShv}(Y) \to \mathsf{PShv}(X)$ is left exact. By the universal property of left Kan extensions, f^*_{pre} is the left adjoint to the presheaf pushforward.

Now, consider the diagram of adjoint pairs relating presheaves and sheaves:

$$\begin{array}{ccc} \mathsf{PShv}(X) & \xleftarrow{f^*_{\mathrm{pre}}} & \mathsf{PShv}(Y) \\ a \downarrow \uparrow i & & a \downarrow \uparrow i \\ \mathsf{Shv}(X) & \underset{f_*}{\overset{f^*}{\leftrightarrows}} & \mathsf{Shv}(Y) \end{array}$$

We define the functor f^* as the composition:

$$f^* = a \circ f^*_{\mathrm{pre}} \circ i.$$

Since f^* is a composition of left adjoints (a and f^*_{pre}), it is left adjoint to f_* on sheaves. Explicitly:

$$\operatorname{Hom}_{\mathsf{Shv}(X)}(f^*G, \mathcal{F}) \simeq \operatorname{Hom}_{\mathsf{PShv}(X)}(f^*_{\mathrm{pre}}(iG), i\mathcal{F}) \simeq \operatorname{Hom}_{\mathsf{PShv}(Y)}(iG, f_*(i\mathcal{F})) \simeq \operatorname{Hom}_{\mathsf{Shv}(Y)}(G, f_*\mathcal{F}).$$

Regarding exactness: By Theorem 5.3.1, the sheafification functor a is left exact (preserves finite limits). The inclusion i preserves limits. Since f^*_{pre} is left exact (as shown above via filtered colimits), the composition f^* is left exact. □

***Remark* 5.4.2.** In fact, the construction f^*_{pre} and f^{pre}_* above can be extended into the general definition of a *morphism of sites*.

A morphism of sites $f: (C, \tau) \to (\mathcal{D}, \sigma)$ is defined not by a functor $C \to \mathcal{D}$, but by a functor in the *inverse* direction:

$$f^{-1}: \mathcal{D} \to C,$$

which is *continuous* in the sense that:

(i) f^{-1} is left exact (preserves finite limits);

(ii) f^{-1} preserves covering families (it sends sieves in σ to sieves in τ).

Given such a continuous functor, we obtain an adjunction on the categories of presheaves and sheaves exactly as we did for topological spaces:

- The *direct image* $f_*: \mathsf{PShv}(C) \to \mathsf{PShv}(\mathcal{D})$ is given by pre-composition: $f_*(F) = F \circ f^{-1}$.
- The *inverse image* $f^*: \mathsf{PShv}(\mathcal{D}) \to \mathsf{PShv}(C)$ is the left Kan extension along f^{-1} (followed by sheafification).

In the context of topological spaces, a continuous map $f: X \to Y$ corresponds to the continuous functor $f^{-1}: \operatorname{Open}(Y) \to \operatorname{Open}(X)$, thus defining a morphism of sites.

Stalks are the special case of sheaf pullback along the inclusion of a single point.

Definition 5.4.3 (Stalk). Let $x \in X$ be a point of a topological space. Consider the one-point space $*$ and the inclusion map $i_x: * \hookrightarrow X$ mapping the unique point to x. For any sheaf F on X, the *stalk* of F at x, denoted by F_x, is defined as the pullback of F along i_x:

$$F_x := i_x^* F.$$

Since $\mathsf{Shv}(*) \simeq \mathsf{Set}$, we view the stalk naturally as a set (or an algebraic structure, depending on the target category).

Using the explicit formula for f^* from Proposition 5.4.1, we can compute this pullback explicitly. Since every open neighborhood U of the point in $*$ is just $*$ itself, and $i_x(U) = \{x\}$, the colimit is taken over open sets $V \subset X$ such that $V \supset \{x\}$. Thus we recover the classical definition:

$$F_x = \operatorname{colim}_{V \ni x} F(V).$$

This perspective explains why stalks commute with pullbacks: the identity $(f^*G)_x \simeq G_{f(x)}$ is simply the functoriality of pullbacks for the composition $* \xrightarrow{x} X \xrightarrow{f} Y$.

We can rephrase the criterion of Proposition 5.3.7 in terms of stalks:

Proposition 5.4.4 (Stalkwise characterization). *Let T be a topological space, let $f: F \to G$ be a map in $\mathsf{PShv}(\operatorname{Open}(T))$, and let $a: \mathsf{PShv}(\operatorname{Open}(T)) \to \mathsf{Shv}(T)$ be the sheafification functor. Then $a(f)$ is an epimorphism/monomorphism/isomorphism in $\mathsf{Shv}(T)$ if and only if, for every $x \in T$, the induced map of stalks $f_x: F_x \to G_x$ is surjective/injective/bijective.*

Proof. By Proposition 5.3.7, $a(f)$ is an epimorphism (resp. monomorphism) in $\mathsf{Shv}(T)$ if and only if f is a τ-local epimorphism (resp. monomorphism). We analyze each case using the properties of the standard topology τ on Top.

1. Epimorphisms ($\Rightarrow$) Suppose f is a τ-local epimorphism. Let $x \in T$ and let $y \in G_x$ be an element of the stalk. By description of the stalk, y is represented by a section $s \in G(U)$ for some open neighborhood U of x, such that $s_x = y$. Since f is a τ-local epimorphism, there exists a τ-covering sieve S on U such that for any $V \to U$ in S, the restriction $s|_V$ is in the image of $f_V: F(V) \to G(V)$. In the standard topology, a sieve is a covering sieve if and only if it contains an open cover. Thus, there exists an open cover $\{U_i\}_{i\in I}$ of U such that for each i, there exists $t_i \in F(U_i)$ with $f(t_i) = s|_{U_i}$. Since $\{U_i\}$ covers U, there exists an index j such that $x \in U_j$. Passing to the stalk at x, we have:

$$f_x((t_j)_x) = (f(t_j))_x = (s|_{U_j})_x = s_x = y.$$

Thus, f_x is surjective.

($\Leftarrow$) Suppose f_x is surjective for all $x \in T$. Let U be an open set and $s \in G(U)$. We must show that the sieve of morphisms $V \to U$ such that $s|_V$ lifts to F is a τ-covering sieve. For every $x \in U$, consider the germ $s_x \in G_x$. Since f_x is surjective, there exists $t_x \in F_x$ such that $f_x(t_x) = s_x$. The germ t_x is represented by a section $\tilde{t} \in F(V_x)$ for some open neighborhood $V_x \subseteq U$ of x. The equality $f_x(t_x) = s_x$ implies that $f(\tilde{t})$ and $s|_{V_x}$ have the same germ at x. Therefore, there exists a possibly smaller open neighborhood $W_x \subseteq V_x$ of x such that $f(\tilde{t}|_{W_x}) = s|_{W_x}$. The collection $\{W_x \hookrightarrow U\}_{x\in U}$ generates a sieve on U. Since $\bigcup_{x\in U} W_x = U$, this is a τ-covering sieve. Thus, f is a τ-local epimorphism.

2. Monomorphisms Recall that f is a τ-local monomorphism if and only if the diagonal $\Delta_f : F \to F \times_G F$ is a τ-local epimorphism. The functor taking a presheaf to its stalk at x commutes with finite limits (it is a filtered colimit). Therefore, there is a natural isomorphism $(F\times_G F)_x \simeq F_x \times_{G_x} F_x$. Under this isomorphism, the stalk of the diagonal, $(\Delta_f)_x$, corresponds to the diagonal of the stalks, $\Delta_{f_x} : F_x \to F_x \times_{G_x} F_x$. Applying the result from part 1 to the morphism Δ_f, we see that f is a τ-local monomorphism if and only if Δ_{f_x} is surjective for all $x \in T$. In the category of sets, the diagonal map $A \to A \times_B A$ is surjective if and only if the map $A \to B$ is injective. Therefore, f is a τ-local monomorphism if and only if f_x is injective for all $x \in T$.

3. Isomorphisms $a(f)$ is an isomorphism if and only if it is both a monomorphism and an epimorphism. Combining the results above, this holds if and only if f_x is both injective and surjective (bijective) for all $x \in T$. □

5.5. Locally Ringed Space

The classical definition of a scheme involves a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings whose stalks are local rings. We collect the vocabulary — ringed spaces, locally ringed spaces, their morphisms — that bridges our functorial approach to this classical picture in the next chapter.

Definition 5.5.1 (Ringed Space). Let k be a ring.

- A *k-ringed space* $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf of k-algebras $\mathcal{O}_X$ on X.
- A *morphism of k-ringed spaces* $(f, \varphi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a map of sheaves of k-algebras $\varphi\colon \mathcal{O}_Y \to f_*(\mathcal{O}_X)$.

We denote by $\mathsf{RingSpace}_k$ the category of k-ringed spaces. When $k = \mathbb{Z}$, we write $\mathsf{RingSpace} = \mathsf{RingSpace}_{\mathbb{Z}}$ for the category of *ringed spaces*.

By the $f^* \dashv f_*$ adjunction, the datum $\varphi\colon \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is equivalent to a map $f^*(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of k-algebras.

Definition 5.5.2 (Local Map). A ring map $\varphi\colon A \to B$ is *local* if it detects units, i.e., if $A^\times = \varphi^{-1}(B^\times)$.

***Remark* 5.5.3.** Let A and B be local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. For a ring map $\varphi\colon A \to B$, the following are equivalent:

(i) φ is local;

(ii) $\mathfrak{m}_A = \varphi^{-1}(\mathfrak{m}_B)$.

(iii) $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

We now introduce locally ringed spaces.

Definition 5.5.4 (Locally Ringed Space). Let k be a ring.

- A *locally k-ringed space* is a k-ringed space $(X, \mathcal{O}_X)$ such that, for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
- If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed k-spaces, a *morphism of locally k-ringed spaces* $(f, \varphi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of k-ringed spaces such that, for each $x \in X$, the map $\varphi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local map of local rings (i.e., $\varphi_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$).

We denote by $\mathsf{LocRingSpace}_k$ the category of locally k-ringed spaces, which is a (non-full!) subcategory of $\mathsf{RingSpace}_k$. When $k = \mathbb{Z}$, we write $\mathsf{LocRingSpace} = \mathsf{LocRingSpace}_{\mathbb{Z}}$ for the category of *locally ringed spaces*.

***Remark* 5.5.5** (Another description of Locally Ringed Spaces). Let $(X, \mathcal{O}_X)$ be a ringed space. For any open subset $U \subset X$ and any section $s \in \mathcal{O}_X(U)$, we denote by U_s the locus where the germ of s is invertible:

$$U_s := \{x \in U \mid s_x \in \mathcal{O}_{X,x}^\times\}.$$

Since the property of being a unit is open in the stalk topology, U_s is an open subset of U.

(i) **Objects:** A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if and only if:
   - $X_0 = \varnothing$.
   - For all open subsets $U \subset X$ and sections $s, t \in \mathcal{O}_X(U)$:

   $$U_{s+t} \subseteq U_s \cup U_t.$$

   (This is the geometric equivalent of the local ring axiom: if a sum is a unit, at least one summand must be a unit).

(ii) **Morphisms:** A morphism (f, φ) in $\mathsf{RingSpace}$ is a morphism in $\mathsf{LocRingSpace}$ if and only if for every open set $V \subset Y$ and every section $t \in \mathcal{O}_Y(V)$:

   $$f^{-1}(V_t) = (f^{-1}(V))_{\varphi(t)}.$$

   (Geometrically: the pullback of the unit locus is the unit locus of the pullback section).

Locally ringed spaces unify many flavors of geometric object. We isolate the relevant pattern by naming spaces built from local models.

Notation 5.5.6. Let k be a ring and let $\mathcal{M}$ be a collection (set) of locally k-ringed spaces, which we call *local models*. We denote by $\mathsf{LocRingSpace}(\mathcal{M})$ the full subcategory of $\mathsf{LocRingSpace}_k$ consisting of objects $(X, \mathcal{O}_X)$ with the following property: every point $x \in X$ admits an open neighborhood U such that the restricted locally ringed space $(U, \mathcal{O}_X|_U)$ is isomorphic to an object in $\mathcal{M}$.

Example 5.5.7 (Manifolds as Locally Ringed Spaces). (i) **Real Manifolds (C^r, Smooth, Analytic).** Let $r \in \mathbb{N} \cup \{\infty, \omega\}$. For any open subset $U \subset \mathbb{R}^n$, denote by C_U^r the sheaf of $\mathbb{R}$-valued functions of class C^r on U (continuous for $r = 0$, smooth for $r = \infty$, real-analytic for $r = \omega$).

Let $\mathcal{D}isk_{C^r}$ be the collection of locally $\mathbb{R}$-ringed spaces consisting of open unit disks in Euclidean spaces:

$$\mathcal{D}isk_{C^r} = \{(D^n, C^r_{D^n}) \mid D^n \subset \mathbb{R}^n \text{ is the open unit disk, } n \in \mathbb{N}\}.$$

The category of C^r-manifolds can be identified with the subcategory of locally ringed spaces locally isomorphic to these models. Specifically, we have an equivalence:

$$\{C^r\text{-manifolds}\} \simeq \{X \in \mathsf{LocRingSpace}(\mathcal{D}isk_{C^r}) \mid X \text{ is Hausdorff and second-countable}\}.$$

In particular:

- $r = 0$ yields *topological manifolds;*
- $r = \infty$ yields *smooth manifolds;*
- $r = \omega$ yields *analytic manifolds.*

See [Wed16] for a sheaf-theoretic introduction to the theory of manifolds.

(ii) **Complex Manifolds.** For an open subset $U \subset \mathbb{C}^n$, denote by $\mathcal{H}_U$ (or simply $\mathcal{O}_U$) the sheaf of holomorphic functions on U. Let $\mathcal{D}isk_{\mathbb{C}}$ be the collection of locally $\mathbb{C}$-ringed spaces $(D^n, \mathcal{H}_{D^n})$ where D^n is the open unit disk in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Then there is an equivalence of categories:

$$\{\text{complex manifolds}\} \simeq \{X \in \mathsf{LocRingSpace}(\mathcal{D}isk_{\mathbb{C}}) \mid X \text{ is Hausdorff and second-countable}\}.$$

(iii) **Generalizations.** More generally, many categories of manifolds—such as manifolds with boundary, manifolds with corners—are equivalent to categories of the form $\mathsf{LocRingSpace}_{\mathbb{R}/\mathbb{C}}(\mathcal{M})$ (usually with additional Hausdorff/second-countable constraints).

***Remark* 5.5.8** (Colimits and Limits of Ringed Spaces)**.** The category $\mathsf{RingSpace}_k$ admits both arbitrary colimits and limits. Let $(X_i, \mathcal{O}_{X_i})_{i\in\mathcal{I}}$ be a diagram of k-ringed spaces. Let $X = \operatorname{colim}_{i\in\mathcal{I}} X_i$ and $Y = \lim_{i\in\mathcal{I}} X_i$ be the colimit and limit in Top, with canonical maps $\iota_j\colon X_j \to X$ and $\pi_j\colon Y \to X_j$. Then the colimit and limit in $\mathsf{RingSpace}_k$ are given by:

$$\operatorname{colim}_{i\in\mathcal{I}}(X_i, \mathcal{O}_{X_i}) = \left(X, \lim_{i\in\mathcal{I}} (\iota_i)_*\mathcal{O}_{X_i}\right),$$
$$\lim_{i\in\mathcal{I}}(X_i, \mathcal{O}_{X_i}) = \left(Y, \operatorname{colim}_{i\in\mathcal{I}} (\pi_i)^*\mathcal{O}_{X_i}\right).$$

Note the reversal of limits and colimits for the structure sheaf: the structure sheaf of the colimit space is the limit of the pushforward sheaves, while the structure sheaf of the limit space is the colimit of the pullback sheaves (in the category of sheaves on Y).

We now discuss colimits in $\mathsf{LocRingSpace}_k$.

Proposition 5.5.9. *The category* $\mathsf{LocRingSpace}_k$ *admits colimits, and the inclusion functor* $\mathsf{LocRingSpace}_k \hookrightarrow \mathsf{RingSpace}_k$ *preserves colimits.*

Proof. Since $\mathsf{LocRingSpace}_k$ is a category with all small colimits if it admits coproducts and coequalizers, it suffices to show that the coproducts and coequalizers formed in $\mathsf{RingSpace}_k$ actually lie in $\mathsf{LocRingSpace}_k$ and satisfy the universal property.

(i) **Coproducts.** Let $\{(X_i, \mathcal{O}_{X_i})\}_{i\in I}$ be a family of locally k-ringed spaces. The coproduct in $\mathsf{RingSpace}_k$ is given by

$$(X, \mathcal{O}_X) = \coprod_{i\in I}^{\mathsf{RS}} (X_i, \mathcal{O}_{X_i}) = \left(\coprod_{i\in I} X_i, \prod_{i\in I} \alpha_{i,*}(\mathcal{O}_{X_i})\right),$$

where X is the disjoint union topological space, and $\alpha_i\colon X_i \hookrightarrow X$ are the canonical inclusions. Explicitly, for any open set $U \subset X$, let $U_i = U \cap X_i$. The structure sheaf is:

$$\mathcal{O}_X(U) = \prod_{i\in I} \mathcal{O}_{X_i}(U_i).$$

Stalks: Let $x \in X$. Then $x \in X_j$ for a unique index j. The stalk is:

$$\mathcal{O}_{X,x} \simeq \mathcal{O}_{X_j,x}.$$

Since $(X_j, \mathcal{O}_{X_j})$ is a locally ringed space, $\mathcal{O}_{X_j,x}$ is a local ring. Thus $(X, \mathcal{O}_X) \in \mathsf{LocRingSpace}_k$.

Universal Property: Consider maps $(f_i, \varphi_i)\colon (X_i, \mathcal{O}_{X_i}) \to (Y, \mathcal{O}_Y)$ in $\mathsf{LocRingSpace}_k$. The unique induced map $(f, \varphi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is defined such that its restriction to each component is (f_i, φ_i). Since the map on stalks φ_x is simply $(\varphi_i)_x$ (where $x \in X_i$), and $(\varphi_i)_x$ is a local map, φ_x is local.

(ii) **Coequalizers.** Consider a pair of morphisms in $\mathsf{LocRingSpace}_k$:

$$(Z, \mathcal{O}_Z) \underset{(g,\psi)}{\overset{(f,\varphi)}{\rightrightarrows}} (Y, \mathcal{O}_Y).$$

Let $h\colon Y \to X$ be the coequalizer of f, g in Top (so X has the quotient topology). The structure sheaf $\mathcal{O}_X$ is defined as the equalizer in the category of sheaves of k-algebras on X:

$$\mathcal{O}_X \xrightarrow{\chi} h_*\mathcal{O}_Y \underset{\mathrm{h}_*(\psi)}{\overset{\mathrm{h}_*(\varphi)}{\rightrightarrows}} h_* f_* \mathcal{O}_Z = h_* g_* \mathcal{O}_Z.$$

Explicitly, for an open set $U \subset X$ (letting $V = h^{-1}(U)$ and $W = f^{-1}(V) = g^{-1}(V)$):

$$\mathcal{O}_X(U) = \{s \in \mathcal{O}_Y(V) \mid \varphi_V(s) = \psi_V(s) \text{ in } \mathcal{O}_Z(W)\}.$$

Verification of Locally Ringed Space: We use Remark 5.5.5. Let $r \in \mathcal{O}_X(U)$. By the definition above, r corresponds to a section $s \in \mathcal{O}_Y(V)$ such that $\varphi(s) = \psi(s)$.

Claim 5.5.10. $h^{-1}(U_r) = V_s$.

Proof of Claim. Let $y \in V$ and $x = h(y)$. The stalk $\mathcal{O}_{X,x}$ is the equalizer of the rings $\mathcal{O}_{Y,y}$ (ranging over $y \in h^{-1}(x)$) and $\mathcal{O}_{Z,z}$. Specifically, the induced map on stalks $\chi_y\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is the inclusion of a subring satisfying the equalizer condition. Crucially, if an element in the subring becomes a unit in the larger ring $\mathcal{O}_{Y,y}$, its inverse must also satisfy the equalizer condition (since maps preserve inverses), so the inverse lies in $\mathcal{O}_{X,x}$. Therefore, χ_y is a local map (it reflects units):

$$r_x \in \mathcal{O}_{X,x}^\times \iff s_y = \chi_y(r_x) \in \mathcal{O}_{Y,y}^\times.$$

This translates to $x \in U_r \iff y \in V_s$, i.e., $h^{-1}(U_r) = V_s$. □

Since Y is a locally ringed space, V_s is open. Because X has the quotient topology, $h^{-1}(U_r) = V_s$ being open implies U_r is open in X. Furthermore, for any $r, r' \in \mathcal{O}_X(U)$ corresponding to $s, s' \in \mathcal{O}_Y(V)$:

$$h^{-1}(U_{r+r'}) = V_{s+s'} \subseteq V_s \cup V_{s'} = h^{-1}(U_r) \cup h^{-1}(U_{r'}) = h^{-1}(U_r \cup U_{r'}).$$

Since h is surjective, taking images yields $U_{r+r'} \subseteq U_r \cup U_{r'}$. Also, $X_0 = \varnothing$ follows from $Y_0 = \varnothing$. Thus $(X, \mathcal{O}_X)$ is a locally ringed space.

Universal Property: Let $(k, \lambda)\colon (X, \mathcal{O}_X) \to (T, \mathcal{O}_T)$ be the unique map in $\mathsf{RingSpace}_k$ induced by a compatible map from Y. For any $x \in X$, the map on stalks $\lambda_x\colon \mathcal{O}_{T,k(x)} \to \mathcal{O}_{X,x}$ fits into a commutative triangle with the local map from Y. Since the map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ reflects units (as shown in the Claim), λ_x must be local.

□

The category $\mathsf{LocRingSpace}_k$ also admits limits, but this is more subtle as the inclusion $\mathsf{LocRingSpace}_k \hookrightarrow \mathsf{RingSpace}_k$ does not preserve them.

5.6. The Comparison Lemma

In this section we explain how to compare the sheaf categories of two related sites.

Definition 5.6.1 (Dense Subcategory). Let $(\mathcal{C}, \tau)$ be a site. A *dense subcategory* of $(\mathcal{C}, \tau)$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ such that every $X \in \mathcal{C}$ admits a τ-cover $(U_i \to X)_{i \in I}$ with $U_i \in \mathcal{D}$.

Proposition 5.6.2 (The "Comparison Lemma"). *Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{D} \subset \mathcal{C}$ a dense subcategory. Let $\tau|_{\mathcal{D}}$ be the induced topology on $\mathcal{D}$. Then the restriction functor $u^* \colon \mathsf{Shv}_\tau(\mathcal{C}) \to \mathsf{Shv}_{\tau|_{\mathcal{D}}}(\mathcal{D})$ is an equivalence of categories.*

Proof. Let $u \colon \mathcal{D} \hookrightarrow \mathcal{C}$ be the inclusion functor. The restriction functor on presheaves $u^* \colon \mathsf{PShv}(\mathcal{C}) \to \mathsf{PShv}(\mathcal{D})$ is given by pre-composition with u. We construct its adjoint, the extension functor u_*, using the right Kan extension.

Step 1: Presheaf Extension (Right Kan Extension). Let $G \in \mathsf{PShv}(\mathcal{D})$. We define its extension $u_*^{\text{pre}} G$ via the right Kan extension along u^{op}:

$$\begin{array}{ccc} \mathcal{D}^{\text{op}} & \xrightarrow{\;G\;} & \mathsf{Set} \\ {\scriptstyle u^{\text{op}}}\big\downarrow & \nearrow{\scriptstyle u_*^{\text{pre}} G := \mathrm{Ran}_{u^{\text{op}}} G} & \\ \mathcal{C}^{\text{op}} & & \end{array}$$

By the pointwise formula for right Kan extensions, for any $X \in \mathcal{C}$, we have:

$$(u_*^{\text{pre}} G)(X) = \lim_{D \in \mathcal{D}_{X/}} G(D).$$

Here $\mathcal{D}_{X/}$ denotes the comma category of objects in $\mathcal{D}$ over X (i.e., morphisms $u(D) \to X$). By the universal property, u_*^{pre} is the right adjoint to u^*.

Step 2: Sheaf Adjunction. Consider the adjunction on sheaves:

$$\begin{array}{ccc} \mathsf{PShv}(\mathcal{C}) & \underset{u^*}{\overset{u_*^{\text{pre}}}{\rightleftarrows}} & \mathsf{PShv}(\mathcal{D}) \\ {\scriptstyle a}\big\downarrow\big\uparrow{\scriptstyle i} & & {\scriptstyle a}\big\downarrow\big\uparrow{\scriptstyle i} \\ \mathsf{Shv}(\mathcal{C}) & \underset{u^*}{\overset{u_*}{\rightleftarrows}} & \mathsf{Shv}(\mathcal{D}) \end{array}$$

Since u^* preserves sheaves (by definition of the induced topology), and u_*^{pre} preserves sheaves (as it is a limit), the restriction $u_* := u_*^{\text{pre}}|_{\mathsf{Shv}(\mathcal{D})}$ gives an adjoint pair $u^* \dashv u_*$ on sheaves.

Step 3: Equivalence. We use the criterion: an adjunction is an equivalence if and only if the right adjoint is fully faithful and the left adjoint is conservative.

(i) u_* **is fully faithful.** It suffices to show that the counit $\varepsilon \colon u^* u_* \to \mathrm{id}_{\mathsf{Shv}(\mathcal{D})}$ is an isomorphism. For any $G \in \mathsf{Shv}(\mathcal{D})$ and $D \in \mathcal{D}$, we verify the formula:

$$(u^* u_* G)(D) = (u_* G)(u(D)) = \lim_{D' \in \mathcal{D}_{/D}} G(D').$$

Since u is fully faithful, the indexing category is the slice category $\mathcal{D}_{/D}$. The terminal object of $\mathcal{D}_{/D}$ is $\mathrm{id}_D \colon D \to D$. Therefore, the limit is simply the value of G at this terminal object:

$$\lim_{D' \in \mathcal{D}_{/D}} G(D') \cong G(D).$$

Thus ε_G is an isomorphism.

(ii) u^* **is conservative.** Let $\varphi\colon F \to G$ be a morphism in $\mathsf{Shv}(C)$ such that $u^*(\varphi)$ is an isomorphism. This means $\varphi_D\colon F(D) \to G(D)$ is a bijection for all $D \in \mathcal{D}$. We show that φ is a τ-local isomorphism (Definition 5.3.5), which for sheaves implies it is an isomorphism.

Fix any $X \in C$. By the density of $\mathcal{D}$, there exists a τ-covering sieve S generated by a family $\{f_i\colon D_i \to X\}_{i\in I}$ where each $D_i \in \mathcal{D}$.

- **τ-local epimorphism:** Let $s \in G(X)$. For each i, consider the restriction $s|_{D_i} \in G(D_i)$. Since φ_{D_i} is surjective, there exists $t_i \in F(D_i)$ such that $\varphi_{D_i}(t_i) = s|_{D_i}$. This means the sieve of morphisms $Y \to X$ over which s lifts to F contains the covering sieve S. Thus, φ is a τ-local epimorphism.
- **τ-local monomorphism:** Let $s, t \in F(X)$ such that $\varphi_X(s) = \varphi_X(t)$. For each i, we have $\varphi_{D_i}(s|_{D_i}) = \varphi_{D_i}(t|_{D_i})$ in $G(D_i)$. Since φ_{D_i} is injective, we must have $s|_{D_i} = t|_{D_i}$. This means the sieve of morphisms $Y \to X$ over which s and t become equal contains the covering sieve S. Thus, the diagonal Δ_φ is a τ-local epimorphism, so φ is a τ-local monomorphism.

Since φ is a τ-local isomorphism between sheaves, it is an isomorphism.

Since u_* is fully faithful and u^* is conservative, they define an equivalence. □

***Remark* 5.6.3.** For the higher categorical version (specifically in the context of ∞-sites and ∞-sheaves), we refer the reader to [Hoy14, Lemma C.3]. The proof presented there follows the same strategy as ours, relying on the right Kan extension along the embedding of the dense subcategory.

We collect a few applications of the Comparison Lemma.

Example 5.6.4 (Comparison Lemma for Presheaves). Let $\mathcal{D} \subset C$ be a full subcategory. Then $\mathcal{D}$ is dense with respect to the indiscrete topology if and only if every object of C is a retract of an object of $\mathcal{D}$. In this case, the Comparison Lemma says that $\mathsf{PShv}(C) \xrightarrow{\sim} \mathsf{PShv}(\mathcal{D})$.

Example 5.6.5 (Basis of a Topological Space). Let T be a topological space and let $\mathcal{B} \subset \mathrm{Open}(T)$ be a basis of its topology. By definition, $\mathcal{B}$ is dense with respect to canonical topology on $\mathrm{Open}(T)$, and the restriction of the canonical topology to $\mathcal{B}$ consists of sieves generated by open coverings in $\mathcal{B}$. Hence, the Comparison Lemma provides an equivalence $\mathsf{Shv}(T) \xrightarrow{\sim} \mathsf{Shv}(\mathcal{B})$.

Example 5.6.6 (Sheaves on Manifolds). Let Man_n be the category of n-dimensional topological manifolds, equipped with the standard topology (induced by the pretopology of open coverings). Every n-manifold admits an open covering by manifolds homeomorphic to $\mathbb{R}^n$, so the full subcategory $\{\mathbb{R}^n\} \subset \mathsf{Man}_n$ is dense, and the Comparison Lemma yields an equivalence $\mathsf{Shv}(\mathsf{Man}_n) \simeq \mathsf{Shv}(\{\mathbb{R}^n\})$.

Example 5.6.7 (Sheaves on $\mathrm{Spec}(R)$). Let R be a ring. Consider the following two full subcategories of the locale $\mathsf{Open}(\mathrm{Spec}(R)) \simeq \mathrm{Open}(\mathrm{Prim}(R)) \simeq \mathsf{Rad}_R$:

(i) $\mathsf{Open}^{\mathrm{pr}}(\mathrm{Spec}(R))$ consists of open subfunctors of the form $D(f)$ with $f \in R$ (called *principal* open subschemes of $\mathrm{Spec}(R)$);

(ii) $\mathsf{Open}^{\mathrm{aff}}(\mathrm{Spec}(R))$ consists of the *affine* open subfunctors of $\mathrm{Spec}(R)$.

We have $\mathsf{Open}^{\mathrm{pr}}(\mathrm{Spec}(R)) \subset \mathsf{Open}^{\mathrm{aff}}(\mathrm{Spec}(R))$, and both subcategories are dense in $\mathsf{Open}(\mathrm{Spec}(R))$, since $D(I) = \bigvee_{f\in I} D(f)$. By the Comparison Lemma, the three categories of sheaves are equivalent. In particular, every sheaf on $\mathsf{Open}^{\mathrm{pr}}(\mathrm{Spec}(R))$ *extends uniquely* to a sheaf on $\mathsf{Open}(\mathrm{Spec}(R))$.

Example 5.6.8 (Sheaves on Presheaves). Let C be a small category, which we view as a full subcategory of $\mathsf{PShv}(C)$ via the Yoneda embedding. Any topology τ on C extends canonically to a topology $\hat{\tau}$ on $\mathsf{PShv}(C)$ as follows: A sieve $\mathcal{R}$ on presheaf F is a $\hat{\tau}$-covering if, for any $y(X) \to F$, the sieve of all $Y \to X$ such that $y(Y) \to y(X) \to F$ is in $\mathcal{R}$ is τ-covering.

Since every object in $\mathsf{PShv}(C)$ is a quotient of a coproduct of representable presheaves, the Comparison Lemma applies to the Yoneda embedding $C \hookrightarrow \mathsf{PShv}(C)$, so that

$$\mathsf{Shv}_{\tau}(C) \simeq \mathsf{Shv}_{\hat{\tau}}(\mathsf{PShv}(C)).$$

Applying this to the category of affine schemes $C = \mathsf{Aff}$ equipped with the Zariski topology τ_{Zar}, the Comparison Lemma yields an equivalence of categories:

$$\mathsf{Shv}^{\mathsf{Cat}}_{\tau_{\mathrm{Zar}}}(\mathsf{Aff}) \simeq \mathsf{Shv}^{\mathsf{Cat}}_{\hat{\tau}_{\mathrm{Zar}}}(\mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})).$$

This equivalence says that, at the level of categorical sheaves, extending from affine schemes to arbitrary algebraic functors and enlarging the topology to $\hat{\tau}_{\mathrm{Zar}}$ leaves the category of sheaves unchanged.

***Remark* 5.6.9** (Comparison Lemma for Stacks). The comparison lemma extends to category-valued sheaves: if $\mathcal{B} \subset C$ is a basis closed under finite limits with the induced topology $\tau|_{\mathcal{B}}$, then restriction induces an equivalence

$$\mathsf{Shv}^{\mathsf{Cat}}_{\tau}(C) \simeq \mathsf{Shv}^{\mathsf{Cat}}_{\tau|_{\mathcal{B}}}(\mathcal{B}).$$

The proof is the same as in the set-valued case, replacing Set by Cat throughout and using that limits of categories agree with limits in Cat.

Looking Ahead

With Grothendieck topologies, sheaves, sheafification, and the Comparison Lemma in hand, we now have the full language of descent. Along the way we set up the Zariski, fppf, and fpqc topologies on Aff, and verified that a Zariski sheaf on affine schemes extends canonically to one on all algebraic functors.

This is exactly the right tool to define schemes. In the next chapter, a scheme is simply a Zariski sheaf $X \in \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ admitting an open cover by affines — all the categorical scaffolding developed here (sheafification, the Comparison Lemma, covering sieves) feeds directly into the theory of Sch.

Chapter 6.

Schemes

Everything in the previous chapters — affine schemes, projective schemes, quasi-coherent modules, Grothendieck topologies — has been building towards this definition. A *scheme* is an algebraic functor that (a) satisfies Zariski descent and (b) is locally affine. In one formula:

$$\text{scheme} = \text{Zariski sheaf} + \text{affine open cover}.$$

Condition (a) says the scheme can be glued from local data; condition (b) says it admits local coordinates, i.e. is built up out of familiar commutative-algebra pieces $\mathrm{Spec}(R)$.

Schemes generalise both affine schemes (which are always schemes) and projective ones (which we saw are coverable by the affines $D(f) \simeq \mathrm{Spec}(A_{(f)})$). But the category of schemes is richer still: it is closed under gluing along open subfunctors, so it contains genuine global objects — compact Riemann surfaces, elliptic curves over $\mathbb{Z}$, moduli spaces — that are not of the form Spec or Proj.

This chapter develops the formal theory. We (i) verify that schemes form a full subcategory $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ closed under immersions, finite limits, cofiltered affine limits, and coproducts; (ii) construct schemes by gluing along locally trivial equivalence relations; (iii) introduce *separatedness* and the quasi-compact / quasi-separated (qcqs) conditions that make finiteness work globally; and (iv) relate the functorial picture to the classical definition of a scheme as a locally ringed space.

6.1. The Category of Schemes

We first verify that schemes form a well-behaved subcategory: closed under immersions, with good properties inherited from the ambient Zariski sheaves. The principle that recurs throughout is that everything about schemes reduces, locally on an affine cover, to commutative algebra.

Definition 6.1.1 (Scheme)**.** A *scheme* is a functor $X : \mathsf{CAlg} \to \mathsf{Set}$ satisfying:

(i) X is a sheaf with respect to the Zariski topology;

(ii) X admits an open cover by affine schemes.

We write $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ for the full subcategory of schemes. For a scheme S, the objects of the slice $\mathsf{Sch}_S := \mathsf{Sch}_{/S}$ are called *S-schemes*; when $S = \mathrm{Spec}(k)$ for a ring k, we abbreviate to *k-schemes*.

Since Sch is only a full subcategory of $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$, properties of the latter need not survive the restriction. We now investigate which constructions and limits do remain in Sch.

Proposition 6.1.2 (Immersions of Schemes)**.** *Let X be a scheme. If $Y \hookrightarrow X$ is an open immersion, a closed immersion, or an immersion, then Y is also a scheme.*

Proof. Choose an affine open cover $(U_i \hookrightarrow X)_{i\in I}$ with each U_i an affine scheme. By stability of open covers under base change, $(U_i \cap Y \hookrightarrow Y)_{i\in I}$ is an open cover of Y. So it suffices to check, in each of the three cases, that (a) every $U_i \cap Y$ admits an affine open cover, and (b) Y is a Zariski sheaf.

The open case. For each i, the open subfunctor $U_i \cap Y \subset U_i$ corresponds to a quasi-coherent radical ideal J_i, so $U_i \cap Y = D(J_i)$. The family $(D(f) \hookrightarrow D(J_i))_{f\in J_i}$ is an affine open cover (each $D(f) \simeq \mathrm{Spec}((R_i)_f)$), and refining the original cover gives an affine open cover $(D(f) \hookrightarrow Y)_{i\in I, f\in J_i}$ of Y.

For Zariski descent of Y: fix a ring R and a Zariski cover $(R \to R_{f_k})_{k\in K}$. Consider the diagram of subfunctor inclusions

$$\begin{array}{ccccc} Y(R) & \longrightarrow & \prod_k Y(R_{f_k}) & \rightrightarrows & \prod_{k,l} Y(R_{f_k f_l}) \\ \downarrow & & \downarrow & & \downarrow \\ X(R) & \longrightarrow & \prod_k X(R_{f_k}) & \rightrightarrows & \prod_{k,l} X(R_{f_k f_l}). \end{array}$$

The bottom row is an equalizer because X is a Zariski sheaf, so the top row is at least a sub-equalizer; we must verify that any compatible family lying in X in fact lies in Y. Take $x \in X(R)$ in the equalizer of the bottom row, and assume $x_k := x|_{R_{f_k}} \in Y(R_{f_k})$ for every k. Pulling back $Y \hookrightarrow X$ along $x\colon \mathrm{Spec}(R) \to X$ yields an open subfunctor of $\mathrm{Spec}(R)$, i.e., an open subscheme $D(I) \subset \mathrm{Spec}(R)$ for some radical ideal $I \subset R$. The hypothesis $x_k \in Y(R_{f_k})$ says $D(I \cdot R_{f_k}) = \mathrm{Spec}(R_{f_k})$, equivalently $I \cdot R_{f_k} = R_{f_k}$ for each k. Since (f_k) generates the unit ideal in R, this forces $I = R$, so $D(I) = \mathrm{Spec}(R)$, which means $x \in Y(R)$.

The closed case. Each $U_i \cap Y \subset U_i$ is a closed subfunctor, hence of the form $V(J_i) = \mathrm{Spec}(R_i/J_i)$ for some ideal $J_i \subset R_i$. In particular $U_i \cap Y$ is itself affine, providing an affine open cover of Y.

For Zariski descent, the same diagram-chase as above applies with V in place of D: pulling back $Y \hookrightarrow X$ along an R-point x of X gives a closed subscheme $V(I) \subset \mathrm{Spec}(R)$. The condition $V(I \cdot R_{f_k}) = \mathrm{Spec}(R_{f_k})$ means $I \cdot R_{f_k} = 0$ for every k. Since vanishing of an ideal is detected by a Zariski cover (modules satisfy Zariski descent), $I = 0$, so $V(I) = \mathrm{Spec}(R)$ and x factors through Y.

The immersion case. By Definition 1.2.51, an immersion $Y \hookrightarrow X$ factors as $Y \hookrightarrow U \hookrightarrow X$ where the first map is closed and the second is open. The open case shows U is a scheme, and applying the closed case to $Y \hookrightarrow U$ shows Y is a scheme. □

Example 6.1.3 (Examples of Schemes). (i) For any ring A, $\mathrm{Spec}(A)$ is a scheme. By Proposition 6.1.2, every quasi-affine scheme is also a scheme.

(ii) For an $\mathbb{N}$-graded ring A, $\mathrm{Proj}(A)$ is a scheme, with affine cover $(D(f) \simeq \mathrm{Spec}(A_{(f)}))_{f\in A_+}$. In particular every (quasi-)projective k-scheme is a scheme.

(iii) The affine line with two origins is a scheme, covered by two copies of $\mathbb{A}^1$.

(iv) For a scheme X and an affine, quasi-affine, projective, or quasi-projective morphism $Y \to X$, the source Y is again a scheme.

(v) For a ring k, a k-module M, and $n \in \mathbb{N}$, the Grassmannian $\mathrm{Gr}_n(M)$ is a scheme.

We turn to gluing schemes — a categorical analogue of gluing topological spaces. We first generalise the notion of equivalence relation to an arbitrary category.

Definition 6.1.4 (Equivalence Relation). Let C be a category. An *equivalence relation* in C is a pair of morphisms $s, t\colon R \rightrightarrows X$ such that, for every $Y \in C$, the induced map

$$(s_*, t_*)\colon \mathrm{Hom}_C(Y, R) \to \mathrm{Hom}_C(Y, X) \times \mathrm{Hom}_C(Y, X)$$

is injective with image an equivalence relation on $\mathrm{Hom}_C(Y, X)$.

If the coequalizer of s, t exists, it is called the *quotient* of X by R, denoted X/R.

***Remark* 6.1.5** (Higher Categorical Analogue: Groupoid Objects). The higher-categorical analogue of an equivalence relation is a *groupoid object*, namely a simplicial object $X_\bullet$ in $\mathcal{C}$

$$\cdots \; X_2 \rightrightarrows X_1 \rightrightarrows X_0$$

satisfying the *groupoid (Kan) condition*: for any $[n] \in \Delta$ and any partition (not necessarily order-preserving) $[n] \simeq \{i_0, \ldots, i_k\} \cup \{i_k, \ldots, i_n\}$, the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{(i_0,\ldots,i_k)^*} & X_k \\ {\scriptstyle (i_k,\ldots,i_n)^*}\downarrow & \lrcorner & \downarrow{\scriptstyle i_k^*} \\ X_{n-k} & \xrightarrow[i_k^*]{} & X_0 \end{array}$$

is a pullback square.

This condition encodes both core properties of groupoids. For order-preserving partitions (such as $[2] = \{0,1\} \cup \{1,2\}$) it gives the *Segal condition* $X_2 \simeq X_1 \times_{X_0} X_1$, ensuring composition. For non-order-preserving partitions it forces invertibility of morphisms.

Specialised to $\mathcal{C} = \mathsf{Set}$: a groupoid object is a classical groupoid, and the image of $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ defines an equivalence relation on $\mathcal{G}_0$ (reflexivity from identities, transitivity from composition, symmetry from inversion). Conversely, an equivalence relation $R \subseteq X \times X$ corresponds bijectively to a groupoid $\mathcal{G}$ in Set with $\mathcal{G}_0 = X$ and $(s,t)\colon \mathcal{G}_1 \hookrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ injective; under this correspondence, the colimit $|\mathcal{G}|$ recovers the quotient X/R.

***Remark* 6.1.6** (Effective Epimorphisms). Any morphism $f : X \to Y$ in a category with pullbacks gives rise to an equivalence relation $X \times_Y X \rightrightarrows X$ (in the higher setting, a groupoid object via the Čech nerve). We call f an *effective epimorphism* if the diagram

$$X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$

is a coequalizer.

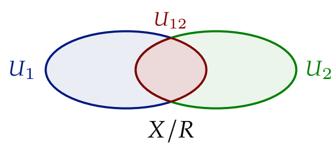
In a 1-topos every epimorphism is effective; this fails in general, e.g. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in CAlg is an epimorphism but not an effective one.

We isolate the relevant class of equivalence relations for gluing schemes.

Definition 6.1.7 (Locally Trivial Relation). An equivalence relation $s,t\colon R \rightrightarrows X$ in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ is *locally trivial* if there exists an open cover $(U_i \hookrightarrow X)_{i\in I}$ such that, for every i, the composition $s^{-1}(U_i) \hookrightarrow R \xrightarrow{t} X$ is an open immersion.

Concretely, gluing two schemes U_1 and U_2 along $U_{12} = U_{21}$ is captured by the following data.

$$\begin{array}{ccc} U_{12} = U_{21} & \xhookrightarrow{\text{open}} & U_1 \\ {\scriptstyle \text{open}}\big\downarrow & \ulcorner & \big\downarrow \\ U_2 & \longrightarrow & X/R \end{array} \quad \rightsquigarrow$$



Here $X = U_1 \sqcup U_2$ and $R = \coprod_{i,j\in\{1,2\}} U_{ij} \rightrightarrows X$, with the convention $U_{ii} = U_i$. If both $U_{12} \hookrightarrow U_1$ and $U_{21} \hookrightarrow U_2$ are open immersions, then R is locally trivial. This motivates:

Definition 6.1.8 (Gluing Datum). A *gluing datum* in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ consists of families $(U_i)_{i\in I}$ and $(U_{ij})_{i,j\in I}$ together with morphisms $U_{ij} \to U_i \times U_j$ satisfying:

- the projections $U_{ij} \hookrightarrow U_i$ and $U_{ij} \hookrightarrow U_j$ are open immersions;
- the diagram $\coprod_{i,j \in I} U_{ij} \rightrightarrows \coprod_{i \in I} U_i$ is an equivalence relation in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$.

***Remark* 6.1.9** (Open Covers and Gluing Data)**.** The notions of open cover and gluing datum are interconvertible:

- If X is the quotient of the equivalence relation given by gluing data $((U_i), (U_{ij}))$, then $(U_i \to X)_{i \in I}$ is an open cover with $U_{ij} \xrightarrow{\sim} U_i \times_X U_j$.
- Conversely, given a Zariski sheaf X and an open cover $(U_i \to X)_{i \in I}$, the families (U_i) and $(U_i \cap U_j)$ form a gluing datum.

We now identify which constructions are preserved by the inclusion $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$.

Theorem 6.1.10 (Limits and Colimits of Schemes)**.** *The full subcategory* $\mathsf{Sch} \subset \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ *is closed under:*

(i) finite limits;

(ii) cofiltered diagrams of affine morphisms;

(iii) arbitrary coproducts;

(iv) quotients by locally trivial equivalence relations.

We first establish a lemma.

Lemma 6.1.11 (Coproducts Are Clopen)**.** *Let* $(X_i)_{i \in I}$ *be a family of Zariski sheaves with coproduct* $X = \coprod_{i \in I} X_i$. *Then each canonical map* $X_i \hookrightarrow X$ *is both an open and a closed immersion. In particular* $(X_i \hookrightarrow X)_{i \in I}$ *is an open cover of* X.

Proof. Decompose $X_i \hookrightarrow X$ as $X_i \hookrightarrow X_i \sqcup \left(\coprod_{j \neq i} X_j\right) = X$. Everything reduces to showing that for any Zariski sheaves X, Y, the inclusion $X \hookrightarrow X \sqcup Y$ is clopen.

Form the pullback in the presheaf category

$$\begin{array}{ccc} X \sqcup^{\mathrm{pre}} \varnothing_Y & \longrightarrow & \mathbb{G}_m \\ \downarrow & \lrcorner & \downarrow \\ X \sqcup^{\mathrm{pre}} Y & \xrightarrow{(1,0)} & \mathbb{A}^1 \end{array}$$

where $\sqcup^{\mathrm{pre}}$ is the presheaf coproduct and $(1,0)$ is the algebraic-functor map taking constant value 1 on X and 0 on Y. The upper-left corner has explicit description

$$\varnothing_Y(R) = \begin{cases} \varnothing, & R \neq 0, \\ Y(0), & R = 0. \end{cases}$$

There is a unique map $\varnothing_Y \to \varnothing = \mathrm{Spec}(0)$, which is an isomorphism on every nonzero ring. Since the empty sieve covers $\mathrm{Spec}(0)$ in the Zariski topology, $\varnothing_Y \to \varnothing$ is a Zariski-local isomorphism, so $a_{\mathrm{Zar}}(\varnothing_Y) = \varnothing$.

Sheafification preserves finite limits, so applying a_{Zar} to the diagram yields a pullback square in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{G}_m \\ \downarrow & \lrcorner & \downarrow \\ X \sqcup Y & \xrightarrow{(1,0)} & \mathbb{A}^1. \end{array}$$

This realises $X \hookrightarrow X \sqcup Y$ as the base change of the open immersion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ along the section $(1,0)$. Geometrically: the open subscheme on which $(1,0)$ is invertible is exactly X, and its complement $V(1,0) = Y$ is the closed locus where it vanishes. Hence $X \hookrightarrow X \sqcup Y$ is simultaneously open and closed. □

By Lemma 6.1.11, condition (2) of Definition 6.1.8 amounts to a locally trivial equivalence relation. Concretely, it unfolds into the following compatibility data, each asserting a factorisation through some $U_{kl} \hookrightarrow U_k \times U_l$:

- *Reflexivity:* the diagonal $U_i \to U_i \times U_i$ factors through U_{ii};
- *Symmetry:* the composite $U_{ij} \to U_i \times U_j \xrightarrow{\sim} U_j \times U_i$ factors through U_{ji};
- *Transitivity:* the composite $U_{ij} \times_{U_j} U_{jk} \to U_i \times U_k$ factors through U_{ik}.

Proof of Theorem 6.1.10. **(1) Finite limits.** It suffices to handle the terminal object and pullbacks. The terminal object is $\operatorname{Spec}(\mathbb{Z})$, a scheme. For pullbacks, let $\alpha\colon X \to Z$ and $\beta\colon Y \to Z$ be morphisms of schemes, and set $P = X \times_Z Y$. Given a pair $x \in |X|$, $y \in |Y|$ with common image $z \in |Z|$, choose an affine open $W \subset Z$ containing z, an affine open $U \subset X$ contained in $\alpha^{-1}(W)$ and containing x, and an affine open $V \subset Y$ contained in $\beta^{-1}(W)$ and containing y. Then $U \times_W V$ is affine (fiber product of affines is affine), and inclusion gives an open subfunctor

$$U \times_W V \hookrightarrow X \times_Z Y = P.$$

Varying (x, y, z) produces an affine open cover of P, so P is a scheme.

(2) Cofiltered limits of affine morphisms. Recall that $f: Y \to X$ is affine iff for any $\operatorname{Spec}(R) \to X$ the pullback $Y \times_X \operatorname{Spec}(R)$ is affine, and there is an equivalence $\operatorname{Spec}\colon \mathsf{CAlg}_X^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}_X$.

Let I be cofiltered and $F\colon I \to \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ a diagram with affine transition maps. Pick any basepoint $x_0 \in I$; cofilteredness gives $\lim_I F \simeq \lim_{I_{/x_0}} F$, so we may assume I has a terminal object $*$. Set $X = F(*)$. The diagram restricts to a diagram in Aff_X, and since $\mathsf{Aff}_X \hookrightarrow \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set})_{/X}$ preserves limits while CAlg admits filtered colimits, the limit is itself an affine X-scheme, hence a scheme.

(3) Arbitrary coproducts. For $X = \coprod_{i \in I} X_i$ with each X_i a scheme: by Lemma 6.1.11 each $X_i \hookrightarrow X$ is an open immersion and these jointly cover X. Refining each X_i by an affine open cover yields one for X, so X is a scheme.

(4) Quotients by locally trivial equivalence relations. Let $R \rightrightarrows X$ be a locally trivial equivalence relation with X a scheme, and form the quotient X/R. Choose an open cover $(U_i \hookrightarrow X)_{i \in I}$ as in Definition 6.1.7, so that for each i the composite $s^{-1}(U_i) \hookrightarrow R \xrightarrow{t} X$ is an open immersion.

Pulling back the inclusion $U_i \hookrightarrow X/R$ along the Zariski-local epimorphism $X \to X/R$ gives the cartesian square

$$\begin{array}{ccc} U_i \times_{X/R} X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ U_i & \longrightarrow & X/R. \end{array}$$

Since $X \times_{X/R} X = R$ (definition of the quotient), $U_i \times_{X/R} X \simeq s^{-1}(U_i)$ via s, with second projection given by t. The locally trivial hypothesis says this base change $s^{-1}(U_i) \xrightarrow{t} X$ is an open immersion. The property "open immersion" descends along Zariski-local epimorphisms, so $U_i \to X/R$ is itself an open immersion. The family $(U_i \to X/R)_{i \in I}$ jointly covers X/R, since (U_i) covers X and $X \to X/R$ is a Zariski-local epimorphism. Refining each U_i by an affine open cover yields an affine open cover of X/R. □

Theorem 6.1.10 thus implies: given a gluing datum in which every U_i and U_{ij} is a scheme, the resulting quotient X is again a scheme.

6.1.1. Examples

Example 6.1.12 (Obtaining Proj by Gluing). Let A be an $\mathbb{N}$-graded ring and $I \subset A_+$ a subset of homogeneous elements with $A_+ \subset \sqrt{(I)}$. The family $(D(f) \subset \mathrm{Proj}(A))_{f \in I}$ is an open cover, with pairwise intersections

$$D(f) \cap D(g) = \mathrm{Spec}(A_{(f)}) \times_{\mathrm{Proj}(A)} \mathrm{Spec}(A_{(g)}) \simeq D(fg).$$

Hence $\mathrm{Proj}(A)$ is the quotient of the equivalence relation

$$\textstyle\coprod_{f,g \in I} \mathrm{Spec}(A_{(fg)}) \rightrightarrows \coprod_{f \in I} \mathrm{Spec}(A_{(f)}).$$

For $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, take $I = \{x_0, \dots, x_n\}$. Then

$$D(x_i) = \mathrm{Spec}\big(\mathbb{Z}[x_0, \dots, \widehat{x_i}, \dots, x_n]\big) \simeq \mathbb{A}^n,$$

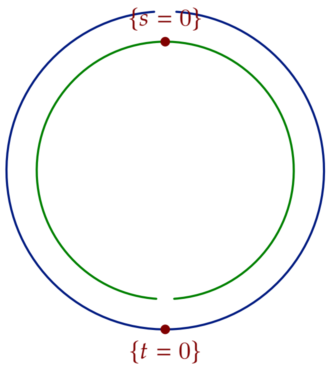
and for $i \neq j$, $D(x_i x_j) \simeq \mathbb{A}^{n-1} \times \mathbb{G}_m$. The presentation becomes

$$\textstyle\coprod_{i<j} \mathbb{A}^{n-1} \times \mathbb{G}_m \rightrightarrows \coprod_{i=0}^{n} \mathbb{A}^n \longrightarrow \mathbb{P}^n.$$

For $n = 1$, this collapses to the pushout

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{t \mapsto t} & \mathbb{A}^1 \\ {\scriptstyle t \mapsto t^{-1}} \downarrow & \ulcorner & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1, \end{array}$$

visualised as gluing two copies of $\mathbb{A}^1$ along $\mathbb{G}_m$ via the identification $t \leftrightarrow t^{-1}$:



Remark **6.1.13** (Group Quotients and Affine Decomposition).

- If a group object G acts on X, then

$$X/G = \mathrm{colim}\Big(X \times G \overset{\mathrm{act}}{\underset{\mathrm{proj}}{\rightrightarrows}} X \Big).$$

  When $X = *$, the resulting object is called a *classifying stack*. In the higher-categorical setting, by Remark 6.1.5, the colimit must be taken as a geometric realisation over the full groupoid object.

- Any scheme is a colimit of affine schemes: take an affine cover and the diagram of pairwise intersections.

Example 6.1.14 (Obtaining $\mathbb{P}^n$ via a Group Action). The canonical action $\mathbb{G}_m \curvearrowright \mathbb{A}^{n+1} - 0$ induces a monomorphism

$$(\mathbb{A}^{n+1} - 0)/\mathbb{G}_m \hookrightarrow \mathbb{P}^n,$$

which is a Zariski-local epimorphism (verified earlier), so in $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ it becomes an isomorphism:

$$(\mathbb{A}^{n+1} - 0)/\mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^n.$$

More generally, for any $\mathbb{N}$-graded ring generated in degree ≤ 1,

$$D(A_+)/\mathbb{G}_m \xrightarrow{\sim} \mathrm{Proj}(A).$$

Example 6.1.15 (Coproducts of Affine Schemes). (i) *Finite coproducts.* The initial scheme is $\varnothing = \mathrm{Spec}(0) = a_{\mathrm{Zar}}(\varnothing^{\mathrm{pre}})$.

For $R, S \in \mathsf{CAlg}$, the product $R \times S$ in CAlg fits into a commutative square (pullback in CAlg, pushout in Aff):

$$\begin{array}{ccc} R \times S & \xrightarrow{(0,1)} & S \\ {\scriptstyle (1,0)}\downarrow & \ulcorner & \downarrow \\ R & \longrightarrow & 0, \end{array}$$

where $R \times S \to R$ and $R \times S \to S$ are localisations at the orthogonal idempotents $e_1 = (1, 0)$ and $e_2 = (0, 1)$ (which satisfy $e_1 + e_2 = 1$). Let $\mathcal{T}$ be the sieve on $R \times S$ generated by $\{R \times S \to R, R \times S \to S\}$. Since (e_1, e_2) generate the unit ideal, $\mathcal{T}$ is a Zariski-covering sieve, so for any $X \in \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$,

$$X(R \times S) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{T}, X) \simeq X(R) \times X(S).$$

This shows $\mathrm{Spec}(R \times S)$ represents the coproduct of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ in the sheaf category, so $\mathsf{Aff} \hookrightarrow \mathsf{Sch}$ preserves finite coproducts.

(ii) *Counterexample for arbitrary coproducts.* The inclusion $\mathsf{Aff} \hookrightarrow \mathsf{Sch}$ does not preserve infinite coproducts. For an infinite family $(\mathrm{Spec}(R_i))_{i \in I}$, the coproduct in Sch is the disjoint union $X = \coprod_{i \in I} \mathrm{Spec}(R_i)$ (a scheme by Theorem 6.1.10, but generally not affine).

If $X \simeq \mathrm{Spec}(A)$ were affine, we would have $A \simeq \mathcal{O}(X) = \prod_{i \in I} R_i$. The idempotents $e_i = (\delta_{ij})_{j \in I} \in A$ (1 in position i, 0 elsewhere) determine standard opens $D(e_i) \simeq \mathrm{Spec}(R_i)$. However, when infinitely many $R_i \neq 0$, the ideal generated by $(e_i)_{i \in I}$ is $\bigoplus_{i \in I} R_i \subsetneq \prod_{i \in I} R_i$ (it does not contain $1 = (1, 1, \ldots)$). Hence

$$\bigcup_{i \in I} D(e_i) \subsetneq \mathrm{Spec}\Big(\prod_{i \in I} R_i\Big),$$

i.e., the affine product contains "extra" points (corresponding to non-principal ultrafilters on I) not present in the coproduct in Sch.

6.2. Separatedness and Quasi-compactness

This section introduces three finiteness/separation conditions on morphisms: *separated*, *quasi-compact*, and *quasi-separated*. They are the scheme-theoretic substitutes for Hausdorff and compactness in topology, but the analogy is indirect: schemes are almost never Hausdorff in the usual topological sense (the underlying space $|X|$ has lots of non-closed points), so the right substitute looks at the diagonal $\Delta_f : Y \to Y \times_X Y$.

The key idea: f is *separated* when Δ_f is a closed immersion — mimicking the topological theorem "f is Hausdorff iff Δ_f is closed". There are two natural strengths, depending on whether Δ_f is required to be a closed immersion (*separated*) or merely an immersion (*locally separated*). The latter turns out to be automatic for morphisms of schemes, so the genuinely restrictive notion is separatedness.

6.2.1. Separatedness

Definition 6.2.1 (Local Separatedness, Separatedness). For a morphism $f\colon Y \to X$ of algebraic functors with diagonal $\Delta_f\colon Y \to Y \times_X Y$:

- f is *locally separated* if Δ_f is an immersion;
- f is *separated* if Δ_f is a closed immersion.

Proposition 6.2.2 (All Scheme Morphisms Are Locally Separated). *Any morphism $f\colon Y \to X$ between schemes is locally separated.*

Proof. Strategy. We exhibit Δ_f as a composition $Y \to W \hookrightarrow Y \times_X Y$ in which the first map is a closed immersion and the second is open. Locally on affine opens $V_{ij} \subset Y$ over $U_i \subset X$, the diagonal of an affine morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the closed immersion dual to the multiplication $B \otimes_A B \to B$; the open W is the union of the affine opens $V_{ij} \times_{U_i} V_{ij} \subset Y \times_X Y$.

Choose an affine open cover $(U_i)_{i\in I}$ of X. For each i, choose an affine open cover $(V_{ij})_{j\in J_i}$ of $f^{-1}(U_i)$, so that each $V_{ij} \to U_i$ is a morphism of affine schemes. For each pair (i, j), the natural open immersion

$$W_{ij} := V_{ij} \times_{U_i} V_{ij} \hookrightarrow f^{-1}(U_i) \times_{U_i} f^{-1}(U_i) \hookrightarrow Y \times_X Y$$

exhibits W_{ij} as an affine open subscheme. Let $|W|$ be the union of all $|W_{ij}|$ inside $|Y \times_X Y|$. Since each $|W_{ij}|$ is open, so is $|W|$. Via the correspondence $\mathsf{Open}(Y \times_X Y) \xrightarrow{\sim} \mathrm{Open}(|Y \times_X Y|)$, there is a unique open subscheme $W \subset Y \times_X Y$ with underlying space $|W|$, and $(W_{ij})_{i,j}$ is an affine open cover of it.

Step 1: Δ_f factors through W. Since $(V_{ij})_{i,j}$ covers Y, it suffices to show $\Delta_f(V_{ij}) \subset W$ for each (i, j). The relative diagonal of the affine morphism $V_{ij} \to U_i$ has image landing in W_{ij}:

$$\Delta_f|_{V_{ij}}\colon V_{ij} \xrightarrow{\Delta_{V_{ij}/U_i}} W_{ij} \hookrightarrow W.$$

Gluing yields a factorisation $\Delta'\colon Y \to W$.

Step 2: $\Delta'\colon Y \to W$ is a closed immersion. Closed immersions are local on the target, so it suffices to check $\Delta'^{-1}(W_{ij}) \to W_{ij}$ is closed for each (i, j). We claim $\Delta'^{-1}(W_{ij}) = V_{ij}$: for $y \in Y$, the condition $\Delta'(y) = (y, y) \in V_{ij} \times_{U_i} V_{ij}$ is equivalent to $y \in V_{ij}$ (the over-U_i constraint is automatic, both projections being $f(y)$). The restricted morphism is then exactly the affine relative diagonal

$$\Delta_{V_{ij}/U_i}\colon V_{ij} \to V_{ij} \times_{U_i} V_{ij}.$$

Writing $V_{ij} = \mathrm{Spec}(B_{ij})$ and $U_i = \mathrm{Spec}(A_i)$, this morphism is dual to the multiplication $B_{ij} \otimes_{A_i} B_{ij} \to B_{ij}$, $b \otimes b' \mapsto bb'$, which is surjective. Hence the corresponding scheme morphism is a closed immersion.

Combining the two steps, Δ_f factors as $Y \xrightarrow{\text{closed}} W \overset{\text{open}}{\hookrightarrow} Y \times_X Y$, so it is an immersion. □

We therefore focus on the genuinely restrictive notion.

***Remark* 6.2.3** (Interpretation of Separatedness). The condition that Δ_f is a closed immersion can be understood as follows.

- In topology, a continuous map $f\colon Y \to X$ has Hausdorff fibers iff the relative diagonal $\Delta_f\colon Y \to Y \times_X Y$ is closed; for Hausdorff base X, this is equivalent to Y itself being Hausdorff over X.
- For schemes, separatedness is checked on affine covers: $f\colon Y \to X$ is separated iff for any affine open $U = \mathrm{Spec}(A) \subset X$ and any affine opens $V_1 = \mathrm{Spec}(B_1)$, $V_2 = \mathrm{Spec}(B_2) \subset f^{-1}(U)$, the intersection $V_1 \cap V_2$ is affine and the natural map $B_1 \otimes_A B_2 \to \mathcal{O}(V_1 \cap V_2)$ is surjective.

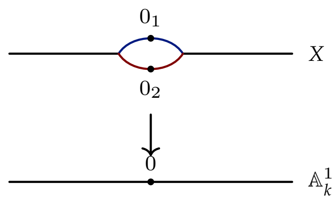
- Intuitively, separatedness rules out "gluing along non-closed subsets". The affine line with two origins is the standard non-example: its two affine charts $U_1, U_2 \simeq \mathbb{A}^1$ are glued along $\mathbb{A}^1 \setminus \{0\}$, which is open but not closed in either chart.

Example 6.2.4. (i) *Affine morphisms are separated.* If $f : Y \to X$ is affine, then $Y \times_X Y \to X$ is also affine (affine morphisms are stable under base change), and Δ_f corresponds to the multiplication $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{O}_Y$, which is surjective; hence Δ_f is a closed immersion.

(ii) *Projective morphisms are separated.* More generally, any projective morphism $f : Y \to X$ is separated; this includes $\mathbb{P}^n_X \to X$ for any scheme X.

(iii) *Open immersions are separated.* If $U \hookrightarrow X$ is an open immersion, $\Delta\colon U \to U \times_X U$ is an isomorphism.

(iv) *Counterexample: affine line with two origins.* Let X be the affine line with two origins, glued from two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$:



The morphism $X \to \mathrm{Spec}(k)$ is not separated: $\Delta\colon X \to X \times_k X$ is an immersion but not closed, because the two origins fail to be separated by disjoint affine neighbourhoods. Equivalently, $U_1 \cap U_2 = \mathbb{A}^1 \setminus \{0\}$ is not closed in either U_i.

Proposition 6.2.5 (Properties of Separated Morphisms)**.**

(i) Consider a commutative triangle of algebraic functors

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & X. & \end{array}$$

If f and g are separated, so is h. If h is a closed immersion and f is separated, then g is a closed immersion. The same statements hold with "separated" replaced by "locally separated".

(ii) Consider a cartesian square of algebraic functors

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X. \end{array}$$

If f is (locally) separated, so is f'. The converse holds if X and Y are Zariski sheaves and g is a Zariski-local epimorphism.

Proof. **(1)** *Cancellation.* Suppose h is a closed immersion and f is separated. Closed immersions are stable under composition and base change, so by Theorem 1.2.53 applied to the property "is a closed immersion" and the sequence $Z \xrightarrow{g} Y \xrightarrow{f} X$ (whose diagonal Δ_f is a closed immersion by hypothesis), we conclude g is a closed immersion.

Composition. Suppose f and g are separated; we show $h = f \circ g$ is too. The diagonal Δ_h factors as Δ_g followed by the canonical inclusion $j \colon Z \times_Y Z \hookrightarrow Z \times_X Z$, where j fits into the cartesian square

$$\begin{array}{ccc} Z \times_Y Z & \xrightarrow{j} & Z \times_X Z \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g \times_X g} \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y. \end{array}$$

Since f is separated, Δ_f is a closed immersion, hence so is its base change j. Composing with the closed immersion Δ_g (since g is separated) gives $\Delta_h = j \circ \Delta_g$ a closed immersion.

The locally separated case is identical, replacing "closed immersion" with "immersion" throughout.

(2) *Forward direction.* The diagonals fit into the cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{\Delta_{f'}} & Y' \times_{X'} Y' \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y \end{array}$$

(the diagonal of a base change is the base change of the diagonal). If Δ_f is a closed immersion, so is $\Delta_{f'}$ by stability under base change. The locally separated case is identical.

Converse, g a Zariski-local epimorphism. Assume f' is separated; we show f is. From the cartesian square above, $\Delta_{f'}$ is the base change of Δ_f along $Y' \times_{X'} Y' \to Y \times_X Y$. The latter map is itself the base change of $X' \to X$, since

$$Y' \times_{X'} Y' = (Y \times_X X') \times_{X'} (Y \times_X X') \simeq (Y \times_X Y) \times_X X'.$$

Zariski-local epimorphisms are stable under base change, so $Y' \times_{X'} Y' \to Y \times_X Y$ is a Zariski-local epimorphism. Closed immersion property descends along Zariski-local epimorphisms (Proposition 1.3.11), so $\Delta_{f'}$ being a closed immersion forces Δ_f to be one. The locally separated case is identical. □

Corollary 6.2.6 (Sections of Morphisms of Schemes). *Let $f : X \to S$ be a morphism of schemes and $s : S \to X$ a section of f. Then s is an immersion. If f is separated, s is a closed immersion.*

Proof. The section s fits into a cartesian square

$$\begin{array}{ccc} S & \xrightarrow{s} & X \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta_f} \\ X & \xrightarrow{(\mathrm{id}_X, s \circ f)} & X \times_S X \end{array}$$

identifying S with the locus $\{x \in X : x = s(f(x))\}$. By Proposition 6.2.2, Δ_f is an immersion (and a closed immersion if f is separated). Stability of (closed) immersions under base change (Proposition 1.2.52) gives the conclusion. □

6.2.2. Quasi-compactness and Quasi-separatedness

Every scheme is a colimit of a diagram of affine schemes and open immersions (Remark 6.1.13); a scheme is *quasi-compact and quasi-separated* (*qcqs*) when such a diagram can be taken to be finite. We now study these finiteness conditions in detail, working at the level of general algebraic functors where they still make sense.

Definition 6.2.7 (Quasi-compact and Quasi-separated).

(i) An algebraic functor X is *quasi-compact* if every Zariski-local epimorphism $\coprod_{i\in I} Y_i \to X$ admits a finite subset $J \subset I$ for which $\coprod_{i\in J} Y_i \to X$ is still a Zariski-local epimorphism.

(ii) A morphism $f: Y \to X$ is *quasi-compact* if for every $Z \to X$ with Z quasi-compact, the pullback $Y \times_X Z$ is quasi-compact.

(iii) An algebraic functor X is *quasi-separated* if for every diagram $Y \to X \leftarrow Z$ with Y, Z quasi-compact, the pullback $Y \times_X Z$ is quasi-compact.

(iv) A morphism $f: Y \to X$ is *quasi-separated* if its diagonal $\Delta_f: Y \to Y \times_X Y$ is quasi-compact.

Proposition 6.2.8 (Characterisations of Quasi-compactness). *For an algebraic functor X, the following are equivalent:*

(i) X is quasi-compact;

(ii) the unique map $X \to \mathrm{Spec}(\mathbb{Z})$ is quasi-compact;

(iii) there exists a Zariski-local epimorphism $\coprod_{i\in I} Y_i \to X$ with I finite and each Y_i affine.

Proof. **(i) ⇒ (iii).** By the Yoneda Lemma, X is the colimit of representables over its category of elements: $X \simeq \mathrm{colim}_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Spec}(R)$. The canonical map

$$p: \coprod_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Spec}(R) \to X$$

is a Zariski-local epimorphism. By (i), it admits a finite subcover, yielding (iii).

(iii) ⇒ (i). Suppose $p: U = \coprod_{i=1}^n U_i \to X$ is a Zariski-local epimorphism with each U_i affine. Let $q: \coprod_{j\in J} Z_j \to X$ be any Zariski-local epimorphism. For each $i \in \{1, \dots, n\}$, the affine U_i is quasi-compact (it is its own singleton finite affine cover), so the pulled-back cover $\coprod_j (Z_j \times_X U_i) \to U_i$ admits a finite refinement indexed by $J_i \subset J$. The finite union $J' = \bigcup_{i=1}^n J_i \subset J$ then refines q to a finite Zariski-local epimorphism, since (U_i) jointly covers X.

(ii) ⇔ (i). If $f: X \to \mathrm{Spec}(\mathbb{Z})$ is quasi-compact, then since $\mathrm{Spec}(\mathbb{Z})$ is affine and hence quasi-compact, the pullback $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}) \simeq X$ is quasi-compact.

Conversely, suppose X is quasi-compact, and let $Z \to \mathrm{Spec}(\mathbb{Z})$ with Z quasi-compact. Choose finite affine covers $U = \coprod_{i\in I} \mathrm{Spec}(A_i) \twoheadrightarrow X$ and $V = \coprod_{j\in J} \mathrm{Spec}(B_j) \twoheadrightarrow Z$ via (iii). Stability of Zariski-local epimorphisms under products gives a Zariski-local epimorphism

$$U \times_{\mathrm{Spec}(\mathbb{Z})} V \simeq \coprod_{(i,j)\in I\times J} \mathrm{Spec}(A_i \otimes_{\mathbb{Z}} B_j) \twoheadrightarrow X \times_{\mathrm{Spec}(\mathbb{Z})} Z.$$

The left-hand side is a finite coproduct of affines, so by (iii) the pullback is quasi-compact. □

Proposition 6.2.9 (Characterisations of Quasi-separatedness). *For an algebraic functor X, the following are equivalent:*

(i) X is quasi-separated;

(ii) the unique map $X \to \mathrm{Spec}(\mathbb{Z})$ is quasi-separated;

(iii) for any $Y \to X$ and $Z \to X$ with Y, Z affine, the fiber product $Y \times_X Z$ is quasi-compact.

Proof. **(i) ⇒ (iii).** Affine schemes are quasi-compact, so this is a special case of the definition.

(iii) ⇒ (ii). We must show the diagonal $\Delta_X \colon X \to X \times X$ is quasi-compact. Take any $W \to X \times X$ with W quasi-compact, and form the pullback $P = W \times_{X \times X} X$. By Proposition 6.2.8(iii), W admits a finite affine cover $\coprod_{i=1}^n U_i \twoheadrightarrow W$; quasi-compactness is preserved by finite coproducts, so it suffices to handle each $P_i = U_i \times_{X \times X} X$ separately.

A map $U_i \to X \times X$ corresponds to a pair $u, v \colon U_i \to X$, and P_i fits into the cartesian square

$$\begin{array}{ccc} P_i & \longrightarrow & U_i \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta_{U_i}} \\ U_i \times_X U_i & \longrightarrow & U_i \times U_i. \end{array}$$

Now (a) $U_i \times_X U_i$ is quasi-compact by hypothesis (iii); (b) Δ_{U_i} is a closed immersion since U_i is affine (the multiplication $A \otimes A \to A$ is surjective); (c) the base change $P_i \to U_i \times_X U_i$ is therefore a closed immersion. A closed subfunctor of a quasi-compact functor is quasi-compact (Proposition 6.2.8(iii)), so P_i is quasi-compact.

(ii) ⇒ (i). For the structure map $X \to \mathrm{Spec}(\mathbb{Z})$, the relative diagonal coincides with the absolute $\Delta_X \colon X \to X \times X$ (since $\mathrm{Spec}(\mathbb{Z})$ is terminal), so (ii) says exactly that Δ_X is quasi-compact. Let $Y, Z \to X$ be quasi-compact; then $Y \times Z$ is quasi-compact (qc is preserved under finite products), and the cartesian square

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \times Z \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\Delta_X} & X \times X \end{array}$$

exhibits $Y \times_X Z$ as the pullback of $Y \times Z$ along Δ_X. Since Δ_X is qc by (ii), this pullback is qc by definition of qc morphism, giving (i). □

Proposition 6.2.10 (Composition of qc/qs Morphisms). *Consider a commutative triangle of algebraic functors*

$$\begin{array}{ccc} & Z & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle h} \\ Y & \xrightarrow{\quad f \quad} & X. \end{array}$$

(i) If f and g are quasi-compact, then h is quasi-compact.

(ii) If f and g are quasi-separated, then h is quasi-separated.

(iii) If h is quasi-compact and f is quasi-separated, then g is quasi-compact.

(iv) If h is quasi-separated, then g is quasi-separated.

(v) If h is quasi-compact and g is a Zariski-local epimorphism, then f is quasi-compact.

(vi) If h is quasi-separated and g is quasi-compact and a Zariski-local epimorphism, then f is quasi-separated.

Proof. We freely use Definition 6.2.7.

(i) Let $W \to X$ with W quasi-compact. Then $W \times_X Y$ is qc since f is qc, and $W \times_X Z \simeq (W \times_X Y) \times_Y Z$ is qc since g is qc.

(ii) The diagonal Δ_h factors as

$$Z \xrightarrow{\Delta_g} Z \times_Y Z \xrightarrow{\iota} Z \times_X Z,$$

where ι is the base change of Δ_f along $g \times_X g \colon Z \times_X Z \to Y \times_X Y$. Since Δ_f is qc (f is qs) and qc is stable under base change, ι is qc; combined with Δ_g qc (g is qs), (i) gives Δ_h qc.

(iii) Factor g through its graph: $g\colon Z \xrightarrow{\Gamma_g} Z \times_X Y \xrightarrow{\mathrm{pr}_Y} Y$. The projection is the base change of h, hence qc. The graph Γ_g fits into the cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\Gamma_g} & Z \times_X Y \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle g \times_X \mathrm{id}} \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y, \end{array}$$

so Γ_g is the base change of Δ_f (qc since f is qs), hence qc. Composition gives g qc.

(iv) The diagonal $\Delta_g\colon Z \to Z \times_Y Z$ is the base change of $\Delta_h\colon Z \to Z \times_X Z$ along the canonical map $Z \times_Y Z \to Z \times_X Z$. By hypothesis Δ_h is qc, and qc is stable under base change, so Δ_g is qc; i.e., g is qs.

(v) Let $W \to X$ with W quasi-compact; we show $W \times_X Y$ is qc. Pulling back $g\colon Z \to Y$ along $W \times_X Y \to Y$ gives a Zariski-local epimorphism

$$W \times_X Z \to W \times_X Y$$

(Zariski-local epimorphisms are stable under base change). The source $W \times_X Z$ is qc since h is qc and W is qc. By Proposition 6.2.8(iii), it admits a finite affine cover; composing with the surjection yields a finite Zariski-local epimorphism from a finite coproduct of affines onto $W \times_X Y$, so $W \times_X Y$ is qc.

(vi) We reduce to (v) applied to the diagonal. First, $g \times_X g\colon Z \times_X Z \to Y \times_X Y$ is qc, being the composition of two base changes of g. Next, $\Delta_h\colon Z \to Z \times_X Z$ is qc by hypothesis. By (i),

$$(g \times_X g) \circ \Delta_h\colon Z \to Y \times_X Y \text{ is qc.}$$

But this composition equals $\Delta_f \circ g$, since both send z to $(g(z), g(z)) \in Y \times_X Y$. Hence $\Delta_f \circ g$ is qc and g is a Zariski-local epimorphism, so (v) applied to the triangle $Z \xrightarrow{g} Y \xrightarrow{\Delta_f} Y \times_X Y$ yields Δ_f qc, i.e., f is qs. □

Proposition 6.2.11 (Base Change of qc/qs Morphisms)**.** *Consider a cartesian square of algebraic functors:*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X. \end{array}$$

If f is quasi-compact (resp. quasi-separated), so is f'. The converse holds if g is a Zariski-local epimorphism.

Proof. Forward direction (qc). For any $W \to X'$ with W qc, the pasting lemma gives $Y' \times_{X'} W \simeq Y \times_X W$, qc since f is qc.

Forward direction (qs). The cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{\Delta_{f'}} & Y' \times_{X'} Y' \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y \end{array}$$

exhibits $\Delta_{f'}$ as the base change of Δ_f along $Y' \times_{X'} Y' \to Y \times_X Y$. Since Δ_f is qc and qc is preserved under base change (just proved), $\Delta_{f'}$ is qc, so f' is qs.

Converse, g a Zariski-local epimorphism (qc). Suppose f' is qc; we show f is qc. Let $W \to X$ with W qc. By Proposition 6.2.8(iii), W admits a finite affine cover $\coprod_k \mathrm{Spec}(A_k) \twoheadrightarrow W$. Base change to Y gives a Zariski-local epimorphism $\coprod_k Y \times_X \mathrm{Spec}(A_k) \twoheadrightarrow Y \times_X W$, so it suffices to show each $Y \times_X \mathrm{Spec}(A_k)$ is qc. We may therefore reduce to the case $W = \mathrm{Spec}(A)$ affine.

The Zariski-local epimorphism g pulls back to a Zariski-local epimorphism $\mathrm{Spec}(A) \times_X X' \to \mathrm{Spec}(A)$, so there exist $f_1, \dots, f_n \in A$ with $(f_1, \dots, f_n) = A$ such that each $\mathrm{Spec}(A_{f_k}) \to \mathrm{Spec}(A) \to X$ factors through X'. For each such k,

$$Y \times_X \mathrm{Spec}(A_{f_k}) \simeq Y' \times_{X'} \mathrm{Spec}(A_{f_k}),$$

which is qc since f' is qc and $\mathrm{Spec}(A_{f_k})$ is affine, hence qc. The finite coproduct $\coprod_k Y \times_X \mathrm{Spec}(A_{f_k}) \twoheadrightarrow Y \times_X \mathrm{Spec}(A)$ is a Zariski-local epimorphism with qc source, so $Y \times_X \mathrm{Spec}(A)$ is qc by Proposition 6.2.8(iii).

Converse, g a Zariski-local epimorphism (qs). From the forward (qs) cartesian square above, $\Delta_{f'}$ is the base change of Δ_f along $Y' \times_{X'} Y' \to Y \times_X Y$, which is itself the base change of $X' \to X$ (using $Y' \times_{X'} Y' \simeq (Y \times_X Y) \times_X X'$). Zariski-local epimorphisms are stable under base change, so $Y' \times_{X'} Y' \to Y \times_X Y$ is a Zariski-local epimorphism. Since $\Delta_{f'}$ is qc by hypothesis, the qc converse just proved (applied to the cartesian square defining $\Delta_{f'}$) gives Δ_f qc, so f is qs. □

Example 6.2.12 (Quasi-compact Immersions). (i) By Proposition 6.2.8(iii), a closed subfunctor of a quasi-compact algebraic functor is itself quasi-compact. It follows that any closed immersion is quasi-compact. In particular, as the terminology suggests, every separated morphism is quasi-separated.

(ii) Let R be a ring and $I \subset R$. The cover $(D(f) \subset D(I))_{f \in I}$ shows that $D(I)$ is quasi-compact iff $\sqrt{(I)} = \sqrt{(J)}$ for some finite $J \subset I$. Hence an open immersion is quasi-compact iff its associated quasi-coherent radical ideal is finitely generated.

(iii) Any monomorphism is quasi-separated, since its diagonal is an isomorphism.

Example 6.2.13 (Quasi-affine and Quasi-projective Schemes). (i) By Example 6.2.12(ii), any quasi-affine scheme is quasi-compact; it is quasi-separated since it is separated (Example 6.2.4).

(ii) Let A be an $\mathbb{N}$-graded ring and $I \subset A$ a homogeneous subset. As in Example 6.2.12(ii), $D(I) \subset \mathrm{Proj}(A)$ is quasi-compact iff the saturated radical ideal generated by I is finitely generated. In particular, every quasi-projective k-scheme is quasi-compact, and is quasi-separated since it is separated.

(iii) By (i), (ii), and Proposition 6.2.11, any quasi-affine or quasi-projective morphism of algebraic functors is quasi-compact and quasi-separated.

For schemes, these finiteness conditions depend only on the underlying topological space and can be tested on any affine open cover.

Proposition 6.2.14 (Quasi-compactness for Schemes). *Let X be a scheme and $(U_i \subset X)_{i \in I}$ an open cover by affine schemes. The following are equivalent:*

(i) X is quasi-compact;

(ii) $|X|$ is quasi-compact as a topological space;

(iii) there exists a finite $J \subset I$ such that $(U_i)_{i \in J}$ already covers X.

Proof. *(i) ⇔ (iii).* The cover $(U_i \hookrightarrow X)_{i \in I}$ is a Zariski-local epimorphism with each U_i affine (hence qc). By Proposition 6.2.8(iii), X is qc iff the cover admits a finite refinement, i.e., a finite J as in (iii).

(ii) ⇔ (iii). The space $|X|$ is the colimit of $(|U_i|)$ in topological spaces (Corollary 4.3.13), and each $|U_i|$ is qc (the prime spectrum of an affine scheme is qc). Such a colimit is qc iff a finite subfamily covers, which is (iii). □

Proposition 6.2.15 (Quasi-separatedness for Schemes). *Let X be a scheme and $(U_i \subset X)_{i \in I}$ an open cover by affine schemes. The following are equivalent:*

(i) X is quasi-separated;

(ii) the intersection of any two quasi-compact open subsets of $|X|$ is quasi-compact;

(iii) for every $i, j \in I$, the intersection $U_i \cap U_j$ is quasi-compact.

Proof. (i) ⇒ (ii). Two qc opens $V, W \subset |X|$ correspond to qc subfunctors of X, and their intersection is the pullback $V \times_X W$, qc by qs of X.

(ii) ⇒ (iii). Each affine U_i is qc, so $U_i \cap U_j$ is qc by hypothesis.

(iii) ⇒ (i). We must show $\Delta_X : X \to X \times X$ is qc. The cover $(U_i \times U_j \hookrightarrow X \times X)_{i,j \in I}$ is a Zariski-local epimorphism, so by Proposition 6.2.11 (qc converse), it suffices to show each base change

$$\Delta_X \times_{X \times X} (U_i \times U_j) : U_i \cap U_j \to U_i \times U_j$$

is qc. The target $U_i \times U_j$ is affine, hence qs, and the source $U_i \cap U_j$ is qc by hypothesis (iii). For any qc $V \to U_i \times U_j$, the fiber product $(U_i \cap U_j) \times_{U_i \times U_j} V$ is qc by qs of $U_i \times U_j$, so the morphism $U_i \cap U_j \to U_i \times U_j$ is qc. □

***Remark* 6.2.16** (QCQS Schemes). Combining Propositions 6.2.14 and 6.2.15: a scheme is qcqs iff it admits a finite affine cover whose pairwise intersections are themselves quasi-compact. This finiteness propagates to many invariants — for instance, the underlying space of a qcqs scheme is spectral, and its category of quasi-coherent modules is compactly generated.

As Example 6.2.13 shows, most schemes of interest are qcqs. Examples failing quasi-compactness include infinite coproducts of nonempty schemes and projective spaces $\mathbb{P}^I_k$ for infinite I (over $k \neq 0$). An infinite-dimensional affine space with doubled origin is quasi-compact but not quasi-separated.

***Remark* 6.2.17** (Tannaka Duality and 1-Affineness). There is a deep conceptual reason why qcqs morphisms are central. An affine scheme $X = \mathrm{Spec}(A)$ is completely determined by its ring of global functions $A \simeq \Gamma(X, \mathcal{O}_X)$; one might say affine schemes are "0-affine".

For a general scheme X, the global section ring loses too much information. However, if X is qcqs, it can be reconstructed from the symmetric monoidal category $\mathsf{QCoh}(X)$. Specifically, Brandenburg [BC14] proved a form of *Tannaka duality*: the functor

$$\{\text{qcqs schemes}\} \to \{\text{symmetric monoidal presentable abelian categories}\}, \quad X \mapsto \mathsf{QCoh}(X)$$

is fully faithful. Thus, at the cost of passing from rings to symmetric monoidal abelian categories, the geometry of qcqs schemes is fully captured.

This phenomenon is formalised by Gaitsgory [Gai15] via 1-*affineness*. In the higher-categorical framework, an algebraic functor X is 1-affine if the ∞-category of quasi-coherent sheaves of categories on X, denoted $1\mathsf{Pr}_X$ (or $\mathsf{ShvCat}(X)$), is equivalent to modules over the symmetric monoidal ∞-category $D(X) := \mathrm{Mod}_X$:

$$1\mathsf{Pr}_X \xrightarrow{\sim} \mathrm{Mod}_{D(X)}(1\mathsf{Pr}).$$

Just as a 0-affine object (affine scheme) is the spectrum of a commutative algebra A (whose modules form Mod_A), a 1-affine object is the "spectrum" of a symmetric monoidal ∞-category $D(X)$ (whose "modules" form $1\mathsf{Pr}_X$). The qcqs condition ensures that a scheme behaves like a 1-affine object: it is determined by its category of modules rather than just its ring of functions.

Scholze and Stefanich generalise 1-affineness to n-affineness in the context of Stefanich rings; see Section B.6.

The following standard definition combines the local "algebraic" finiteness properties of Definition 1.4.1 with the global "topological" finiteness properties of this section.

Definition 6.2.18 (Morphism of Finite Presentation/Type). Let $f : X \to S$ be a morphism of algebraic functors.

(i) f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

(ii) f is *of finite type* if it is locally of finite type and quasi-compact.

For example, every quasi-projective morphism is of finite type.

6.3. Schemes as Locally Ringed Spaces

We now show that every scheme can be canonically regarded as a locally ringed space (Definition 5.5.4), and that this construction induces an equivalence onto a natural full subcategory of LocRingSpace.

Let X be an algebraic functor. Denote by $\mathcal{O}_X$ the presheaf of functions on $\mathsf{Open}(X) \simeq \mathrm{Open}(|X|)$, given by

$$\mathcal{O}_X(U) = \mathcal{O}(U) = \mathrm{Hom}_{\mathsf{Fun}(\mathsf{CAlg},\mathsf{Set})}(U, \mathbb{A}^1).$$

By Corollary 3.3.4, $\mathcal{O}_X$ is a sheaf of rings on $|X|$. This yields a functor

$$(|-|, \mathcal{O}_{(-)})\colon \mathsf{Fun}(\mathsf{CAlg},\mathsf{Set}) \longrightarrow \mathsf{RingSpace}, \quad X \mapsto (|X|, \mathcal{O}_X). \tag{6.1}$$

***Warning* 6.3.1** (Ambiguity of Notation $\mathcal{O}_X$). The symbol $\mathcal{O}_X$ is overloaded:

- In Notation 3.1.19, $\mathcal{O}_X$ denotes the initial object of CAlg_X (the category of quasi-coherent algebras on X); it serves as the monoidal unit of Mod_X.
- Above, $\mathcal{O}_X$ denotes the sheaf of rings $\mathcal{O}|_{\mathsf{Open}(X)}$ on $|X|$.

These are two restrictions of the global structure functor $\mathcal{O}\colon \mathsf{Fun}(\mathsf{CAlg},\mathsf{Set})^{\mathrm{op}} \longrightarrow \mathsf{CAlg}$ to X. We will see below that they agree when X is a scheme.

To justify that this functor produces the "correct" geometry, we first inspect the affine case. Recall from Example 5.6.7 the description of sheaves on $\mathrm{Spec}(R)$.

Example 6.3.2 (Prime Spectra as Locally Ringed Spaces). Let $X = \mathrm{Spec}(R)$. The ringed space $(|X|, \mathcal{O}_X)$ has the explicit description:

- $|X| = \mathrm{Prim}(R)$, the prime spectrum;
- under the equivalence $\mathsf{Shv}(\mathrm{Prim}(R)) \simeq \mathsf{Shv}(\mathsf{Open}^{\mathrm{pr}}(\mathrm{Spec}(R)))$, we have $\mathcal{O}_X(D(f)) = R_f$.

For a prime $\mathfrak{p} \in \mathrm{Prim}(R)$, the stalk is the local ring

$$\mathcal{O}_{X,\mathfrak{p}} = \mathrm{colim}_{\mathfrak{p}\in|D(f)|}\, \mathcal{O}_X(D(f)) = \mathrm{colim}_{f\notin\mathfrak{p}}\, R_f = R_{\mathfrak{p}}.$$

Since $R_{\mathfrak{p}}$ is local, $(\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is in fact a locally ringed space.

For any ring map $\varphi\colon R \to S$, the induced map $(\mathrm{Prim}(S), \mathcal{O}_{\mathrm{Spec}(S)}) \to (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a morphism of locally ringed spaces: at any $\mathfrak{q} \in \mathrm{Prim}(S)$, the map on stalks is the local homomorphism $\varphi_{\mathfrak{q}}\colon R_{\varphi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$.

Proposition 6.3.3 (Factorisation through Sheaves). *There is a unique factorisation*

$$\begin{array}{ccc}
\mathsf{Fun}(\mathsf{CAlg},\mathsf{Set}) & \xrightarrow{\Phi := (|-|, \mathcal{O}_{(-)})} & \mathsf{RingSpace} \\
{\scriptstyle a_{\mathrm{Zar}}}\big\downarrow & & \big\uparrow \\
\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{CAlg}^{\mathrm{op}}) & \overset{\bar{\Phi}}{\dashrightarrow} & \mathsf{LocRingSpace}.
\end{array}$$

In particular, the locally ringed space associated to an algebraic functor depends only on its associated Zariski sheaf.

Proof. We construct Φ and verify the factorisation in three steps.

Step 1: Construction of the locally ringed space. Since LocRingSpace is cocomplete (Proposition 5.5.9), the universal property of presheaves (Theorem 1.1.10(v)) extends $R \mapsto (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ uniquely along the Yoneda embedding to a colimit-preserving functor

$$\Phi := (|-|, \mathcal{O}_{(-)}) \colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \longrightarrow \mathsf{LocRingSpace}.$$

Explicitly, using the canonical presentation $X \simeq \mathrm{colim}_{(R,x)\in \mathrm{El}(X)} \mathrm{Spec}(R)$,

$$(|X|, \mathcal{O}_{|X|}) := \mathrm{colim}^{\mathsf{RingSpace}}_{(R,x)\in \mathrm{El}(X)} (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)}).$$

By Proposition 5.5.9, LocRingSpace $\hookrightarrow$ RingSpace preserves colimits, so $(|X|, \mathcal{O}_{|X|})$ is in fact a locally ringed space. Concretely, the structure sheaf is the limit of pushforwards

$$\mathcal{O}_{|X|} := \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} (\alpha_x)_* \mathcal{O}_{\mathrm{Spec}(R)},$$

which is itself a sheaf on $|X|$ (a limit of sheaves).

Step 2: Identification with the algebraic structure sheaf. We show the topologically constructed sheaf $\mathcal{O}_{|X|}$ coincides with the algebraically defined presheaf $\mathcal{O}_X$ ($\mathcal{O}_X(U) = \mathrm{Hom}_{\mathsf{Fun}}(U, \mathbb{A}^1)$).

Let $U \subset |X|$ be open, defining an open subfunctor $U \subset X$. Since colimits are stable under base change,

$$U \simeq \mathrm{colim}_{(R,x)\in \mathrm{El}(X)} x^{-1}(U),$$

where $x^{-1}(U) = \alpha_x^{-1}(U) \subset \mathrm{Spec}(R)$. Computing the algebraic side:

$$\begin{aligned}
\mathcal{O}_X(U) &= \mathrm{Hom}_{\mathsf{Fun}(\mathsf{CAlg},\mathsf{Set})}(U, \mathbb{A}^1) \\
&\simeq \mathrm{Hom}_{\mathsf{Fun}}\left(\mathrm{colim}_{(R,x)} x^{-1}(U), \mathbb{A}^1\right) \\
&\simeq \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Fun}}(x^{-1}(U), \mathbb{A}^1) \\
&\simeq \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathcal{O}_{\mathrm{Spec}(R)}(U_R),
\end{aligned}$$

which is exactly the section $\mathcal{O}_{|X|}(U)$ from Step 1. Hence $\mathcal{O}_X \simeq \mathcal{O}_{|X|}$ as sheaves on $|X|$. This shows in particular that $\mathcal{O}_X$ is a sheaf, and that $(|X|, \mathcal{O}_X)$ is a locally ringed space.

Step 3: Factorisation through Zariski sheafification. By construction, Φ is the left Kan extension along Aff $\hookrightarrow$ Fun(CAlg, Set) of $R \mapsto (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. Affine descent says this functor inverts every Zariski-covering family in the locally ringed space sense, so Φ inverts all Zariski-local isomorphisms. Since a_{Zar} is the universal such inversion, Φ factors uniquely through a_{Zar}. □

Theorem 6.3.4 (Adjunction between Rings and Locally Ringed Spaces). *The global section functor*

$$\Gamma \colon \mathsf{LocRingSpace} \longrightarrow \mathsf{CAlg}^{\mathrm{op}}, \quad (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$$

has a right adjoint given by the prime spectrum functor

$$\mathrm{Prim} \colon \mathsf{CAlg}^{\mathrm{op}} \longrightarrow \mathsf{LocRingSpace}, \quad R \mapsto (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)}).$$

***Remark* 6.3.5.** Since $\mathcal{O}(\mathrm{Spec}(R)) \simeq R$, the counit of this adjunction is an isomorphism, so Prim is fully faithful.

Proof. We must show that for every ring R and locally ringed space $X = (T, \mathcal{O}_T)$ there is a natural bijection

$$\mathrm{Hom}_{\mathsf{CAlg}}(R, \mathcal{O}_T(T)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{LocRingSpace}}\big((T, \mathcal{O}_T), (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})\big).$$

Construction $\psi \mapsto (f, \varphi)$. Given $\psi \colon R \to \mathcal{O}_T(T)$, we build a morphism (f, φ) of locally ringed spaces. For $x \in T$, consider the composite $R \xrightarrow{\psi} \mathcal{O}_T(T) \to \mathcal{O}_{T,x}$, and let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_{T,x}$. Define $f \colon T \to \mathrm{Prim}(R)$ by

$$f(x) := \{r \in R : \psi(r)_x \in \mathfrak{m}_x\},$$

clearly a prime ideal.

Step 1: f is continuous. For a basic open $D(g) \subset \mathrm{Prim}(R)$,

$$f^{-1}(D(g)) = \{x \in T : g \notin f(x)\} = \{x \in T : \psi(g)_x \in \mathcal{O}_{T,x}^{\times}\},$$

which is the open locus $T_{\psi(g)}$ where the section $\psi(g) \in \mathcal{O}_T(T)$ is invertible.

Step 2: Construction of the sheaf morphism φ. The sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ is determined by its values $\mathcal{O}_{\mathrm{Spec}(R)}(D(g)) \simeq R_g$ on the basis of principal opens. The map $\psi(g)|_{T_{\psi(g)}}$ is invertible by definition, so the universal property of localisation produces a unique factorisation

$$\begin{array}{ccccc} R & \xrightarrow{\psi} & \mathcal{O}_T(T) & \xrightarrow{\mathrm{res}} & \mathcal{O}_T(T_{\psi(g)}) \\ \downarrow & & & \nearrow_{\exists!\,\varphi_{D(g)}} & \\ R_g & & & & \end{array}$$

The compatibility on intersections $D(g) \cap D(h) = D(gh)$ follows from uniqueness of the localisation, giving a sheaf morphism $\varphi \colon \mathcal{O}_{\mathrm{Spec}(R)} \to f_* \mathcal{O}_T$.

Step 3: (f, φ) is a morphism of locally ringed spaces. For $x \in T$ with $\mathfrak{p} = f(x)$, the induced map on stalks

$$\varphi_x \colon R_{\mathfrak{p}} \to \mathcal{O}_{T,x}, \quad \frac{a}{s} \mapsto \psi(a)_x \cdot \psi(s)_x^{-1}$$

is well-defined: $s \notin \mathfrak{p}$ means $\psi(s)_x \notin \mathfrak{m}_x$, hence $\psi(s)_x \in \mathcal{O}_{T,x}^{\times}$. To see locality, note that $\frac{a}{s} \in \mathfrak{p}R_{\mathfrak{p}}$ requires $a \in \mathfrak{p}$, hence $\psi(a)_x \in \mathfrak{m}_x$, so $\varphi_x(\frac{a}{s}) \in \mathfrak{m}_x$.

Verification of bijection. Let $\Theta \colon (h, \theta) \mapsto \theta_{\mathrm{Spec}(R)} \colon R \simeq \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Prim}(R)) \to \mathcal{O}_T(T)$ be the candidate inverse, and $\Lambda \colon \psi \mapsto (f, \varphi)$ as above.

$\Theta \circ \Lambda = \mathrm{id}$. Setting $g = 1$ in Step 2, the unique factorisation $R_1 \simeq R \to \mathcal{O}_T(T)$ recovers exactly ψ.

$\Lambda \circ \Theta = \mathrm{id}$. Given (h, θ), set $\psi = \Theta(h, \theta)$ and $(\tilde{h}, \tilde{\theta}) = \Lambda(\psi)$. For locality: by hypothesis, $\theta_x \colon R_{h(x)} \to \mathcal{O}_{T,x}$ is local, meaning $r \in h(x)$ iff $\theta_x(r) \in \mathfrak{m}_x$ iff $\psi(r)_x \in \mathfrak{m}_x$. Hence $h(x) = \tilde{h}(x)$, so $h = \tilde{h}$. Since h and $\tilde{h}$ agree, and the sheaf maps θ and $\tilde{\theta}$ are determined on the basis $\{D(g)\}$ by the global section map $\psi = \Theta(h, \theta)$ via Step 2 (a property of the localisation construction), $\theta = \tilde{\theta}$. □

Recall from Notation 5.5.6 that for a collection $\mathcal{M}$ of (geometric) objects, $\mathsf{LocRingSpace}(\mathcal{M})$ denotes locally ringed spaces that locally look like objects in $\mathcal{M}$.

Theorem 6.3.6 (Schemes as Locally Ringed Spaces). *Let $\mathcal{P}rim$ be the collection of locally ringed spaces $(\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for R any ring. Then the realisation functor $(|-|, \mathcal{O}) \colon \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}) \to \mathsf{LocRingSpace}$ restricts to an equivalence of categories*

$$\mathsf{Sch} \xrightarrow{\sim} \mathsf{LocRingSpace}(\mathcal{P}rim).$$

Proof. By the realisation–nerve adjunction (Theorem 1.1.10(vi)), the colimit-preserving functor $\Phi = (|-|, \mathcal{O})$ has a right adjoint

$$G\colon \mathsf{LocRingSpace} \to \mathsf{Fun}(\mathsf{CAlg}, \mathsf{Set}), \quad G(T, \mathcal{O}_T)(R) = \mathrm{Hom}_{\mathsf{LocRingSpace}}\big((\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)}), (T, \mathcal{O}_T)\big).$$

Let η and ε denote the unit and counit. We show η_X is an isomorphism for every scheme X and ε_Y is an isomorphism for every $Y \in \mathsf{LocRingSpace}(\mathcal{P}rim)$.

Step 1: Full faithfulness (η_X is an isomorphism on schemes).

Affine base case. If $X = \mathrm{Spec}(R)$, then $\Phi(X) \simeq (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$, and η_X corresponds under Theorem 6.3.4 (and Remark 6.3.5) to the isomorphism $R \simeq \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$.

Descent step. Let X be a general scheme with affine open cover $(U_i)_{i\in I}$. We first observe that $G(Y)$ is a Zariski sheaf for any locally ringed space Y. This holds because affine schemes satisfy Zariski descent in LocRingSpace: for any standard Zariski cover $(\mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R))$, the induced map on geometric realisations

$$\coprod_{i,j} |\mathrm{Spec}(R_{f_i f_j})| \rightrightarrows \coprod_i |\mathrm{Spec}(R_{f_i})| \to |\mathrm{Spec}(R)|$$

is a coequaliser in LocRingSpace, and $\mathrm{Hom}_{\mathsf{LocRingSpace}}(-, Y)$ sends colimits to limits.

Both X (by definition) and $G(\Phi(X))$ (by the previous paragraph) are Zariski sheaves. Consider the commutative diagram with rows the resulting equaliser diagrams:

$$\begin{array}{ccccc}
X & \longrightarrow & \prod_i U_i & \rightrightarrows & \prod_{i,j} U_i \cap U_j \\
\downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \prod \eta_{U_i}} & & \downarrow{\scriptstyle \prod \eta_{U_i \cap U_j}} \\
G(\Phi(X)) & \longrightarrow & \prod_i G(\Phi(U_i)) & \rightrightarrows & \prod_{i,j} G(\Phi(U_i \cap U_j)).
\end{array}$$

Each η_{U_i} is an isomorphism by the affine base case. If X is separated, each $U_i \cap U_j$ is affine, so the third column is also componentwise an isomorphism, and a diagram chase forces η_X to be an isomorphism.

For general X: each $U_i \cap U_j$ is separated (since $U_i \cap U_j \hookrightarrow U_i$ is an open immersion, hence separated, and separatedness is preserved under base change to U_j). By the separated case just established, $\eta_{U_i \cap U_j}$ is an isomorphism. Hence η_X is an isomorphism.

Step 2: Essential surjectivity (ε_Y is an isomorphism on $\mathsf{LocRingSpace}(\mathcal{P}rim)$). Let $Y = (T, \mathcal{O}_T) \in \mathsf{LocRingSpace}(\mathcal{P}rim)$, with cover $(U_i)_{i\in I}$ of T where each $U_i \simeq \Phi(\mathrm{Spec}(R_i))$.

Set $X := G(Y)$. Right adjoints preserve limits, hence monomorphisms; in particular G takes open immersions in LocRingSpace to monomorphisms in Fun(CAlg, Set), and in fact to open subfunctors (which are detected by specific universal properties preserved by G). From Step 1 applied to affines, $G(U_i) \simeq G(\Phi(\mathrm{Spec}\, R_i)) \simeq \mathrm{Spec}\, R_i$, so X admits an affine open cover by the $\mathrm{Spec}\, R_i$, hence is a scheme.

Now consider $\varepsilon_Y : \Phi(X) \to Y$. Since Φ preserves colimits, $\Phi(X)$ is glued from $\Phi(G(U_i))$. Locally on U_i,

$$\varepsilon_{U_i} : \Phi(G(U_i)) \simeq \Phi(\mathrm{Spec}(R_i)) \xrightarrow{\sim} U_i$$

is an isomorphism (it is the counit of the affine adjunction, an isomorphism by Remark 6.3.5). Since ε_Y is locally an isomorphism in LocRingSpace, it is a global isomorphism. □

6.4. Quasi-Coherent Modules on Schemes

For an algebraic functor X, the category Mod_X of quasi-coherent modules (Definition 3.1.15) is by construction a limit of categories Mod_R over the category of elements, with transition functors given by base change f^* (Example 3.1.4).

For any ring R, Mod_R is a Grothendieck abelian category, and a limit of abelian categories with *exact* transition functors is again abelian. However, for a general ring map $\varphi\colon R \to S$, the base change $\varphi^* = (-)\otimes_R S$ is only right exact, not exact. Consequently, for a general algebraic functor X, the category Mod_X need not be abelian (kernels may not exist or behave well).

The situation improves dramatically for schemes. The key point: a scheme is glued from affines along open immersions, and an open immersion of affines $j\colon \mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(A)$ comes from a flat ring map $A \to B$ (e.g., a localisation $B = A_f$). Hence the gluing functor $j^*\colon \mathrm{Mod}_A \to \mathrm{Mod}_B$ is exact, not merely right exact.

Using Zariski descent for quasi-coherent modules (Proposition 3.3.3), we can describe Mod_X locally; since the gluing functors are exact, we obtain favourable abelian-categorical properties.

Theorem 6.4.1 (Abelian Property of Mod_X)**.** *Let X be a scheme. Then:*

(i) Mod_X *is an abelian category.*

(ii) For any open immersion $j\colon U \hookrightarrow X$, the restriction functor $j^*\colon \mathrm{Mod}_X \to \mathrm{Mod}_U$ is exact.

(iii) For any open cover $(j_i\colon U_i \hookrightarrow X)_{i\in I}$, the family $(j_i^*)_{i\in I}$ is jointly exact and conservative. Consequently, a morphism f in Mod_X is a monomorphism (resp. epimorphism, isomorphism) iff each restriction $f|_{U_i}$ is.

Proof. **(1) Local presentation as a limit of abelian categories with exact transitions.** Choose an affine open cover $(\mathrm{Spec}(R_i))_{i\in I}$ of X, and write $U_i = \mathrm{Spec}(R_i)$. By Zariski descent (Proposition 3.3.3),

$$\mathrm{Mod}_X \xrightarrow{\sim} \lim\Big(\prod_i \mathrm{Mod}_{U_i} \rightrightarrows \prod_{i,j} \mathrm{Mod}_{U_i\cap U_j} \rightrightarrows \prod_{i,j,k} \mathrm{Mod}_{U_i\cap U_j\cap U_k}\Big).$$

Each pairwise intersection $U_i \cap U_j$ is open in the affine U_i, so admits an affine open cover by principal opens; iterating, we may refine the limit to one indexed by a diagram of affines in which every transition map comes from an open immersion $\mathrm{Spec}(A_f) \hookrightarrow \mathrm{Spec}(A)$.

Each such transition functor is base change along a localisation $A \to A_f$, which is flat. Hence $\mathrm{Mod}_A \to \mathrm{Mod}_{A_f}$, $M \mapsto M\otimes_A A_f$, is exact. A limit of abelian categories along exact functors is abelian, with kernels, cokernels, and exact sequences computed termwise.

(2) Exactness of j^*. Let $j\colon U \hookrightarrow X$ be an open immersion. Choose an affine open cover $(\mathrm{Spec}(R_i))_i$ of X; each intersection $U \cap \mathrm{Spec}(R_i)$ is open in $\mathrm{Spec}(R_i)$ and so admits an affine cover by principal opens $\mathrm{Spec}((R_i)_{f_{i,k}})_k$. Together these give an affine open cover of U, and the descent presentation of j^* reduces to base change along the family of flat localisations $R_i \to (R_i)_{f_{i,k}}$. Each such base change is exact, hence so is j^*.

(3) Conservativity and joint exactness. The restriction $\prod_i j_i^*\colon \mathrm{Mod}_X \to \prod_i \mathrm{Mod}_{U_i}$ is the projection from the descent limit (Proposition 3.3.3) onto its first factor; in particular, a morphism in Mod_X is an isomorphism iff each component restriction is, so $\prod_i j_i^*$ is conservative. Each j_i^* is exact by (2), so the product is exact. An exact and conservative functor preserves and detects monomorphisms, epimorphisms, and exact sequences. ∎

Looking Ahead

The theory of schemes is now set up: Sch as a full subcategory of $\mathsf{Shv}_{\mathrm{Zar}}(\mathsf{Aff})$ closed under immersions and many categorical constructions; gluing via locally trivial equivalence relations; separatedness, quasi-compactness, quasi-separatedness, and finite presentation/finite type as the natural finiteness conditions; the classical locally-ringed-space picture recovered as an equivalent viewpoint; and quasi-coherent modules forming an abelian category with good local properties.

With the foundations complete, Part II of these notes turns to the *geometry* of schemes — qualitative properties of schemes and their morphisms. The next chapter introduces the basic toolkit: local properties (reduced, noetherian, regular, normal), global properties (irreducible, integral, qcqs), and the dimension theory that underlies most geometric reasoning on schemes. From there, Part II will go on to study divisors, blow-ups, smoothness, properness, and other core geometric constructions.

Part II.

Algebraic Geometry II (Geometry of Schemes)

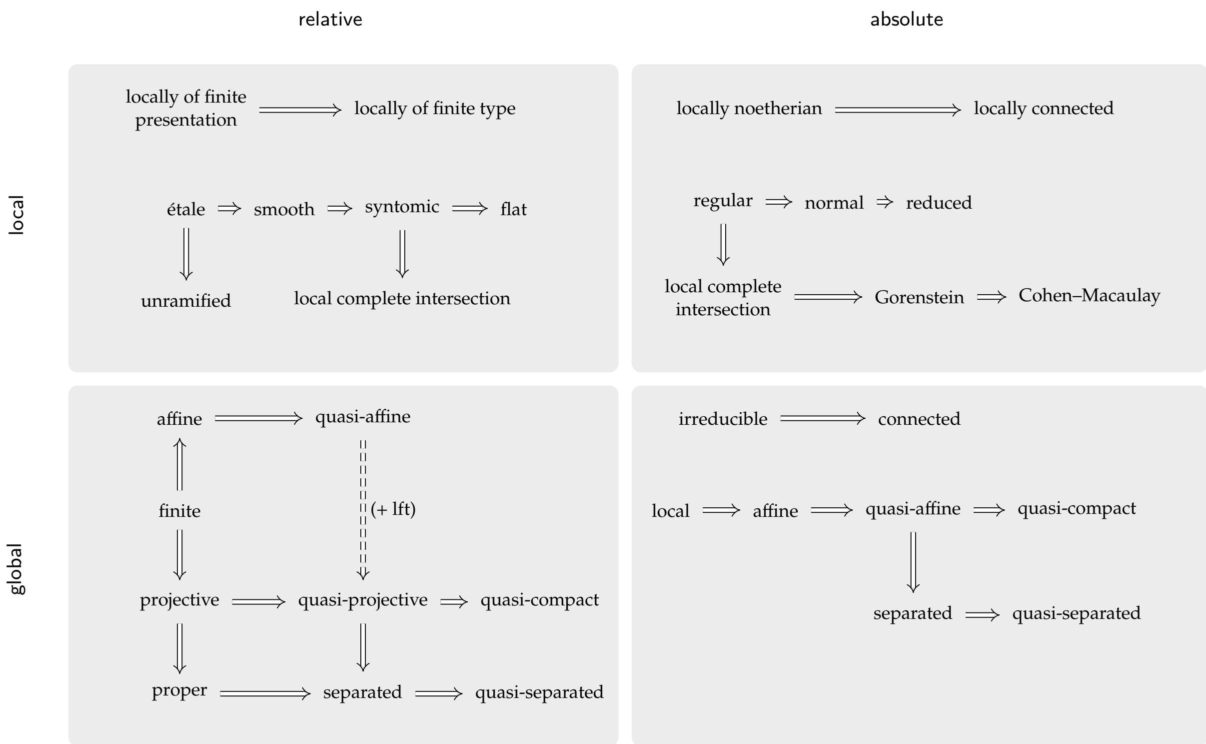
Chapter 7.

Properties of Schemes

To navigate the category of schemes one classifies its objects and morphisms by *properties*, organised along two orthogonal axes. A property is *local* when it can be checked in arbitrarily small Zariski neighbourhoods of every point, and *global* otherwise; it is *absolute* when it pertains to a scheme on its own, and *relative* when it pertains to a morphism.

Local properties of schemes are entirely controlled by commutative algebra: they are determined by their restrictions to affine opens. Typical examples are "locally of finite presentation" (Definition 6.2.18), "reduced" (Definition 7.2.1), "locally noetherian" (Definition 7.3.3), regularity, and normality. Affineness, connectedness, and quasi-compactness are global. On the relative side, projectivity is a basic example, depending intrinsically on the base.

The following chart lists the properties we will meet in this chapter, together with the standard implications among them. It is organised by the two orthogonal axes "local / global" and "relative / absolute"; each arrow is a one-way implication that we will justify later in the chapter.



Below the diagram, we record three convenient abbreviations that mix local and global:

of finite presentation	=	lfp + qcqs
of finite type	=	lft + qc
noetherian	=	locally noetherian + qc
integral	=	reduced + irreducible

***Remark* 7.0.1** (From Relative to Absolute)**.** Every relative property has an underlying absolute version, since every scheme is canonically a $\mathrm{Spec}(\mathbb{Z})$-scheme. The absolute versions of "affine", "quasi-affine", "quasi-compact", "quasi-separated", and "separated" are obtained in this way and are well-behaved. For other relative properties such as smoothness or properness the resulting absolute property is too restrictive to be useful (e.g. a nonempty $\mathbb{C}$-scheme can never be smooth or proper over $\mathrm{Spec}(\mathbb{Z})$).

7.1. Local Properties

The bridge from commutative algebra to scheme theory is a single bootstrapping principle: any reasonable property of rings extends uniquely to schemes via Zariski covers. We now make "reasonable" precise.

Definition 7.1.1 (Local Property of Rings)**.** A property P of (commutative) rings is *local* if:

(i) for every R with property P and every $f \in R$, the localisation R_f has property P;

(ii) whenever $f_1, \dots, f_n \in R$ generate the unit ideal and each R_{f_i} has property P, the ring R has property P.

Definition 7.1.2 (Local Property of Schemes)**.** A property P of schemes is *local* if:

(i) for every scheme X with property P and every open subscheme $U \subset X$, the scheme U has property P;

(ii) whenever $(U_i \subset X)_{i \in I}$ is an open cover and each U_i has property P, the scheme X has property P.

***Remark* 7.1.3** (Sheaf-Theoretic Reading)**.** For each scheme X the truth value of a local property of schemes assembles into a presheaf $P\colon \mathrm{Open}(X)^{\mathrm{op}} \to \{\bot \to \top\}$ on the poset of open subsets: condition (1) is functoriality, and condition (2) says that this presheaf is a sheaf for the canonical (Zariski) topology on $\mathrm{Open}(X)$.

***Remark* 7.1.4** (Local Properties as Zariski Sheaves)**.** There is a synthetic way to view Definition 7.1.1. A property P of rings determines a $\{0 \to 1\}$-valued predicate on $\mathrm{Ob}(\mathsf{CAlg})$. The two conditions then say:

- condition (i) demands that P be *monotone along principal localisations* $R \to R_f$;
- condition (ii) is the descent (sheaf) condition for principal Zariski covers $(R \to R_{f_i})_i$ with $(f_1, \dots, f_n) = R$.

Together these are precisely the data of a Zariski sheaf on $\mathsf{Aff} = \mathsf{CAlg}^{\mathrm{op}}$ valued in the propositional poset $\{0 \to 1\}$. From this vantage point Proposition 7.1.5 below is no longer ad hoc: it is the Comparison Lemma (Proposition 5.6.2) applied to the basis $\mathsf{Aff} \subset \mathsf{Sch}$ of affine schemes. A Zariski sheaf on Aff extends uniquely along Zariski descent to a Zariski sheaf on Sch, giving the asserted unique extension to schemes.

The following bootstrapping result is the engine that produces local properties of schemes from properties of rings.

Proposition 7.1.5 (Extending Absolute Local Properties)**.** *Every local property P of rings extends uniquely to a local property $\widehat{P}$ of schemes such that $\widehat{P}(\mathrm{Spec}(R)) \iff P(R)$.*

Proof. Define $\widehat{P}$ on schemes by declaring $\widehat{P}(X)$ to hold precisely when there exists an affine open cover $(\mathrm{Spec}(R_i) \hookrightarrow X)_{i \in I}$ with $P(R_i)$ for every i. We must check that this is compatible with P on affines and that it satisfies the two local-property axioms; uniqueness will then be automatic.

Compatibility on affines reduces to the converse direction $\widehat{P}(\mathrm{Spec}(R)) \Rightarrow P(R)$, the forward implication being witnessed by the singleton cover. So suppose $\widehat{P}(\mathrm{Spec}(R))$ is witnessed by an affine cover $(\mathrm{Spec}(S_j) \hookrightarrow \mathrm{Spec}(R))_{j\in J}$ with $P(S_j)$ for all j. The principal opens $\{D(f)\}_{f\in R}$ form a basis of $\mathrm{Spec}(R)$, so each $\mathrm{Spec}(S_j)$ is itself covered by basic opens $D(f) \subset \mathrm{Spec}(S_j)$; concretely, such an f is the image under $S_j \to R$ of some $g \in S_j$, and $R_f \simeq (S_j)_g$. By Definition 7.1.1(i), P propagates to each such R_f. We thus obtain a basic-open refinement $\{D(f_\lambda)\}_\lambda$ of the original cover with $P(R_{f_\lambda})$ for every λ. Quasi-compactness of $\mathrm{Spec}(R)$ extracts a finite subfamily $f_1, \ldots, f_n$ that already covers $\mathrm{Spec}(R)$, equivalently $(f_1, \ldots, f_n) = R$, and Definition 7.1.1(ii) then yields $P(R)$.

For stability under restriction to opens, suppose $\widehat{P}(X)$ is witnessed by an affine cover $(\mathrm{Spec}(R_i))_i$ and let $U \subset X$ be open. Each intersection $U \cap \mathrm{Spec}(R_i)$ is an open subfunctor of $\mathrm{Spec}(R_i)$, hence of the form $D(J_i)$ for some radical ideal $J_i \subset R_i$. Pick a generating set $\{f_{i,\alpha}\}$ of J_i. The principal opens $\{D(f_{i,\alpha})\}_{i,\alpha}$ then cover U by affines $\mathrm{Spec}((R_i)_{f_{i,\alpha}})$, each of which has property P by Definition 7.1.1(i); hence $\widehat{P}(U)$.

Stability under gluing is immediate. If $(U_i \subset X)_{i\in I}$ is an open cover with $\widehat{P}(U_i)$ for each i, picking affine open covers $(\mathrm{Spec}(R_{ij}))_j$ of each U_i with $P(R_{ij})$ produces a combined affine open cover $(\mathrm{Spec}(R_{ij}))_{i,j}$ of X witnessing $\widehat{P}(X)$.

Finally, uniqueness: any local property Q of schemes with $Q(\mathrm{Spec}(R)) \iff P(R)$ must satisfy $Q(X) \iff Q(\mathrm{Spec}(R_i))$ for all $i \iff P(R_i)$ for all i on every affine cover, and this is precisely the defining condition of $\widehat{P}(X)$. □

We now turn to relative versions. For morphisms $f : Y \to X$ there are two distinct "axes" along which a property can be local: along the source Y and along the base X.

Definition 7.1.6 (Locality for Morphisms of Schemes). Let P be a property of morphisms of schemes.

(i) P is *local on the source* if for every morphism $f : Y \to X$:
- if f has property P and $V \subset Y$ is open, then $f|_V : V \to X$ has property P;
- if $(V_i \subset Y)_{i\in I}$ is an open cover and each $f|_{V_i}$ has property P, then so does f.

(ii) P is *local on the base* if for every morphism $f : Y \to X$:
- if f has property P and $U \subset X$ is open, then the pullback $f_U : Y \times_X U \to U$ has property P;
- if $(U_i \subset X)_{i\in I}$ is an open cover and each f_{U_i} has property P, then so does f.

(iii) P is *local* if it is local both on the source and on the base.

(iv) P is *stable under base change* if for every cartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

the property $P(f)$ implies $P(f')$.

(v) P is *stable under composition* if it contains all isomorphisms and if $g \circ f$ has property P whenever both f and g do.

Definition 7.1.6 has an evident analogue for morphisms of rings (just apply Spec). The relative version of the extension proposition is then:

Proposition 7.1.7 (Extending Relative Local Properties). *Let P be a property of morphisms of (commutative) rings.*

(i) If P is local on the base, then P extends uniquely to a property $\widehat{P}$ of affine *morphisms of schemes that is local on the base.*

(ii) If P is local, then P extends uniquely to a local property $\widehat{P}$ of morphisms of schemes.

In both cases, stability under base change and stability under composition are inherited by $\widehat{P}$.

Proof. For (1), recall that a morphism $f : Y \to X$ of schemes is *affine* if for every map $\operatorname{Spec}(R) \to X$ the pullback $Y \times_X \operatorname{Spec}(R)$ is an affine scheme (Proposition 1.3.11). Define

$$\widehat{P}(f) \quad :\Longleftrightarrow \quad \text{there is an affine open cover } (\operatorname{Spec}(R_i) \hookrightarrow X)_i \text{ with } P\big(R_i \to \Gamma(Y \times_X \operatorname{Spec}(R_i))\big) \text{ for all } i.$$

That this is well defined—that is, independent of the cover—and locally on the base reduces, exactly as in the proof of Proposition 7.1.5, to a Zariski-cover refinement applied to the base X: given two cover witnesses, refine both to a common cover by basic opens of one of them, and use locality on the base for P to compare. The cover-refining argument transports verbatim because the affineness of f ensures each pullback is again Spec of the relevant localisation.

For (2), if P is also local on the source, the same construction performed simultaneously on the source yields a property local on both axes; uniqueness again follows by reducing any morphism to its restrictions to affine opens of source and base.

The inheritance of stability under base change and composition is straightforward: each axiom for $\widehat{P}$ can be verified on an affine cover using the corresponding axiom for P. □

***Remark* 7.1.8** (Zariski-Local versus Local on the Base)**.** A property of R-algebras is *Zariski-local* in the sense of Definition 1.3.8 if and only if it is local on the base in the sense of Definition 7.1.6. Both notions express the same descent statement: locality of the property along Zariski covers of the target.

***Remark* 7.1.9** (Source-Locality Often Suffices)**.** Most relative properties of practical interest are stable under base change and composition. For such P, locality on the source implies locality on the base: given a base cover $(U_i \subset X)_i$, pull it back to $(Y \times_X U_i \subset Y)_i$, an open cover of Y. If P holds for each restriction $f_{U_i} = f|_{Y \times_X U_i}$ then it holds for f by source-locality, while the converse is base change.

Example 7.1.10 (Local Finiteness Conditions)**.** "Locally of finite presentation" and "locally of finite type" (Definition 1.4.1) are local properties of morphisms of rings, so by Proposition 7.1.7 they extend uniquely to local properties of morphisms of schemes. Unwinding the construction: $f : Y \to X$ is locally of finite presentation (resp. finite type) iff every $y \in |Y|$ admits affine open neighbourhoods $V \ni y$ in Y and $U \ni f(y)$ in X with $f(V) \subset U$ such that $\Gamma(V)$ is a finitely presented (resp. finitely generated) $\Gamma(U)$-algebra.

Example 7.1.11 (Properties Local on the Base)**.** The following properties of morphisms of schemes are local on the base, but generally *not* on the source:

(i) affine;

(ii) quasi-affine;

(iii) finite;

(iv) quasi-compact;

(v) quasi-separated;

(vi) separated;

(vii) of finite presentation;

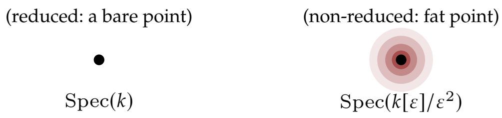
(viii) of finite type.

All eight are stable under base change and composition. Properties that fail to be local on the base are typically global in flavour: "projective" and "quasi-projective" are the standard examples. Properness, the local-on-base sibling of projectivity, repairs this defect.

7.2. Reduced Schemes

A ring R is *reduced* when its nilradical vanishes: $\mathrm{Nil}(R) = \sqrt{0} = 0$. The *reduction* $R^{\mathrm{red}} := R/\mathrm{Nil}(R)$ is the universal reduced quotient. We extend both notions to schemes.

Pictorially, $\mathrm{Spec}(k)$ is a single point; $\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ is that same point together with a *first-order infinitesimal thickening*, invisible to classical set-theoretic geometry but carrying nonzero nilpotents. Reducedness is precisely the absence of such thickenings.



Definition 7.2.1 (Reduced Scheme). A scheme X is *reduced* if there exists an affine open cover $(\mathrm{Spec}(R_i) \hookrightarrow X)_{i\in I}$ such that each R_i is reduced.

Lemma 7.2.2 (Reduced is Local for Rings). *The property "reduced" of rings is local in the sense of Definition 7.1.1.*

Proof. For (i), localisation commutes with the nilradical: $\mathrm{Nil}(R_f) = \mathrm{Nil}(R) \cdot R_f$, so $\mathrm{Nil}(R) = 0$ forces $\mathrm{Nil}(R_f) = 0$.

For (ii), suppose $(f_1, \dots, f_n) = R$ and each R_{f_i} is reduced. Given a nilpotent $a \in R$, its image in each R_{f_i} is also nilpotent, hence zero by reducedness. Now Zariski descent for elements applies: when (f_i) generates the unit ideal the canonical map $R \hookrightarrow \prod_i R_{f_i}$ is injective, so $a = 0$. □

Proposition 7.2.3 (Reduced is Local for Schemes). *The property "reduced" is a local property of schemes. Moreover,* $\mathrm{Spec}(R)$ *is reduced if and only if R is reduced.*

Proof. Apply Proposition 7.1.5 to Lemma 7.2.2. □

***Remark* 7.2.4** (Global Sections of Reduced Schemes). If X is reduced and $(\mathrm{Spec}(R_i))_{i\in I}$ is any affine open cover, then $\Gamma(X, \mathcal{O}_X)$ embeds into $\prod_{i\in I} R_i$ via the restriction maps. Each R_i is reduced and a subring of a reduced ring is reduced, so $\Gamma(X, \mathcal{O}_X)$ is reduced.

Example 7.2.5 (Reduced Proj). Let A be an $\mathbb{N}$-graded ring and $I \subset A_+$ a homogeneous subset with $A_+ \subset \sqrt{(I)}$. By Theorem 2.3.6 the affines $\mathrm{Spec}(A_{(f)})$ with $f \in I$ cover $\mathrm{Proj}(A)$, so reducedness of $\mathrm{Proj}(A)$ amounts to reducedness of every $A_{(f)}$ with $f \in I$.

A clean sufficient condition is that A itself be reduced: then each localisation A_f is reduced, and so is its degree-zero part $A_{(f)}$. In particular, if R is reduced then $\mathbb{P}^n_R$ is reduced for every $n \geq -1$.

Recall that on any scheme X we have an embedding $\mathsf{Rad}_X \hookrightarrow \mathsf{Id}_X$ of posets, picking out those quasi-coherent ideals I whose restriction $I(x_i) \subset R_i$ is radical for some (equivalently, every) affine open cover $(\mathrm{Spec}(R_i) \hookrightarrow X)_i$. Indeed, radicality is local by Proposition 3.2.1, and for affine $X = \mathrm{Spec}(R)$ the embedding $\mathsf{Rad}_R \hookrightarrow \mathsf{Id}_R$ is tautological.

Proposition 7.2.6 (Classification of Reduced Closed Subschemes). *Let X be a scheme. A closed subscheme $Z \hookrightarrow X$ is reduced if and only if the quasi-coherent ideal $I \in \mathsf{Id}_X$ with $V(I) = Z$ is radical.*

Proof. The question is local on X and reduces to the affine case: for $\operatorname{Spec}(R)$ affine and $J \subset R$ an ideal, the closed subscheme $\operatorname{Spec}(R/J)$ is reduced iff R/J is reduced (Proposition 7.2.3) iff $J = \sqrt{J}$.

To globalise, pick any affine cover $(\operatorname{Spec}(R_i) \hookrightarrow X)_i$. The pullback $Z \cap \operatorname{Spec}(R_i)$ corresponds to an ideal $I(x_i) \subset R_i$, and Z is reduced iff each $\operatorname{Spec}(R_i/I(x_i))$ is reduced (locality of "reduced" for schemes), iff each $I(x_i)$ is radical, iff $I \in \mathsf{Rad}_X$ (definition of $\mathsf{Rad}_X \hookrightarrow \mathsf{Id}_X$). □

Corollary 7.2.7 (Closed-Reduced Bijection). *For any scheme X the composition*

$$\mathsf{Closed}^{\mathrm{red}}(X) \hookrightarrow \mathsf{Closed}(X) \xrightarrow{|-|} \mathsf{Closed}(|X|)$$

is a bijection. In particular, the second map is surjective: every closed subset of $|X|$ is the underlying set of a unique reduced closed subscheme.

Proof. Three identifications combine. First, Proposition 7.2.6 gives $\mathsf{Closed}^{\mathrm{red}}(X) \simeq \mathsf{Rad}_X$, since reducedness of a closed subscheme corresponds to radicality of its defining ideal. Second, the classification of closed subfunctors (Proposition 3.2.1), applied locally on an affine cover, gives $\mathsf{Closed}(X) \simeq \mathsf{Id}_X$. Finally, the realisation $|-|$ identifies Rad_X with $\mathsf{Closed}(|X|)$: the locale of radical ideals of a ring is spatial (Theorem 4.0.1), and realisation commutes with Zariski gluing. □

***Remark* 7.2.8** (Subscheme Diagram). Combining Corollary 7.2.7 with the classification of open and closed subfunctors (Proposition 3.2.1) and the complementarity $U \leftrightarrow X \setminus U$ gives the following commutative diagram of order isomorphisms, where "c" denotes complementation:

$$\begin{array}{ccc}
\mathsf{Closed}(|X|)^{\mathrm{op}} & \xrightarrow[\sim]{c} & \mathrm{Open}(|X|) \\
\sim \updownarrow & & \updownarrow \sim \\
\mathsf{Closed}^{\mathrm{red}}(X)^{\mathrm{op}} & \xleftarrow{D} & \mathsf{Rad}_X \\
\big\downarrow |-| & & \big\downarrow \uparrow \sqrt{\cdot} \\
\mathsf{Closed}(X)^{\mathrm{op}} & \xleftarrow{V} & \mathsf{Id}_X
\end{array}$$

Example 7.2.9 (Reduced Closed Subschemes of Proj). Let A be an $\mathbb{N}$-graded ring. By Corollary 7.2.7 and Example 7.2.5, reduced closed subschemes of $\operatorname{Proj}(A)$ are in bijection with closed subsets of $|\operatorname{Proj}(A)|$, hence with saturated radical homogeneous ideals $I \subset A$ via the projective Nullstellensatz (Corollary 2.3.28):

$$\{\text{saturated radical homogeneous ideals of } A\} \xrightarrow{\sim} \mathsf{Closed}^{\mathrm{red}}(\operatorname{Proj}(A)), \qquad I \mapsto V(I).$$

Definition 7.2.10 (Reduction of a Scheme). Let X be a scheme. The *reduction* of X is the closed subscheme

$$X^{\mathrm{red}} := V\big(\sqrt{0}\big) \subset X$$

where $\sqrt{0} \in \mathsf{Rad}_X$ is the radical ideal corresponding to the entire underlying space $|X|$.

Proposition 7.2.11 (Reduction is a Right Adjoint). *Let $\mathsf{Sch}^{\mathrm{red}} \subset \mathsf{Sch}$ denote the full subcategory of reduced schemes. Then the inclusion $\mathsf{Sch}^{\mathrm{red}} \hookrightarrow \mathsf{Sch}$ has a right adjoint given by $X \mapsto X^{\mathrm{red}}$.*

Proof. Concretely, we must show that for every reduced scheme Y and every morphism $f : Y \to X$, the postcomposition map

$$\mathrm{Hom}_{\mathsf{Sch}}(Y, X^{\mathrm{red}}) \longrightarrow \mathrm{Hom}_{\mathsf{Sch}}(Y, X)$$

induced by the closed immersion $X^{\mathrm{red}} \hookrightarrow X$ is a bijection: f factors uniquely through $X^{\mathrm{red}} = V(\sqrt{0})$.

Such a factorisation is equivalent to vanishing of the pullback ideal $f^*(\sqrt{0}) \in \mathsf{Id}_Y$. Compute this pullback locally: on an affine open $\mathrm{Spec}(S) \hookrightarrow Y$, it equals $\sqrt{0_S} \subset S$. Since Y is reduced, every S in any affine cover is reduced, so $\sqrt{0_S} = 0$. Locality of Id_Y then promotes this to $f^*(\sqrt{0}) = 0$ globally, producing the desired unique factorisation. □

***Remark* 7.2.12** (Same Underlying Space, Smaller Structure Sheaf). The closed immersion $X^{\mathrm{red}} \hookrightarrow X$ is an isomorphism on underlying topological spaces, since the open complement is $D(\sqrt{0}) = D(0) = \emptyset$. As a locally ringed space, X^{red} has the form $(|X|, \mathcal{O}_X^{\mathrm{red}})$ where $\mathcal{O}_X^{\mathrm{red}}(U) = \Gamma(U^{\mathrm{red}}, \mathcal{O}_{U^{\mathrm{red}}})$. The sheaf $\mathcal{O}_X^{\mathrm{red}}$ is the sheafification of the presheaf $U \mapsto \Gamma(U, \mathcal{O}_X)^{\mathrm{red}}$: on affine opens both presheaves agree, and "reduce-then-sheafify" equals "sheafify-then-take-the-corresponding-quotient".

7.3. Noetherian Schemes

A ring R is *noetherian* when every ascending chain of ideals stabilises; passing through V, this is equivalent to the descending chain condition (DCC) on closed subschemes of $\mathrm{Spec}(R)$. We extend both formulations to schemes.

Proposition 7.3.1 (Characterisations of Noetherian Rings). *For a ring R, the following are equivalent:*

(i) R is noetherian.

(ii) Every submodule of a finitely generated R-module is finitely generated.

(iii) The category $\mathrm{Mod}_R^{\mathrm{ft}}$ of finitely generated R-modules is abelian and the inclusion $\mathrm{Mod}_R^{\mathrm{ft}} \hookrightarrow \mathrm{Mod}_R$ is exact.

(iv) Every R-module of finite type is of finite presentation: $\mathrm{Mod}_R^{\mathrm{ft}} = \mathrm{Mod}_R^{\mathrm{fp}}$.

(v) Every R-algebra of finite type is of finite presentation: $\mathsf{CAlg}_R^{\mathrm{ft}} = \mathsf{CAlg}_R^{\mathrm{fp}}$.

Proof. We establish the chain $(1) \Leftrightarrow (2) \Leftrightarrow (3)$, $(2) \Leftrightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

For $(1) \Rightarrow (2)$, take M finitely generated and $N \subset M$. Choose a surjection $R^n \twoheadrightarrow M$ and pull N back to a submodule $N' \subset R^n$; it suffices to show N' is finitely generated. Induct on n: the short exact sequence $0 \to R^{n-1} \to R^n \to R \to 0$ gives a short exact sequence $0 \to N' \cap R^{n-1} \to N' \to J \to 0$ with $J \subset R$ an ideal. ACC on ideals makes J finitely generated, the inductive hypothesis makes $N' \cap R^{n-1}$ finitely generated, and hence so is N'. The converse $(2) \Rightarrow (1)$ is the case $M = R$.

The equivalence $(2) \Leftrightarrow (3)$ unfolds as follows: that $\mathrm{Mod}_R^{\mathrm{ft}} \hookrightarrow \mathrm{Mod}_R$ is exact is the statement that finitely generated submodules and quotients of finitely generated modules are again finitely generated; the quotient direction is automatic, and the submodule direction is exactly (2).

For $(2) \Leftrightarrow (4)$: if M is of finite type, choose a surjection $R^n \twoheadrightarrow M$. Its kernel is finitely generated by (2), so M is finitely presented. The reverse runs the same argument backwards.

For $(4) \Rightarrow (5)$, let A be a finite-type R-algebra, presented as $R[x_1, \dots, x_n] \twoheadrightarrow A$. The kernel is an ideal in $R[x_1, \dots, x_n]$, which is noetherian by Hilbert's basis theorem (a consequence of (4) applied iteratively to free modules); hence the kernel is finitely generated and A is finitely presented.

For $(5) \Rightarrow (1)$, let $I \subset R$ be an arbitrary ideal. The quotient R/I is a finite-type R-algebra (presented by $R[\,] \twoheadrightarrow R/I$), so by (5) it is finitely presented over R. A finite presentation of R/I as an R-algebra exhibits I as the ideal generated by finitely many elements. Hence ACC on ideals. □

***Remark* 7.3.2** (Hilbert's Basis Theorem). Condition (5) of Proposition 7.3.1 says that finitely generated algebras over a noetherian ring are again noetherian. The special case $R[x_1, \dots, x_n]$ noetherian (when R is) is the classical *Hilbert basis theorem*.

Definition 7.3.3 (Locally Noetherian, Noetherian Scheme). A scheme X is *locally noetherian* if there is an affine open cover $(\mathrm{Spec}(R_i) \hookrightarrow X)_{i\in I}$ with each R_i noetherian. A scheme is *noetherian* if it is locally noetherian and quasi-compact.

Lemma 7.3.4 (Noetherian is Local for Rings). *The property "noetherian" of rings is local in the sense of Definition 7.1.1.*

Proof. For (i), localisations of noetherian rings are noetherian: ideals of R_f correspond to ideals of R disjoint from $\{f^n\}_{n\geq 0}$, and ACC is preserved under this correspondence.

For (ii), suppose $(f_1, \dots, f_n) = R$ and each R_{f_i} is noetherian. Given an ascending chain $J_0 \subset J_1 \subset \cdots$ in R, each localised chain $J_0 R_{f_i} \subset J_1 R_{f_i} \subset \cdots$ stabilises at some index N_i. Let $N = \max_i N_i$. For $k \geq N$ the quotient J_k/J_N vanishes after every localisation $R \to R_{f_i}$, and since (f_i) generates the unit ideal, Zariski descent for modules (Example 3.1.14) forces $J_k/J_N = 0$, i.e. $J_k = J_N$. □

Proposition 7.3.5 (Locally Noetherian is Local for Schemes). *"Locally noetherian" is a local property of schemes. Moreover,* $\mathrm{Spec}(R)$ *is locally noetherian if and only if R is noetherian.*

Proof. Apply Proposition 7.1.5 to Lemma 7.3.4. □

Example 7.3.6 (Noetherian Proj). Let A be an $\mathbb{N}$-graded ring and $I \subset A_+$ a homogeneous subset with $A_+ \subset \sqrt{(I)}$. Then:

- $\mathrm{Proj}(A)$ is *locally noetherian* iff each $A_{(f)}$ with $f \in I$ is noetherian;
- $\mathrm{Proj}(A)$ is moreover *quasi-compact* iff some *finite* such I exists.

If A itself is noetherian, both conditions are automatic: A_+ is then finitely generated as an ideal (so we may take I finite), and each $A_{(f)}$ is the degree-zero part of the noetherian localisation A_f, hence noetherian. Thus, when A is noetherian $\mathrm{Proj}(A)$ is noetherian. In particular, $\mathbb{P}^n_R$ is noetherian whenever R is, for every $n \geq -1$.

***Remark* 7.3.7** (Coherent Rings). A ring R is *coherent* if the inclusion $\mathrm{Mod}_R^{\mathrm{fp}} \hookrightarrow \mathrm{Mod}_R$ is closed under kernels — equivalently, $\mathrm{Mod}_R^{\mathrm{fp}}$ is an abelian subcategory of Mod_R, and the inclusion is exact. Noetherian rings are coherent by Proposition 7.3.1(3), but the converse fails: every valuation ring is coherent, and so is the polynomial ring $R[x_i \mid i \in I]$ for any noetherian R and any (possibly infinite) set I. "Coherent" is local, so by Proposition 7.1.5 it extends to a local property of schemes.

Proposition 7.3.8 (Noetherian iff DCC on Closed Subschemes). *A scheme X is noetherian if and only if every descending chain $Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ of closed subschemes in X stabilises.*

Proof. Suppose first that X is noetherian, and pick a finite affine open cover $(\mathrm{Spec}(R_i))_{i=1}^n$ with each R_i noetherian. Given a descending chain $(Z_k)_k$ of closed subschemes, each restriction $Z_k \cap \mathrm{Spec}(R_i)$ corresponds to an ascending chain of ideals in R_i, which stabilises at some index N_i. Setting $N = \max_i N_i$, every restriction $Z_k \cap \mathrm{Spec}(R_i)$ for $k \geq N$ already equals $Z_N \cap \mathrm{Spec}(R_i)$, and Zariski descent for closed subschemes promotes this to $Z_k = Z_N$ globally.

Conversely, assume DCC for closed subschemes of X. We must show X is quasi-compact and that every affine open $\mathrm{Spec}(R) \subset X$ has R noetherian.

For quasi-compactness, let $(U_\alpha)_{\alpha\in A}$ be any open cover, with closed complements $Z_\alpha = X \setminus U_\alpha$. Suppose for contradiction that no finite subfamily of (U_α) covers X, i.e. for every finite $F \subset A$ the intersection

$Z_F := \bigcap_{\alpha\in F} Z_\alpha$ is nonempty. Build a strictly descending sequence inductively. Pick any $\alpha_0 \in A$ and set $F_0 = \{\alpha_0\}$, so $Z_{F_0} \neq \emptyset$. Having constructed F_k with $Z_{F_k} \neq \emptyset$, pick a point $p \in Z_{F_k}$. Since (U_α) covers X, there exists α_{k+1} with $p \in U_{\alpha_{k+1}}$, equivalently $p \notin Z_{\alpha_{k+1}}$. Then $F_{k+1} := F_k \cup \{\alpha_{k+1}\}$ satisfies $Z_{F_{k+1}} \subsetneq Z_{F_k}$, since the point $p \in Z_{F_k}$ is excluded from $Z_{F_{k+1}}$. The resulting strictly descending chain $Z_{F_0} \supsetneq Z_{F_1} \supsetneq \cdots$ contradicts DCC. Hence some finite subfamily covers X.

For noetherianity of affine pieces, let $\mathrm{Spec}(R) \subset X$ be an affine open. An ascending chain of ideals $J_0 \subset J_1 \subset \cdots$ in R produces a descending chain of closed subschemes $V(J_0) \supset V(J_1) \supset \cdots$ inside $\mathrm{Spec}(R)$. Adjoining the fixed complement $X \setminus \mathrm{Spec}(R)$ extends this to a descending chain of closed subschemes of X, which stabilises by hypothesis; therefore the original chain in R stabilises. □

Definition 7.3.9 (Noetherian Space). A topological space T is *noetherian* if every descending sequence of closed subsets $Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ in T stabilises.

***Remark* 7.3.10** (Noetherian Implies Quasi-Compact). A noetherian space T is quasi-compact: any open cover gives by complementation a descending chain of closed sets under "remove one more open"; DCC forces the chain to stabilise, which happens exactly when a finite subcover has been reached. Moreover, any subspace of T is noetherian (the DCC restricts), hence also quasi-compact.

***Remark* 7.3.11** (Underlying Space of Noetherian Scheme). By Proposition 7.3.8 and Corollary 7.2.7, if X is a noetherian scheme then the underlying space $|X|$ is a noetherian space. The converse fails: for k a field, the ring $R = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ is non-noetherian (the chain $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$ never stabilises), yet $|\mathrm{Spec}(R)|$ is a single point since $R^{\mathrm{red}} = k$.

***Remark* 7.3.12** (Noetherian Induction). To prove that every closed subset Z of a noetherian space T has a property P, it suffices to show that Z has P whenever every proper closed $W \subsetneq Z$ does. Indeed, if some closed subset failed P, the family of closed counterexamples would, by DCC, contain a minimal element Z_0; every proper closed $W \subsetneq Z_0$ would then have P, forcing Z_0 itself to have P by hypothesis — contradiction. The analogous statement holds for closed subschemes of a noetherian scheme (by Proposition 7.3.8). This method of proof is called *noetherian induction*.

Proposition 7.3.13 (Finite Type versus Finite Presentation). *Let X be locally noetherian.*

(i) X is quasi-separated.

(ii) If $f : Y \to X$ is locally of finite type, then Y is locally noetherian and f is locally of finite presentation.

(iii) If X is noetherian and $f : Y \to X$ is of finite type, then Y is noetherian and f is of finite presentation.

Proof. For (1), quasi-separatedness asks that for every pair of affine opens $U, V \subset X$ the intersection $U \cap V$ be quasi-compact. Localising on $U = \mathrm{Spec}(R)$ with R noetherian, $|U|$ is a noetherian space (Remark 7.3.11), so any open $|U \cap V| \subset |U|$ is again noetherian (Remark 7.3.10), in particular quasi-compact.

For (2), both claims are local on X, so we may reduce to $X = \mathrm{Spec}(R)$ with R noetherian. Cover Y by affines $\mathrm{Spec}(A_i)$ with each A_i a finitely generated R-algebra; by Hilbert's basis theorem each A_i is noetherian, so Y is locally noetherian, and by Proposition 7.3.1(5) each A_i is finitely presented, so f is locally of finite presentation.

For (3), combine (2) with the fact that quasi-compactness propagates: a finite-type morphism with quasi-compact target has quasi-compact source. □

7.4. Irreducible Components

A topological space T is *irreducible* if it is nonempty and any two nonempty opens meet (equivalently, T is not a union of two proper closed subsets). For sober spaces — e.g. underlying spaces of schemes — irreducibility is equivalent to the existence of a *generic point* η with $\overline{\{\eta\}} = T$. A scheme X is irreducible iff $|X|$ is (Definition 4.3.8).

Definition 7.4.1 (Irreducible Components). (i) The *irreducible components* of a topological space T are the maximal irreducible subsets of T.

(ii) The *irreducible components* of a scheme X are the irreducible components of $|X|$.

***Remark* 7.4.2** (Basic Facts on Irreducible Components). Let T be a topological space.

(i) By Zorn's lemma, every irreducible subset of T is contained in a maximal one, i.e. in an irreducible component. Since singletons are irreducible, T is the union of its irreducible components.

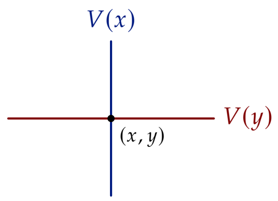
(ii) A subset $A \subset T$ is irreducible iff its closure $\overline{A}$ is. Consequently, every irreducible component of T is closed.

Example 7.4.3 (Irreducible Components of Affine Schemes). Let R be a ring. The bijection $\mathsf{Rad}_R \xrightarrow{\sim} \mathsf{Closed}(\mathrm{Prim}(R))$, $I \mapsto V(I) = \{\mathfrak{p} \mid I \subset \mathfrak{p}\}$ (Theorem 4.0.1), restricts to an inclusion-reversing bijection between primes and irreducible closed subsets. Consequently, the irreducible components of $\mathrm{Spec}(R)$ correspond to the *minimal* primes of R, and $\mathrm{Spec}(R)$ is irreducible iff R has a unique minimal prime, iff $\mathrm{Nil}(R)$ is prime.

Example 7.4.4 (Irreducible Components of Proj). Let A be an $\mathbb{N}$-graded ring. By the projective Nullstellensatz (Corollary 2.3.28), the closed subsets of $|\mathrm{Proj}(A)|$ correspond to saturated radical homogeneous ideals $I \subset A$. Under this bijection, irreducible closed subsets correspond to saturated homogeneous *prime* ideals, and hence irreducible components correspond to the minimal such. In particular, $\mathrm{Proj}(A)$ is irreducible iff $\mathrm{Nil}(A)^{\mathrm{sat}}$ is prime.

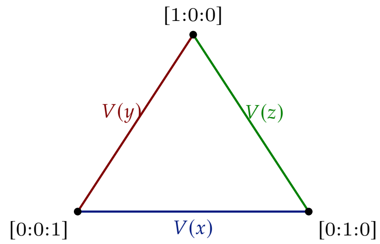
Example 7.4.5 (Two Concrete Examples). Let k be a field.

(i) The affine k-scheme $\mathrm{Spec}(k[x,y]/(xy))$ has exactly two irreducible components, corresponding to the minimal primes (x) and (y) in $k[x,y]/(xy)$. These are the two coordinate axes; they meet only at the origin (x,y).



(ii) The projective k-scheme $\mathrm{Proj}(k[x,y,z]/(xyz))$ has three irreducible components, corresponding to the minimal saturated prime ideals (x), (y), (z). These are three lines in $\mathbb{P}^2_k$ arranged in a triangle;

each pair meets at a single point away from the third component.



Proposition 7.4.6 (Noetherian Spaces Have Finitely Many Components). *A noetherian topological space has only finitely many irreducible components. In particular, a noetherian scheme has only finitely many irreducible components.*

Proof. We show by noetherian induction (Remark 7.3.12) on closed subsets $Z \subset T$ that every Z has only finitely many irreducible components; the case $Z = T$ then yields the proposition.

So fix Z and assume the conclusion for every proper closed $W \subsetneq Z$. If Z itself is irreducible there is nothing to prove. Otherwise, write $Z = Z_1 \cup Z_2$ with both pieces proper closed. Each Z_j has finitely many irreducible components by the inductive hypothesis. Any irreducible subset of Z lies entirely in Z_1 or in Z_2 (else its decomposition would contradict irreducibility), so the irreducible components of Z are precisely the maximal elements of the union of components of Z_1 and of Z_2 — a finite set. □

Definition 7.4.7 (Integral Scheme, Function Field). A scheme X is *integral* if it is irreducible and reduced. The *function field* (or *field of rational functions*) of an integral scheme X is the residue field $\kappa(\eta)$ at the generic point $\eta \in |X|$.

Example 7.4.8 (Integral Affine Schemes). A scheme $\mathrm{Spec}(R)$ is integral iff R is an integral domain, in which case the function field is $\mathrm{Frac}(R)$. Indeed, $\mathrm{Spec}(R)$ is reduced iff R is reduced (Proposition 7.2.3), and irreducible iff $\mathrm{Nil}(R)$ is prime (Example 7.4.3); together these say (0) is prime, i.e. R is an integral domain.

Proposition 7.4.9 (Properties of Integral Schemes). *Let X be an integral scheme with generic point η.*

(i) For every nonempty open subscheme $U \subset X$, the restriction map $\mathcal{O}_X(U) \to \kappa(\eta)$ is injective.

(ii) For every nonempty affine open $U \subset X$, the map $\mathcal{O}_X(U) \to \kappa(\eta)$ exhibits $\kappa(\eta)$ as the fraction field of $\mathcal{O}_X(U)$.

(iii) For every point $x \in |X|$, the map $\mathcal{O}_{X,x} \to \kappa(\eta)$ exhibits $\kappa(\eta)$ as the fraction field of the local ring $\mathcal{O}_{X,x}$.

Proof. We start with the affine case (2). Let $U = \mathrm{Spec}(R)$ be nonempty. Irreducibility of X forces $\eta \in U$, and U inherits both irreducibility (any nonempty open of an irreducible space is irreducible) and reducedness (locality of reducedness, Proposition 7.2.3). Hence U is integral, so by Example 7.4.8 R is a domain with $\kappa(\eta) = \mathrm{Frac}(R)$.

For (1), pick an affine $V = \mathrm{Spec}(R) \subset U$ and observe that $\mathcal{O}_X(U) \to R$ is injective: cover U by such affines $V_\alpha = \mathrm{Spec}(R_\alpha)$ (each R_α a domain by (2)); reducedness gives the embedding $\mathcal{O}_X(U) \hookrightarrow \prod_\alpha R_\alpha$ (Remark 7.2.4). Postcomposing with $R_\alpha \hookrightarrow \mathrm{Frac}(R_\alpha) = \kappa(\eta)$ yields the injection $\mathcal{O}_X(U) \hookrightarrow \kappa(\eta)$.

For (3), fix $x \in |X|$ and an affine open $\mathrm{Spec}(R) \ni x$ corresponding to a prime $\mathfrak{p}_x \subset R$. The stalk $\mathcal{O}_{X,x} = R_{\mathfrak{p}_x}$ is a localisation of the domain R; its fraction field equals $\mathrm{Frac}(R) = \kappa(\eta)$, since localisations of a domain share the same fraction field. □

Example 7.4.10 (Integral Proj). Let A be an integral $\mathbb{N}$-graded ring with $A_+ \neq 0$. Then $\operatorname{Proj}(A)$ is integral: reducedness comes from Example 7.2.5, and irreducibility from Example 7.4.4 (the hypothesis $A_+ \neq 0$ forces $(0)^{\mathrm{sat}} = (0)$ in the integral ring A, so $(0)^{\mathrm{sat}}$ is prime).

By Proposition 7.4.9(2), the function field of $\operatorname{Proj}(A)$ equals $\operatorname{Frac}(A_{(f)})$ for any nonzero homogeneous $f \in A_+$, which one identifies with $\operatorname{Frac}(A)_0$, the degree-zero part of the fraction field of A. As a special case, for R a domain with fraction field K and any $n \geq 0$, $\mathbb{P}^n_R$ is integral with function field $K(x_1, \dots, x_n)$.

***Remark* 7.4.11** (Sheaf of Rational Functions). For any scheme X, one defines the *sheaf of rational functions* $\mathcal{K}_X$ on $|X|$ as the sheafification of the presheaf

$$U \longmapsto \mathcal{O}_X(U)[S_U^{-1}],$$

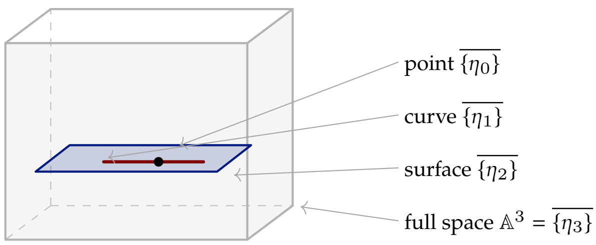
where $S_U \subset \mathcal{O}_X(U)$ is the multiplicative subset of all sections f such that $f|_V$ is a non-zero-divisor in $\mathcal{O}_X(V)$ for every open $V \subset U$. Sheafification being left exact, the canonical map $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$ is a monomorphism. If X is integral with generic point η, then $\mathcal{K}_X$ is the constant sheaf with value $\kappa(\eta)$. If X is locally noetherian, one can identify

$$\mathcal{K}_X(X) = \operatorname{colim}_{U \subset X} \mathcal{O}_X(U),$$

where the colimit ranges over open subschemes $U \subset X$ whose closure is X (the so-called *schematically dense* opens).

7.5. Regular and Normal Schemes: Dimension Theory

Geometrically, the dimension of a space is the length of the longest nested chain of "subspaces of decreasing size" it admits: a 3-dimensional space contains a surface, contains a curve, contains a point, in four strict layers. For schemes, where there is no a priori notion of subspace beyond closed subschemes, Krull translated this idea into pure topology: count the longest chain of irreducible closed subsets. The choice of irreducible closed sets (rather than arbitrary closed subsets, or open ones) is forced by sobriety: in a sober space such chains are exactly chains of points under specialisation, so each layer of the chain has a unique generic point and dimension counts "how deep specialisation can run". The picture below in $\mathbb{A}^3$ illustrates the prototypical chain of length 3.



$$\{\text{point}\} \subsetneq \text{curve} \subsetneq \text{surface} \subsetneq \mathbb{A}^3$$

codimensions $3,\ 2,\ 1,\ 0$ (a chain of length $3 = \dim \mathbb{A}^3$)

Regularity and normality are local conditions on schemes that refine "reduced"; both are most naturally explained through Krull dimension. We collect the dimension-theoretic background here. The deeper theorems — Krull's principal ideal theorem and the dimension formula — are quoted from commutative algebra [Mat89, ch. 5].

Definition 7.5.1 (Chains and Length in a Poset). Let $(P, \leq)$ be a poset. A *chain of length n* in P is a sequence $x_0 \lneq x_1 \lneq \cdots \lneq x_n$ of strictly increasing elements. The *length* of P is

$$\ell(P) := \sup\{n \in \mathbb{N} \mid \text{there exists a chain of length } n \text{ in } P\} \in \mathbb{N} \cup \{\infty\}.$$

Definition 7.5.2 (Krull Dimension, Codimension). Let T be a topological space, ordered by inclusion on irreducible closed subsets.

(i) The *Krull dimension* of T is $\dim(T) := \ell(\{\text{irreducible closed subsets of } T\})$.

(ii) For $K \subset T$ irreducible closed, the *codimension* of K in T is

$$\operatorname{codim}(K, T) := \ell(\{\text{irreducible closed subsets of } T \text{ containing } K\}) \in \mathbb{N} \cup \{\infty\}$$

(the poset is nonempty since it contains K).

For a scheme X, set $\dim(X) := \dim(|X|)$ and $\operatorname{codim}(K, X) := \operatorname{codim}(K, |X|)$ for $K \subset |X|$ irreducible closed; for a ring R, $\dim(R) := \dim(\operatorname{Spec}(R))$, which equals the length of the longest chain of primes in R.

***Remark* 7.5.3** (Notational Convention for the Rest of the Chapter). Krull dimension and codimension are intrinsically topological invariants: rigorously they are computed on $|X|$, not on X itself. To keep notation light we will write $\dim(X)$ and $\operatorname{codim}(K, X)$ in place of $\dim(|X|)$ and $\operatorname{codim}(K, |X|)$ throughout this and the remaining sections of the chapter, and similarly speak of an "irreducible closed subset $K \subset X$" meaning a closed subscheme whose underlying space is irreducible (equivalently, an irreducible closed subset of $|X|$). The context makes the intended reading unambiguous.

***Remark* 7.5.4** (Specialisation Order). Given $x, y \in T$, write $x \rightsquigarrow y$ and say that *x specialises to y* if $y \in \overline{\{x\}}$; this defines a preorder on the underlying set of T, which becomes a partial order precisely when T is T_0 (in particular when T is sober). For any sober space the closure map is an isomorphism of posets

$$(T, \rightsquigarrow) \xrightarrow{\sim} (\{\text{irreducible closed subsets of } T\}, \supset), \qquad x \mapsto \overline{\{x\}}.$$

Definition 7.5.2 can therefore be rephrased in terms of the poset of points: for example,

$$\dim(T) = \ell(T, \rightsquigarrow) = \sup_{x \in T} \operatorname{codim}(x, T).$$

***Remark* 7.5.5** (Dimension is Local). (i) If $(T_i \subset T)_{i \in I}$ is either an open covering or a closed covering, then $\dim(T) = \sup_i \dim(T_i)$: any irreducible chain in T has its maximum in some T_i, and irreducible closed subsets of T_i are also irreducible in T.

(ii) If $T' \subset T$ is any subspace, then $\dim(T') \leq \dim(T)$.

(iii) For a scheme X and $x \in X$, one has $\operatorname{codim}(x, X) = \dim(\mathcal{O}_{X,x})$ (exercise).

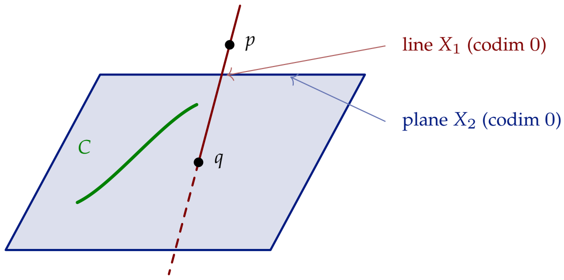
***Remark* 7.5.6** (Naive Desiderata Fail). Two innocent-looking desiderata about dimension are *false* in general:

(i) $\dim(U) = \dim(X)$ for every nonempty open $U \subset X$;

(ii) $\dim(V(f)) = \dim(X) - 1$ for every $f \in \mathcal{O}(X)$.

Both already fail for $X = \operatorname{Spec}(k[x]_{(x)})$, the spectrum of a local DVR with $\dim X = 1$ (the unique nontrivial chain of primes is $(0) \subsetneq (x)$). Removing the closed point gives the field $U = \operatorname{Spec} k(x)$ with $\dim U = 0$, breaking (1); choosing f to be a unit in $\mathcal{O}(X)$ produces $V(f) = \emptyset$, breaking (2). Both desiderata do hold in controlled forms — under flatness, or for integral schemes of finite type over a field — as the theorems below make precise.

The same subtlety persists when irreducible components meet. The figure below shows $X = X_1 \cup X_2$ with X_1 a line meeting X_2 a plane transversally at a single point: the codimension of a closed subset of X depends genuinely on which irreducible component of X contains it.



Codimension in $X = X_1 \cup X_2$ depends on which component the subset sits inside:
- $C \subset X_2$: a curve in the plane, so $\operatorname{codim}(C, X) = 1$;
- $p \in X_1$: a point on the line away from the plane, so $\operatorname{codim}(p, X) = 1$;
- $q \in X_1 \cap X_2$: codim 1 in X_1 but codim 2 in X_2, so $\operatorname{codim}(q, X) = 2$.

We turn from these pathologies to the simplest positive computation: the dimension of a PID. This will be the workhorse base case for everything that follows.

Example 7.5.7 (Dimension of Principal Ideal Domains). For a principal ideal domain R,

$$\dim(R) = \begin{cases} 0 & \text{if } R \text{ is a field,} \\ 1 & \text{otherwise.} \end{cases}$$

Indeed, in a PID every nonzero prime is maximal (generated by an irreducible element), so the longest chain of primes is $(0) \subsetneq (\pi)$. Consequently, if X is a scheme all of whose local rings are PIDs, then $\dim(X) \leq 1$.

We quote the fundamental trio of noetherian dimension theorems.

Theorem 7.5.8 (Krull's Principal Ideal Theorem). *Let R be a noetherian ring, $X = \operatorname{Spec}(R)$, and $f \in R$. For any irreducible component K of $V(f)$,*

$$\operatorname{codim}(K, X) = \begin{cases} 1 & \text{if } f \text{ is not a zero-divisor in } R^{\mathrm{red}}, \\ 0 & \text{otherwise.} \end{cases}$$

***Remark* 7.5.9** (Local Form: A Proof Sketch). Localising at the generic point of an irreducible component of $V(f)$ reduces Theorem 7.5.8 to the following local statement, whose proof we sketch since it conveys the geometric content of the theorem.

Claim. Let $(R, \mathfrak{m})$ be a noetherian local ring, $f \in \mathfrak{m}$ a non-zero-divisor with $\sqrt{(f)} = \mathfrak{m}$. Then $\dim R \leq 1$.

The geometric idea is that $\sqrt{(f)} = \mathfrak{m}$ makes $R/(f)$ artinian (a 0-dimensional "thickened point"), forcing chains of primes through $\mathfrak{m}$ to have length at most one. The proof formalises this through symbolic powers and two applications of Nakayama's lemma.

Let $\mathfrak{p} \subsetneq \mathfrak{m}$ be a prime; we will show $\mathfrak{p}$ is minimal in R, so the chain $\mathfrak{p} \subsetneq \mathfrak{m}$ has length one.

For each $n \geq 0$ define the *symbolic power* $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$. Note that $f \notin \mathfrak{p}$, since $f \in \mathfrak{p}$ would give $\sqrt{(f)} \subseteq \mathfrak{p} \subsetneq \mathfrak{m}$, contradicting $\sqrt{(f)} = \mathfrak{m}$. By the definition of symbolic powers via the localisation $R_{\mathfrak{p}}$, this means $f \cdot r \in \mathfrak{p}^{(n)} \Rightarrow r \in \mathfrak{p}^{(n)}$ for every $r \in R$.

Step 1: the symbolic powers stabilise. Since $R/(f)$ is artinian, the descending chain of ideals $\big(\mathfrak{p}^{(n)} + (f)\big)/(f)$ in $R/(f)$ stabilises at some index n_0. Translating back to R: for $n \geq n_0$,

$$\mathfrak{p}^{(n_0)} \subseteq \mathfrak{p}^{(n_0+1)} + (f),$$

i.e. every $a \in \mathfrak{p}^{(n_0)}$ can be written $a = b + fc$ with $b \in \mathfrak{p}^{(n_0+1)}$ and $c \in R$. Then $fc = a - b \in \mathfrak{p}^{(n_0)}$, and since $f \notin \mathfrak{p}$ we obtain $c \in \mathfrak{p}^{(n_0)}$. Hence $\mathfrak{p}^{(n_0)} = \mathfrak{p}^{(n_0+1)} + f \cdot \mathfrak{p}^{(n_0)}$.

This expresses the finitely generated R-module $M := \mathfrak{p}^{(n_0)}/\mathfrak{p}^{(n_0+1)}$ as $M = (f)\cdot M$, with $f \in \mathfrak{m}$. Nakayama's lemma applied to M over the local ring R yields $M = 0$, i.e. $\mathfrak{p}^{(n_0)} = \mathfrak{p}^{(n_0+1)}$. By induction, $\mathfrak{p}^{(n)} = \mathfrak{p}^{(n_0)}$ for every $n \geq n_0$.

Step 2: $\mathfrak{p}R_\mathfrak{p}$ is nilpotent. Localising at $\mathfrak{p}$ converts the equality $\mathfrak{p}^{(n_0)} = \mathfrak{p}^{(n_0+1)}$ (whose definition was already in $R_\mathfrak{p}$) into $(\mathfrak{p}R_\mathfrak{p})^{n_0} = (\mathfrak{p}R_\mathfrak{p})^{n_0+1}$. The finitely generated $R_\mathfrak{p}$-module $(\mathfrak{p}R_\mathfrak{p})^{n_0}$ is now fixed under multiplication by the maximal ideal $\mathfrak{p}R_\mathfrak{p}$, so a second application of Nakayama (this time over the local ring $R_\mathfrak{p}$) yields $(\mathfrak{p}R_\mathfrak{p})^{n_0} = 0$.

Hence $\mathfrak{p}R_\mathfrak{p}$ is nilpotent, $\dim R_\mathfrak{p} = 0$, and $\mathfrak{p}$ is a minimal prime of R. Since this holds for every $\mathfrak{p} \subsetneq \mathfrak{m}$, the longest chain of primes ending at $\mathfrak{m}$ has length ≤ 1, i.e. $\dim R \leq 1$.

The full theorem follows: at the generic point of an irreducible component K of $V(f)$, the local ring $R_\mathfrak{p}$ (where $\mathfrak{p}$ is the corresponding prime) satisfies the hypotheses of the claim, and $\operatorname{codim}(K, X) = \dim(R_\mathfrak{p})$ is then ≤ 1. The dichotomy 0 versus 1 depending on whether f is a zero-divisor in R^{red} comes from whether $\mathfrak{p}$ is itself minimal in R (in which case f vanishes on the whole component, hence is a zero-divisor mod nilpotents).

Theorem 7.5.10 (Krull's Height Theorem). *Let R be noetherian, $f_1, \dots, f_n \in R$, and $Y = V(f_1, \dots, f_n) \subset X = \operatorname{Spec}(R)$. For every irreducible closed subset $K \subset Y$,*

$$\operatorname{codim}(K, X) \leq \operatorname{codim}(K, Y) + n.$$

The case where K is an irreducible component of Y (so $\operatorname{codim}(K, Y) = 0$) recovers the more common form $\operatorname{codim}(K, X) \leq n$.

Theorem 7.5.11 (Converse to Height). *Let R be noetherian and $K \subset \operatorname{Spec}(R)$ an irreducible closed subset of codimension n. Then there exist $f_1, \dots, f_n \in R$ such that K is an irreducible component of $V(f_1, \dots, f_n)$.*

Corollary 7.5.12 (Codimension via Generators). *Let R be noetherian, $X = \operatorname{Spec}(R)$, and $\mathfrak{p} \subset R$ a prime ideal. Then*

$$\operatorname{codim}\big(V(\mathfrak{p}), X\big) \leq (\textit{minimum number of generators of } \mathfrak{p} \textit{ as a radical ideal}) \leq \infty.$$

Proof. Apply Theorem 7.5.10 with $f_1, \dots, f_n$ a set of radical-ideal generators of $\mathfrak{p}$. □

Theorem 7.5.13 (Dimension Formula). *Let $f : Y \to X$ be a morphism of locally noetherian schemes, $y \in Y$, and $x = f(y) \in X$. Write $Y_x := \operatorname{Spec}(\kappa(x)) \times_X Y$ for the fibre, containing y canonically. Then*

$$\operatorname{codim}(y, Y) \;\leq\; \operatorname{codim}(x, X) + \operatorname{codim}(y, Y_x),$$

with equality whenever f is flat *at y (i.e. the induced ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat); this property is known as "going down".*

Corollary 7.5.14 (Dimension Under Polynomial Extension). *If X is a locally noetherian scheme, then $\dim(X \times \mathbb{A}^n) = \dim(X) + n$.*

Proof. By induction it suffices to treat $n = 1$, and we may localise to $X = \operatorname{Spec}(R)$. The map $R \hookrightarrow R[t]$ is flat, so Theorem 7.5.13 applies to $\operatorname{Spec}(R[t]) \to \operatorname{Spec}(R)$ with equality. The fibre over a point $x \in \operatorname{Spec}(R)$ is $\operatorname{Spec}(\kappa(x)[t])$, which has dimension 1 by Example 7.5.7 since $\kappa(x)[t]$ is a PID. Taking the supremum over x gives $\dim(R[t]) = \dim(R) + 1$. □

Example 7.5.15 (Dimension of Affine and Projective Bundles). Let X be a locally noetherian scheme and V a vector bundle of rank $n \geq 1$ over X. Then

$$\dim(\mathbb{A}(V)) = \dim(X) + n, \qquad \dim(\mathbb{P}(V)) = \dim(X) + n - 1.$$

Both claims are local on X (Remark 7.5.5), so we may assume V is free, whence $\mathbb{A}(V) \simeq X \times \mathbb{A}^n$ and $\mathbb{P}(V) \simeq X \times \mathbb{P}^{n-1}$. The $\mathbb{A}$-case is Corollary 7.5.14; the $\mathbb{P}$-case reduces to the affine case via the standard cover of $\mathbb{P}^{n-1}$ by copies of $\mathbb{A}^{n-1}$.

Theorem 7.5.16 (Dimension of Integral k-Schemes of Finite Type). *Let k be a field and X an integral k-scheme of finite type, with generic point η. Then*

(i) $\dim(X) = \mathrm{tr.deg}\big(\kappa(\eta)/k\big)$.

(ii) For every irreducible closed $K \subset X$, $\dim(X) = \dim(K) + \mathrm{codim}(K, X)$.

Example 7.5.17 (Failure of $\dim + \mathrm{codim}$ in General). The equality $\dim(K) + \mathrm{codim}(K, X) = \dim(X)$ fails outside the finite-type-over-a-field setting. Take $X = \mathrm{Spec}(\mathbb{Z}_{(p)}[x])$, which has $\dim(X) = 2$ via the chain $(0) \subsetneq (p) \subsetneq (p, x)$. The closed subscheme $K = V(px - 1) = \mathrm{Spec}(\mathbb{Z}_{(p)}[x]/(px - 1)) = \mathrm{Spec}(\mathbb{Z}_{(p)}[1/p]) = \mathrm{Spec}(\mathbb{Q})$ has $\dim(K) = 0$. Yet Krull's principal ideal theorem gives $\mathrm{codim}(K, X) = 1$, since $px - 1$ is a non-zerodivisor. Hence $\dim(K) + \mathrm{codim}(K, X) = 1 < 2 = \dim(X)$.

Proposition 7.5.18 (Dimension of a Noetherian Local Ring). *Let $(R, \mathfrak{m})$ be a noetherian local ring. Then*

$$\dim(R) = \min\big\{n \in \mathbb{N} \;\big|\; \textit{there exist } f_1, \ldots, f_n \in \mathfrak{m} \textit{ with } \sqrt{(f_1, \ldots, f_n)} = \mathfrak{m}\big\}.$$

Proof. For the upper bound, any tuple $(f_1, \ldots, f_n)$ with $\sqrt{(f_1, \ldots, f_n)} = \mathfrak{m}$ makes $V(\mathfrak{m})$ an irreducible component of $V(f_1, \ldots, f_n)$, so Theorem 7.5.10 gives $\dim(R) = \mathrm{codim}(V(\mathfrak{m}), \mathrm{Spec}(R)) \leq n$. The minimum on the right is therefore at least $\dim(R)$.

For the matching lower bound, apply Theorem 7.5.11 to $K = V(\mathfrak{m})$, which has codimension $d = \dim(R)$, to obtain $f_1, \ldots, f_d \in R$ with $V(\mathfrak{m})$ an irreducible component of $V(f_1, \ldots, f_d)$. Since $V(\mathfrak{m})$ is the unique closed point of $\mathrm{Spec}(R)$, we must have $V(f_1, \ldots, f_d) = V(\mathfrak{m})$, i.e. $\sqrt{(f_1, \ldots, f_d)} = \mathfrak{m}$. Hence the minimum is at most d. □

7.6. Regular Schemes and the Cotangent Space

We now introduce three geometric conditions on local rings — *regular*, *Cohen–Macaulay*, and *normal* — that form a strict hierarchy of "nice" singularity behaviour at the closed point. The table summarises what to remember:

Condition on $(R, \mathfrak{m})$	**Geometric meaning**	**Clean algebraic criterion**
Regular	"smooth point" (manifold-like)	$\mathfrak{m}$ is generated by $\dim(R)$ elements, equivalently $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = \dim(R)$
Cohen–Macaulay	"no hidden embedded primes" (the unmixed condition)	$\mathrm{depth}(R) = \dim(R)$, i.e. $\mathfrak{m}$ contains a regular sequence of length $\dim(R)$
Normal	"only codim-≥2 singularities" (Hartogs extension holds)	R is integrally closed in $\mathrm{Frac}(R)$, equivalently $R_\mathfrak{p}$ a DVR in codim 1 and $R = \bigcap_{\mathrm{codim}\,1} R_\mathfrak{p}$ (Serre)

Every regular ring is Cohen–Macaulay (Remark 7.6.24) and normal (Remark 7.7.2); in dimension one all three coincide (Proposition 7.7.7); in higher dimensions the implications are strict. This section develops the regular and Cohen–Macaulay conditions; the next section treats normality.

Our approach is *geometric*: the Zariski cotangent space $T_x^*X = \mathfrak{m}_x/\mathfrak{m}_x^2$ counts tangent directions at x, and regularity is the algebraic reflection of the classical "smooth point" condition that this count match the Krull dimension.

Tangent Space via the Functor of Points

There is an evocative way to package the Zariski tangent space, using the *ring of dual numbers* $k[\varepsilon] := k[t]/(t^2)$. An R-valued point $a + b\varepsilon \in \mathbb{A}^n(R[\varepsilon])$ has $a \in R^n$ recording a "base point" and $b \in R^n$ recording a "tangent direction". This upgrades to a functorial definition of tangent space for any algebraic k-functor.

Definition 7.6.1 (Tangent Space (Functor of Points)). Let X be an algebraic k-functor and $x \in X(k)$ a k-point. For each k-algebra R, the projection $\pi\colon R[\varepsilon] \twoheadrightarrow R$, $\varepsilon \mapsto 0$, induces $X(\pi)\colon X(R[\varepsilon]) \to X(R)$; write $x_R \in X(R)$ for the image of x under $X(k \to R)$. The *tangent space to X at x* is the algebraic k-functor

$$T_xX\colon \mathsf{CAlg}_k \to \mathsf{Set}, \qquad T_xX(R) \;:=\; X(\pi)^{-1}(x_R),$$

defined by the pullback

$$\begin{array}{ccc} T_xX(R) & \longrightarrow & X(R[\varepsilon]) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle X(\pi)} \\ \{x_R\} & \hookrightarrow & X(R). \end{array}$$

In words, $T_xX(R)$ collects the "infinitesimal deformations" of x_R: those $R[\varepsilon]$-points of X that degenerate to x_R when ε is set to zero.

For closed subfunctors of affine space the tangent space has a completely explicit description via derivatives, thanks to the following algebraic identity.

Lemma 7.6.2 (Dual-Number Evaluation). *Let $f \in k[x_1, \ldots, x_n]$, $a \in k^n$, R a k-algebra, and $b \in R^n$. Then in $R[\varepsilon]$,*

$$f(a + b\varepsilon) \;=\; f(a) \;+\; \Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot b_i\Big)\varepsilon,$$

where $\frac{\partial f}{\partial x_i}$ is the formal partial derivative.

Proof. Both sides are k-linear in f (the left via the evaluation homomorphism $\mathrm{ev}_{a+b\varepsilon}$, the right via linearity of $\partial/\partial x_i$), so it suffices to check on monomials $f = \prod_i x_i^{m_i}$. Substituting $x_i = a_i + b_i\varepsilon$ and using $\varepsilon^2 = 0$, the binomial theorem gives $(a_i + b_i\varepsilon)^{m_i} = a_i^{m_i} + m_i a_i^{m_i - 1} b_i \varepsilon$. Multiplying the factors and discarding all terms of order ≥ 2 in ε yields the claimed identity, the ε-coefficient being exactly $\sum_j \frac{\partial f}{\partial x_j}(a) \cdot b_j$. □

***Remark* 7.6.3** (Derivatives Are Algebraic). The proof of Lemma 7.6.2 is purely algebraic: partial derivatives appear *not* because we are doing analytic Taylor expansion, but because the algebraic relation $\varepsilon^2 = 0$ automatically kills every higher-order term, leaving the first-order information — which is precisely what formal partial derivatives record. Formal partial derivatives are an algebraic consequence of $\varepsilon^2 = 0$, not an analytic input.

Corollary 7.6.4 (Tangent Space via Jacobian). *Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ and $a \in V(f_1, \dots, f_m)(k)$. Then, as k-functors,*

$$T_a V(f_1, \dots, f_m) = \ker \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix},$$

the kernel of the Jacobian matrix interpreted as a k-linear map $\mathbb{A}^n \to \mathbb{A}^m$. In particular, for a single equation f, $T_a V(f) = \ker\big(\nabla f(a) \colon \mathbb{A}^n \to \mathbb{A}^1\big)$.

Proof. For any k-algebra R, $T_a V(f_1, \dots, f_m)(R)$ consists of those $b \in R^n$ such that $f_j(a + b\varepsilon) = 0$ in $R[\varepsilon]$ for every j. By Lemma 7.6.2 this becomes $f_j(a) + (\nabla f_j(a) \cdot b)\varepsilon = 0$; the constant term vanishes since $a \in V(f_j)(k)$, so the condition reduces to $\nabla f_j(a) \cdot b = 0$, i.e. $b \in \ker(\mathrm{Jac}(a))(R)$. The identification holds for every R, hence as k-functors. □

***Remark* 7.6.5** (Intrinsic Description via Derivations). The Jacobian description above depends on a choice of embedding into $\mathbb{A}^n$. There is an intrinsic reformulation. Let $X = \mathrm{Spec}(A)$ be affine and $x \colon A \to k$ a k-point, corresponding to the maximal ideal $\mathfrak{m} = \ker(x)$. A lift $\tilde{x} \colon A \to k[\varepsilon]$ with $\tilde{x} \bmod \varepsilon = x$ necessarily has the form $\tilde{x}(a) = x(a) + D(a)\varepsilon$ for a k-linear map $D \colon A \to k$. The requirement that $\tilde{x}$ be a ring homomorphism translates into the *Leibniz rule*

$$D(ab) = x(a)D(b) + x(b)D(a),$$

making D a *k-derivation* of A valued in k (with k viewed as an A-module via x). Since $D(1) = 0$ and $D(\mathfrak{m}^2) = 0$ by Leibniz, D is determined by its restriction to $\mathfrak{m}/\mathfrak{m}^2$, giving

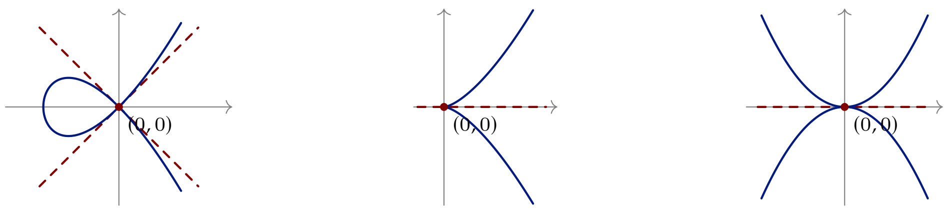
$$T_x X \simeq \mathrm{Der}_k(A, k) \simeq (\mathfrak{m}/\mathfrak{m}^2)^\vee.$$

This reconciles the functorial definition with Definition 7.6.7 below.

Example 7.6.6 (Plane Curve Singularities: Node, Cusp, Tacnode). Assume $\mathrm{char}(k) \neq 2, 3$. Consider the plane curves

$$N = V(y^2 - x^3 - x^2), \qquad C = V(y^2 - x^3), \qquad T = V(y^2 - x^4)$$

in $\mathbb{A}^2_k$. Using Corollary 7.6.4, the tangent space at (a, b) is the kernel of $\nabla f(a, b)$ where f is the defining polynomial.



Node $N : y^2 = x^2(x+1)$ Cusp $C : y^2 = x^3$ Tacnode $T : y^2 = x^4$

For the node N, $f(x, y) = y^2 - x^3 - x^2$ gives $\nabla f = (-3x^2 - 2x,\ 2y)$. The gradient vanishes iff $x(3x+2) = 0$ and $y = 0$, i.e. at $(0, 0)$ or at $(-2/3, 0)$. Only the first lies on N, since substituting $(-2/3, 0)$ into the equation gives $0 - (-2/3)^3 - (-2/3)^2 = 8/27 - 4/9 = -4/27 \neq 0$. Hence $(0, 0)$ is the unique singular point of N, with $T_{(0,0)}N = k^2$ (two-dimensional), and all other points are smooth with $\dim T_{(a,b)}N = 1$.

For the cusp C, $f(x, y) = y^2 - x^3$ gives $\nabla f = (-3x^2,\ 2y)$, which vanishes only at the origin. At $(0, 0) \in C$, $T_{(0,0)}C = k^2$; elsewhere the curve is smooth.

For the tacnode T, $f(x, y) = y^2 - x^4$ gives $\nabla f = (-4x^3,\ 2y)$, vanishing only at the origin. Again $(0, 0)$ is the sole singularity with $T_{(0,0)}T = k^2$.

All three curves thus have a single singular point at the origin with two-dimensional tangent space; the Zariski tangent space alone cannot distinguish them. The actual geometric types differ: the node consists of two smooth branches meeting transversally, the cusp is one branch "returning on itself", and the tacnode is two branches tangent to the x-axis. Finer invariants (multiplicity, Milnor number, intersection numbers) are needed to tell them apart.

The Zariski Cotangent Space (Stalk Viewpoint)

The functor-of-points tangent space of Definition 7.6.1 is the most natural definition, since it works for any algebraic functor. For a locally ringed space $(X, \mathcal{O}_X)$ there is an equivalent, purely stalk-theoretic reformulation expressing the tangent space at x in terms of the local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$. The bridge is the intrinsic description of Remark 7.6.5: a tangent vector is a k-derivation of $\mathcal{O}_{X,x}$, equivalently a $\kappa(x)$-linear functional on $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Definition 7.6.7 (Zariski Cotangent and Tangent Space — Stalk Version)**.** Let $(X, \mathcal{O}_X)$ be a locally ringed space and $x \in X$. The *Zariski cotangent space* at x is the $\kappa(x)$-vector space

$$T_x^*X \;:=\; \mathfrak{m}_x/\mathfrak{m}_x^2 \;\simeq\; \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x),$$

with the canonical *differential* $d\colon \mathfrak{m}_x \twoheadrightarrow T_x^*X$, $f \mapsto df$. The *Zariski tangent space* is $T_xX := (T_x^*X)^\vee$.

***Remark* 7.6.8** (Reconciling the Two Viewpoints)**.** Definition 7.6.1 and Definition 7.6.7 agree for an affine scheme $X = \mathrm{Spec}(A)$ and a k-point $x\colon A \to k$: the functorial T_xX applied to k is canonically $\mathrm{Der}_k(A, k) \simeq (\mathfrak{m}/\mathfrak{m}^2)^\vee$ (Remark 7.6.5), recovering the dual of the stalk version. The stalk definition is what generalises cleanly to locally ringed spaces beyond the functor-of-points world (e.g. classical C^∞-manifolds); the dual-numbers definition generalises in the opposite direction, to higher and derived algebraic geometry.

***Remark* 7.6.9** (Recovering the Classical (Co)Tangent Spaces)**.** For a smooth C^∞-manifold X, viewing $(X, \mathcal{O}_X^\infty)$ as a locally ringed space recovers the classical (co)tangent spaces of differential geometry (Example 7.6.10).

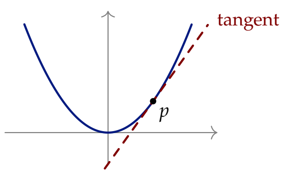
Example 7.6.10 (Tangent Vectors via Curves)**.** On a C^∞-manifold one usually represents tangent vectors by smooth curves:

$$\{\gamma\colon (-1,1) \to X \text{ smooth}, \gamma(0) = x\} \twoheadrightarrow T_xX, \quad \gamma \mapsto \big([f] \mapsto (f \circ \gamma)'(0)\big).$$

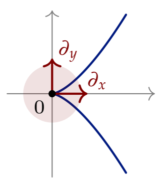
For general schemes there may be no such curves, but Definition 7.6.7 (or equivalently Definition 7.6.1) provides an algebraic replacement that agrees with this construction in the smooth case.

Regular Local Rings

Geometrically, regularity is the algebraic counterpart of *smoothness*: the tangent space has the "correct" dimension matching $\dim X$. The figure shows a smooth point of an affine plane curve with its tangent line, contrasted with the singular cusp $y^2 = x^3$ at the origin where "too many" tangent directions exist.



Smooth: $\dim T_p = 1 = \dim X$



Cusp: $\dim T_0 = 2 > 1 = \dim X$

The fundamental bound on Zariski cotangent dimension is the following.

Proposition 7.6.11 (Nakayama Bound)**.** *Let $(R, \mathfrak{m})$ be a noetherian local ring with residue field κ. Then*

$$\dim(R) \;\leq\; \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2).$$

Proof. By Nakayama's lemma, $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ equals the minimal number of generators of $\mathfrak{m}$. The bound is then Corollary 7.5.12 applied to $\mathfrak{p} = \mathfrak{m}$. □

Definition 7.6.12 (Regular Local Ring)**.** A noetherian local ring $(R, \mathfrak{m})$ is *regular* if equality holds in Proposition 7.6.11: $\dim(R) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Equivalently, $\mathfrak{m}$ can be generated by $\dim(R)$ elements.

Example 7.6.13 ($\mathbb{A}^n_k$ at the Origin)**.** For a field k, the local ring $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ has $\dim(R) = n$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ generated by exactly n elements. Hence R is regular — the affine space $\mathbb{A}^n_k$ is smooth at every closed point, as expected.

The following standard structural results on regular local rings are quoted from commutative algebra; the proofs are nontrivial, but the statements are what one uses in practice.

Theorem 7.6.14 (Regular Local Rings: Standard Facts)**.** *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d with residue field $\kappa = R/\mathfrak{m}$. Then:*

(i) the associated graded

$$\mathrm{gr}_\mathfrak{m}(R) \;=\; \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \;\simeq\; \mathrm{Sym}_\kappa(\mathfrak{m}/\mathfrak{m}^2) \;\simeq\; \kappa[t_1, \ldots, t_d]$$

is a polynomial ring;

(ii) (Auslander–Buchsbaum) R is a unique factorisation domain;

(iii) for every prime $\mathfrak{p} \subset R$, the localisation $R_\mathfrak{p}$ is regular local.

All three statements are nontrivial [Mat89, §§19–20]. Morally: (1) is the "infinitesimal polynomial ring" picture at a smooth point; (2) is algebraic smoothness manifesting as unique factorisation; (3) is exactly what licences the global definition of a regular scheme.

Theorem 7.6.15 (Serre's Homological Characterisation of Regularity)**.** *For a noetherian ring R and $n \in \mathbb{N} \cup \{\infty\}$, the following are equivalent:*

(i) R is regular of Krull dimension $\leq n$;

(ii) every R-module has a finite projective resolution of length $\leq n$;

(iii) every cyclic quotient R/I has a finite projective resolution of length $\leq n$.

See [Mat89, Theorem 19.2]. Condition (ii) is what makes regularity manifestly preserved under localisation, giving Theorem 7.6.14(3).

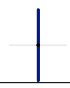
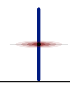
Regular Rings and Regular Schemes

Definition 7.6.16 (Regular Ring, Regular Scheme). A ring R is *regular* if it is noetherian and every localisation $R_{\mathfrak{p}}$ is regular local. A scheme X is *regular* if it is locally noetherian and every stalk $\mathcal{O}_{X,x}$ is regular local.

***Remark* 7.6.17** (Regularity is Local). "Regular" is a local property of rings: localisation preservation (Theorem 7.6.14(3)) gives axiom (i) of Definition 7.1.1, and gluing from a Zariski cover reduces to a stalk-by-stalk check. By Proposition 7.1.5, regularity extends uniquely to a local property of schemes, equivalent to the stalk formulation above.

Cohen–Macaulay and Depth

Regularity is too restrictive in practice; Cohen–Macaulay-ness relaxes it by replacing "$\mathfrak{m}$ generated by $\dim(R)$ elements" with "$\mathfrak{m}$ contains a regular sequence of length $\dim(R)$". The geometric content is best seen by contrast with dimension. The Krull dimension counts how many irreducible closed subsets of $\operatorname{Spec}(R)$ can be nested through the closed point; the depth instead counts the longest sequence of cuts in $\mathfrak{m}$ that genuinely lower the dimension at each step — each cut a non-zerodivisor on what remains, refusing to be absorbed into existing components. When these counts match, every component of $\operatorname{Spec}(R)$ contributes its full expected dimension and nothing hides at lower depth: this is the Cohen–Macaulay condition. Failure of CM signals *embedded primes*, low-dimensional components stuck inside higher-dimensional ones, invisible to topology but visible to depth. The two-dimensional table below records the mechanism in a concrete case.

ideal J	$V(J)$	**k-basis of $k[x,y]/J$**	**geometric content**
(x)	y-axis	$1, y, y^2, \ldots$	 values along the axis
(x^2, y)	origin	$1, x$	value + ∂_x at origin
$(x^2, xy) = (x) \cap (x^2, y)$	y-axis	$1, x, y, y^2, \ldots$	 axis values + ∂_x at origin

The third row decomposes as the union of the first two: $V(x^2, xy) = V(x) \cup V(x^2, y)$, mirroring the primary decomposition $(x^2, xy) = (x) \cap (x^2, y)$. The minimal prime (x) gives the clean y-axis; the embedded prime $(x, y) \supsetneq (x)$ is witnessed by the fat point $V(x^2, y)$, geometrically a thickening of the origin in the x-direction. The third scheme is *not* Cohen–Macaulay: every element of $\mathfrak{m} = (x, y)$ is annihilated by something supported at the embedded prime, so $\operatorname{depth} = 0 < 1 = \dim$.

We now formalise the two notions made informal above — "cuts that lower dimension" and "regular sequence" — and define depth and Cohen–Macaulay-ness precisely.

Definition 7.6.18 (System of Parameters). A *system of parameters* in a noetherian local ring $(R, \mathfrak{m})$ of dimension d is a sequence $f_1, \ldots, f_d \in \mathfrak{m}$ with $\sqrt{(f_1, \ldots, f_d)} = \mathfrak{m}$. By Proposition 7.5.18, d is the minimum length of such a sequence, and systems of parameters always exist.

Definition 7.6.19 (Regular Sequence). A sequence $f_1, \ldots, f_n \in \mathfrak{m}$ is a *regular sequence* if each f_i is a non-zerodivisor in $R/(f_1, \ldots, f_{i-1})$ (the case $i = 1$ asking that f_1 be a non-zerodivisor in R).

Corollary 7.6.20 (Non-Zerodivisors Drop Dimension). *If $f \in \mathfrak{m}$ is a non-zerodivisor in R^{red}, then* $\dim(R/f) = \dim(R) - 1$.

Proof. The upper bound $\dim(R/f) \leq \dim(R) - 1$ is Krull's principal ideal theorem (Theorem 7.5.8). For the matching lower bound, lift any system of parameters of R/f to elements of R; together with f this collection generates $\mathfrak{m}$ up to radical, so by Proposition 7.5.18, $\dim(R) \leq \dim(R/f) + 1$. □

Corollary 7.6.21 (Regular Sequences Drop Dimension Additively). *For a regular sequence $f_1, \ldots, f_n \in \mathfrak{m}$, $\dim(R/(f_1, \ldots, f_n)) = \dim(R) - n$.*

Proof. Iterate Corollary 7.6.20. □

Definition 7.6.22 (Depth, Cohen–Macaulay). Let $(R, \mathfrak{m})$ be a noetherian local ring.

(i) The *depth* of R is the maximum length of a regular sequence in $\mathfrak{m}$:

$$\operatorname{depth}(R) \ := \ \max\{n \mid \exists \text{ regular sequence } f_1, \ldots, f_n \in \mathfrak{m}\}.$$

(ii) R is *Cohen–Macaulay (CM)* if $\operatorname{depth}(R) = \dim(R)$.

By Corollary 7.6.21, $\operatorname{depth}(R) \leq \dim(R)$ always.

Proposition 7.6.23 (Regular Parameters Are a Regular Sequence). *If $(R, \mathfrak{m})$ is regular of dimension d with $\mathfrak{m} = (f_1, \ldots, f_d)$, then $(f_1, \ldots, f_d)$ is a regular sequence.*

Proof. This is a nontrivial consequence of Krull's intersection theorem combined with induction on d; see [Mat89, Theorem 14.2]. □

***Remark* 7.6.24** (Regular ⇒ Cohen–Macaulay). By Proposition 7.6.23, a regular local ring has a regular sequence of length $\dim(R)$ in its maximal ideal, so $\operatorname{depth}(R) \geq \dim(R)$. Combined with the reverse inequality (Definition 7.6.22), equality holds: regular implies Cohen–Macaulay.

Example 7.6.25 (The Cross: Cohen–Macaulay but Not Regular). Consider the local ring $R = (k[x, y]/(xy))_{(x,y)}$, obtained by localising the coordinate ring of the cross $V(xy) \subset \mathbb{A}^2_k$ at the origin. Geometrically R captures the infinitesimal neighbourhood of the singular point where the two coordinate axes meet; one has $\dim(R) = 1$.

This ring is not regular: the maximal ideal $\mathfrak{m} = (x, y)$ requires two generators, since x and y live in distinct minimal primes (the two axes) and no single element of $\mathfrak{m}$ generates $\mathfrak{m}$. Equivalently, $\mathfrak{m}/\mathfrak{m}^2$ has dimension $2 > 1 = \dim(R)$.

Yet R is Cohen–Macaulay, because the line $V(x - y)$ meets the cross transversally: it crosses both axes only at the origin. Algebraically, the element $x - y$ is a non-zerodivisor in R. To see this, note that $R = (k[x, y]/(xy))_{(x,y)}$ is a localisation of $k[x, y]/(xy)$, and the minimal primes of the latter are (x) and (y). Since $x - y$ lies in neither (x) nor (y) (as $-y \notin (x)$ and $x \notin (y)$), it is a non-zerodivisor in $k[x, y]/(xy)$, hence also in any localisation. Thus $(x - y)$ is a regular sequence of length 1 in $\mathfrak{m}$, and $\operatorname{depth}(R) \geq 1 = \dim(R)$, forcing equality. The cross is singular but has no "hidden embedded components".

Example 7.6.26 (The Cuspidal Cubic). Let k be a field with $\operatorname{char}(k) \neq 2, 3$ and $X = \operatorname{Spec}(k[x, y]/(y^2 - x^3))$. We compute $T^*_p X$ at two points.

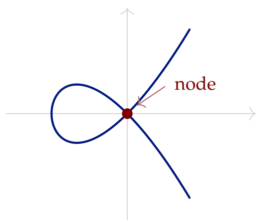
At the singular point $p = (0, 0)$, $\mathfrak{m} = (x, y)$ and the relation $y^2 - x^3$ lies in $\mathfrak{m}^2$, so $\mathfrak{m}/\mathfrak{m}^2 \simeq k\{x, y\}$ is two-dimensional. Hence $\dim T^*_{(0,0)} X = 2 > 1 = \dim X$, so X is not regular at the origin.

At the smooth point $p = (1, 1)$, substitute $z = x - 1$, $w = y - 1$. The relation $y^2 = x^3$ becomes $(1 + w)^2 = (1 + z)^3$, i.e. $w^2 + 2w = z^3 + 3z^2 + 3z$. Modulo $(z, w)^2$ this reduces to $2w = 3z$, and since $2, 3 \in k^\times$ this eliminates w in favour of z. So $T^*_{(1,1)} X$ has basis $\{z\}$ and $\dim T^*_{(1,1)} X = 1 = \dim X$, confirming that X is regular at $(1, 1)$.

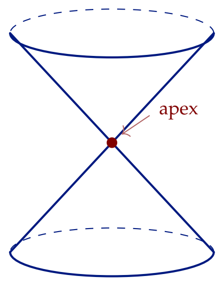
7.7. Normal Schemes

Normality is a weaker cousin of regularity, defined by integral-closedness of all local rings. Geometrically, a normal scheme is one whose only singularities are "small" — of codimension at least 2 — the codimension-1 part being forced to be regular. Two complementary intuitions make this concrete.

The first is *Hartogs extension*: on a normal noetherian scheme X, every rational function regular on the complement of a codimension-≥ 2 closed subset already extends regularly across that subset. This is the algebraic shadow of the classical Hartogs phenomenon for holomorphic functions on $\mathbb{C}^n$ $(n \geq 2)$: codimension-2 "holes" cannot truly remove regularity. The second is *Serre's criterion* Theorem 7.7.3: normality is exactly the condition that codimension-1 localisations are DVRs (so the scheme is regular in codimension 1) and that global sections can be reconstructed from these codimension-1 DVRs. The two viewpoints are equivalent, and together they explain why normal singularities are "mild": nodes and cusps (codimension-1 singular loci of curves) violate normality, but the apex of a quadric cone (a codimension-2 point) does not. The figure below contrasts the two cases.



Nodal curve $V(y^2 - x^3 - x^2)$
singular locus = origin (codim 1)
not normal



Quadric cone $V(xy - z^2) \subset \mathbb{A}^3$
singular locus = apex (codim 2)
normal

The nodal curve fails normality because its singular point is codimension 1 in a 1-dimensional scheme; the quadric cone passes because its apex is codimension 2 in a 2-dimensional scheme, leaving every codim-1 local ring a DVR.

In dimension 1, the codim-≥ 2 allowance is empty, so normal coincides with regular — this is Proposition 7.7.7 below.

The algebraic definition of normality goes through integral closure of rings: a domain A is normal when no "hidden algebraic functions" live just outside A in its fraction field. Concretely, integral elements over A are roots of monic polynomials with coefficients in A, and the condition asks that no such root in $\mathrm{Frac}(A) \setminus A$ exist. The connection to codimension-1 regularity in Serre's criterion runs through valuation theory: a codim-1 failure of regularity always produces a non-trivial integral element, while codim-≥ 2 failures do not.

Definition 7.7.1 (Integrally Closed Extension, Integrally Closed Domain). (i) A ring extension $A \subset B$ is *integrally closed* if every $b \in B$ satisfying a monic polynomial $f \in A[x]$ with $f(b) = 0$ already lies in A.

(ii) A domain A is *integrally closed* (or *normal*) if A is integrally closed in its fraction field $\mathrm{Frac}(A)$.

***Remark* 7.7.2** (UFDs and Regular Local Rings Are Normal). Every UFD is integrally closed in its fraction field. By Theorem 7.6.14(2), every regular local ring is a UFD, hence normal.

Theorem 7.7.3 (Serre's Criterion for Normality). *Let A be a noetherian domain. The following are equivalent:*

(i) A is integrally closed;

(ii) *for every prime* $\mathfrak{p} \subset A$ *of codimension* 1*, the localisation* $A_{\mathfrak{p}}$ *is a DVR, and* $A = \bigcap_{\mathrm{codim}(\mathfrak{p})=1} A_{\mathfrak{p}}$ *inside* $\mathrm{Frac}(A)$.

Proof. See [Mat89, Theorem 11.5]; this is Serre's $(R_1) + (S_2)$ criterion. □

Definition 7.7.4 (Normal Scheme). A scheme X is *normal* if for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.

Definition 7.7.5 (Valuation Ring, DVR). A *valuation ring* is a domain R in which divisibility is a total order. A *discrete valuation ring (DVR)* is a noetherian valuation ring that is not a field; equivalently, a local PID that is not a field.

Theorem 7.7.6 (Krull's Intersection Theorem). *Let* $(R, \mathfrak{m})$ *be a noetherian local ring. Then* $\bigcap_{n \geq 0} \mathfrak{m}^n = 0$*. See [Mat89, Theorem 8.10].*

Proposition 7.7.7 (Regular Local of Dimension 1 = DVR). *Let* $(R, \mathfrak{m})$ *be a noetherian local ring of dimension* 1*. Then* R *is regular if and only if* R *is a DVR.*

Proof. Suppose R is a DVR. Then $\mathfrak{m} = (\pi)$ is principal, so R is regular of dimension 1.

Conversely, suppose R is regular of dimension 1. By Theorem 7.6.14(2), R is a UFD — in particular a domain — so $|\mathrm{Spec}(R)| = \{(0), \mathfrak{m}\}$ and $\mathfrak{m} = (\pi)$ for a single uniformiser π. We must show every nonzero $a \in R$ has a unique expression $a = u\pi^n$ with $u \in R^\times$ and $n \geq 0$. By Theorem 7.7.6, $\bigcap_n \mathfrak{m}^n = 0$, so there is a maximal n with $a \in \mathfrak{m}^n$; write $a = u\pi^n$. If $u \in \mathfrak{m}$, then $a = u\pi^n \in \mathfrak{m}^{n+1}$, contradicting maximality of n; hence $u \notin \mathfrak{m}$, and since R is local this forces $u \in R^\times$. Uniqueness of n and u follow by the same maximality argument. Thus R is a DVR. □

Example 7.7.8 (Regular Local Rings of Dimension ≤ 1). (i) R is regular of dimension 0 iff R is a field.

(ii) R is regular of dimension 1 iff R is a DVR (Proposition 7.7.7).

When a 1-dimensional integral scheme fails to be normal, the integral closure construction produces a regular replacement covering it, the *normalisation*. The next example carries this out by hand for the cusp, illustrating how a rational function "invented" on X can become regular on the cover.

Example 7.7.9 (Normalisation of the Cuspidal Cubic). Consider the cuspidal cubic $X = \mathrm{Spec}(A)$ with $A = k[x,y]/(y^2 - x^3)$, an integral 1-dimensional k-scheme with a unique singular point at the origin (Example 7.6.26). The defining polynomial $y^2 - x^3$ is irreducible, so A is a domain; its fraction field $K = \mathrm{Frac}(A)$ is generated by x and y subject to $y^2 = x^3$.

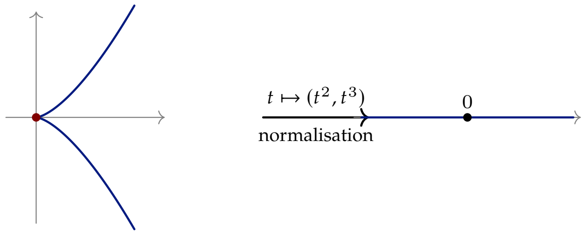
Set $t := y/x \in K$. The relation $y^2 = x^3$ rewrites as

$$t^2 = \frac{y^2}{x^2} = \frac{x^3}{x^2} = x,$$

so t satisfies the monic polynomial $T^2 - x = 0$ over A, hence is integral over A. Moreover $y = tx = t \cdot t^2 = t^3$. Thus inside K we have $A \subseteq A[t] = k[t]$ with $x = t^2, y = t^3$. Since $k[t]$ is a PID it is integrally closed, and any integral closure of A in K must contain t, so

$$\widetilde{A} = k[t], \qquad \widetilde{X} = \mathbb{A}^1_k \xrightarrow{t \mapsto (t^2, t^3)} X.$$

Geometrically, the normalisation map separates the cusp: $\mathbb{A}^1_k$ is regular everywhere, while X has its A_2-singularity at the origin; the map is bijective on points but fails to be an isomorphism precisely there, where $t = 0$ pulls back the cusp to the regular point $0 \in \mathbb{A}^1_k$. The cusp is, in this sense, "parametrised away" by the new variable $t = y/x$ — which is rational on X but regular on $\widetilde{X}$.



$X = V(y^2 - x^3)$
cusp at origin

$\widetilde{X} = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$
regular everywhere

***Remark* 7.7.10** (Regular versus Normal)**.** In dimension 1, regular and normal coincide (Proposition 7.7.7). In higher dimensions normal is strictly weaker than regular, and the quadric cone $V(xy - z^2) \subset \mathbb{A}^3$ pictured at the start of this section is the prototypical witness: it is normal everywhere but not regular at the apex $(0,0,0)$, where the maximal ideal $\mathfrak{m} = (x, y, z)$ requires three generators while the local ring has dimension 2. This singularity is known as the A_1 *singularity* or *ordinary double point*. Normality tolerates such isolated codim-≥ 2 singularities, ruling out only codim-1 ones such as nodes and cusps.

Looking Ahead

Chapters 1–6 built the foundational categorical machinery; this chapter added the first substantial geometric vocabulary: local versus global properties, reducedness, noetherianity, irreducibility, integrality, dimension (in the Krull sense), and the three main notions of smoothness — regular, Cohen–Macaulay, normal — with the hierarchy

$$\text{regular} \Rightarrow \text{Cohen–Macaulay} \quad \text{and} \quad \text{regular} \Rightarrow \text{normal}$$

becoming an equivalence in dimension ≤ 1. Zariski tangent spaces and Nakayama's lemma provide the concrete tools for detecting regularity at a point, and Serre's criteria (homological for regularity, $R_1 + S_2$ for normality) reduce these conditions to data that are computable in practice.

Part II of these notes moves beyond properties of individual schemes to the geometry of *morphisms*: flat, smooth, étale, unramified, syntomic, proper, projective. Alongside these, we will study divisors (Weil and Cartier), blow-ups, Chern classes, and the beginning of intersection theory. The local-to-global technology built here — extension of properties along Zariski covers, the interplay between Mod_X and subfunctors, dimension-theoretic control — will carry through unchanged into these more sophisticated chapters.

A categorical reformulation deepens the dual-numbers picture and points beyond the static setting. Given an A-module M, the *trivial square-zero extension* $A \oplus M$ is the ring with multiplication $(a, m)(a', m') = (aa', am' + a'm)$ and canonical projection to A; a k-derivation $D\colon A \to M$ is precisely a section of this projection in the slice $(\mathsf{CAlg}_k)_{/A}$, giving $\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_{(\mathsf{CAlg}_k)_{/A}}(A, A \oplus M)$. Taking $M = k$ recovers the tangent space at a k-point; taking $M = \Omega_{A/k}$ gives the universal derivation $d\colon A \to \Omega_{A/k}$. In the derived refinement of algebraic geometry (animated rings, ∞-categories), the static Kähler module $\Omega_{A/k}$ is replaced by the *cotangent complex* $\mathbb{L}_{A/k}$, characterised by the analogous mapping-space equivalence; concretely, $\mathbb{L}_{-/k}$ is the left-derived (animated) extension of $\Omega_{-/k}$. The cotangent complex governs deformation theory and obstruction theory uniformly across smooth, singular, and derived settings, and will reappear when we discuss smoothness via lifting properties.

Part III.

Appendix

Appendix A.

Sheaf Cohomology as Sheafification

Note: This appendix is not part of the original lecture course; it grew out of discussions on understanding sheaf cohomology from a derived/animated perspective. The treatment follows Lurie's Spectral Algebraic Geometry *[Lur18] and Mathew's work on Galois groups in stable homotopy theory [Mat16].*

A.1. Conventions

Throughout this appendix we work in the derived setting and *drop the* R*-prefix* on all functors. Concretely:

- All limits, colimits, and tensor products are derived. The symbol $\otimes$ stands for derived tensor product; the classical tensor product is recovered as $\pi_0(- \otimes -)$.
- $\Gamma(X, \mathcal{F})$ denotes derived global sections; the classical Γ is $H^0(X, \mathcal{F}) := \pi_0 \Gamma(X, \mathcal{F})$.
- $\mathrm{Hom}_R(M, N)$ denotes derived Hom, so $\mathrm{Ext}^i_R(M, N) = \pi_{-i} \mathrm{Hom}_R(M, N)$.
- For a stable ∞-category C, the symbol Mod_R refers to the ∞-category of (left) module spectra over a connective $\mathbb{E}_\infty$-ring R (equivalently, $\mathsf{D}(R)$ when R is discrete).
- Presheaves take values in a stable presentable category $\mathcal{D}$ — typically $\mathsf{D}(\mathbb{Z})$, Sp, or Mod_R for a base ring.

The classical 1-categorical theory is recovered by passing to π_0 at the end.

A.2. The Thesis: Cohomology = Sheafification + Global Sections

The core observation of this appendix is captured in a single line. For a scheme X and a presheaf $\mathcal{F} \in \mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathcal{D})$, set

$$\boxed{\Gamma(X, \mathcal{F}) := (L\mathcal{F})(X),}$$

where $L\colon \mathsf{PShv}(\mathsf{Sch}, \mathcal{D}) \to \mathsf{Shv}_\tau(\mathsf{Sch}, \mathcal{D})$ is sheafification with respect to a Grothendieck topology τ (we focus on the Zariski topology). All cohomological information about $\mathcal{F}$ is contained in this single derived object.

Three immediate consequences illustrate the formalism:

(i) *Mayer–Vietoris.* For an open cover $X = U \cup V$, the sheaf condition is exactly the fiber square

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(U \cap V, \mathcal{F}). \end{array}$$

(ii) *Čech cohomology.* For an open cover $\mathfrak{U} = (U_i \to X)_{i \in I}$, write

$$U_n := \coprod_{(i_0,\dots,i_n) \in I^{n+1}} U_{i_0} \cap \cdots \cap U_{i_n}.$$

The sheaf condition along $\mathfrak{U}$ unfolds into the equivalence

$$\Gamma(X, \mathcal{F}) \xrightarrow{\sim} \lim_{[n] \in \Delta} \Gamma(U_n, \mathcal{F}).$$

We call the right-hand side the *derived Čech complex* of $\mathfrak{U}$.

(iii) *Pushforward and base change.* For a morphism $f\colon Y \to X$, the pushforward $f_*\colon \mathsf{Shv}_\tau(Y, \mathcal{D}) \to \mathsf{Shv}_\tau(X, \mathcal{D})$ is determined by $\Gamma(X, f_*\mathcal{F}) \simeq \Gamma(Y, \mathcal{F})$. Higher direct images are then $H^i(X, f_*\mathcal{F}) = \pi_{-i} f_*\mathcal{F}$ in the appropriate t-structure.

These are the standard cohomological tools, derived without spectral sequences, injective resolutions, or flabby/soft sheaves: everything is encoded in the sheaf condition together with the universal property of sheafification.

***Remark* A.2.1** (Where classical Čech can fail). The classical *naïve* Čech complex of a cover is the cosimplicial *abelian group* $[n] \mapsto H^0(U_n, \mathcal{F})$ — the degree-zero truncation of $\Gamma(U_n, \mathcal{F})$. When some $\Gamma(U_n, \mathcal{F})$ has nonzero higher cohomology, this truncation drops information, and naïve Čech cohomology disagrees with sheaf cohomology. We will see in §Section A.5 that the two agree precisely when each $\Gamma(U_n, \mathcal{F})$ is concentrated in degree zero.

A.3. Quasi-coherent Sheaves are Sheaves

For a scheme X, recall (§3) that the category of quasi-coherent modules is the limit

$$\mathrm{Mod}_X = \lim_{(R, x\colon \mathrm{Spec}(R) \to X)} \mathrm{Mod}_R$$

with derived base change as transition functors. For an R-module M, the associated presheaf on CAlg_R is

$$\mathcal{F}_M\colon \mathsf{CAlg}_R \to \mathcal{D}, \qquad A \mapsto M \otimes_R A.$$

The main theorem of this appendix is:

Theorem A.3.1 (Quasi-coherent Modules are Zariski Sheaves). *For any ring R and any R-module M, the presheaf $\mathcal{F}_M$ is a Zariski sheaf on CAlg_R. Equivalently, $L\mathcal{F}_M \simeq \mathcal{F}_M$, and consequently*

$$\Gamma(\mathrm{Spec}(R), \mathcal{F}_M) \simeq \mathcal{F}_M(R) = M.$$

Since M is concentrated in degree zero (for $M \in \mathrm{Mod}_R^{\heartsuit}$), the higher cohomology vanishes:

Corollary A.3.2 (Affine Cohomology Vanishing). *For an affine scheme $X = \mathrm{Spec}(R)$ and a discrete quasi-coherent module M,*

$$H^i(X, \mathcal{F}_M) = 0 \quad \textit{for all } i > 0.$$

The strategy for proving Theorem A.3.1 is to establish the stronger statement that $\mathcal{F}_M$ satisfies *faithfully flat* descent. This is the content of the next section, via Mathew's framework of descendable algebras.

A.4. Faithfully Flat Descent

Throughout this section, C is a stable presentable symmetric monoidal category. In our application $C = \mathrm{Mod}_R$.

A.4.1. Descendable Algebras

Definition A.4.1 (Thick tensor ideal). For $A \in \mathsf{CAlg}(C)$, the *thick tensor ideal generated by* A, denoted $\langle A \rangle \subset C$, is the smallest full subcategory containing A and closed under finite (co)limits, retracts, and tensor products $- \otimes X$ for arbitrary $X \in C$.

Definition A.4.2 (Descendable algebra, [Mat16, Def. 3.18]). $A \in \mathsf{CAlg}(C)$ is *descendable* if $\mathbb{1}_C \in \langle A \rangle$.

The cobar construction associated to A is the cosimplicial algebra

$$A^{\otimes \bullet} \colon \Delta \to \mathsf{CAlg}(C), \qquad [n] \mapsto A^{\otimes(n+1)}.$$

Definition A.4.3 (Pro-constant cosimplicial object). A cosimplicial object $M^\bullet \colon \Delta \to C$ is *pro-constant* if its filtered diagram of partial totalizations

$$n \mapsto \mathrm{Tot}_{\leq n}(M^\bullet) := \lim_{m \in \Delta_{\leq n}} M^m$$

is pro-equivalent (in $\mathsf{Pro}(C)$) to a constant pro-system.

Example A.4.4 (Split cosimplicial objects). If $M^\bullet$ is split (i.e., admits a coaugmentation that is a section in homotopy at each level), then $\mathrm{Tot}_{\leq n}(M^\bullet)$ stabilizes for $n \gg 0$, so $M^\bullet$ is automatically pro-constant.

Example A.4.5 (Tensoring with pro-constants). In any stable symmetric monoidal category where $\otimes$ preserves limits in each variable, tensoring with a pro-constant cosimplicial object remains pro-constant, and the limit commutes with the tensor:

$$\left(\lim_{\Delta} M^\bullet\right) \otimes X \simeq \lim_{\Delta}(M^\bullet \otimes X).$$

This is automatic for $C = \mathrm{Mod}_R$, since $\otimes_R$ is exact.

The two key theorems of Mathew connect descendability with pro-constancy:

Theorem A.4.6 (Mathew, [Mat16, Prop. 3.20]). *$A \in \mathsf{CAlg}(C)$ is descendable iff the cobar $A^{\otimes \bullet}$ is pro-constant with limit $\mathbb{1}_C$.*

Proof sketch. ($\Rightarrow$) Let $\mathcal{X} \subset C$ be the full subcategory of X such that $X \otimes A^{\otimes \bullet}$ is pro-constant with limit X. Example A.4.5 shows $\mathcal{X}$ is closed under finite (co)limits, retracts, and tensor products. By Example A.4.4, $A \in \mathcal{X}$ — the cosimplicial object $A \otimes A^{\otimes \bullet}$ is split via the obvious extra degeneracy. Hence $\langle A \rangle \subset \mathcal{X}$, in particular $\mathbb{1} \in \mathcal{X}$.

($\Leftarrow$) If $A^{\otimes \bullet}$ is pro-constant with limit $\mathbb{1}$, some $\mathrm{Tot}_{\leq n}(A^{\otimes \bullet})$ admits $\mathbb{1}$ as a retract. But $\mathrm{Tot}_{\leq n}(A^{\otimes \bullet})$ is a finite limit of $A, A^{\otimes 2}, \ldots$, hence in $\langle A \rangle$. □

Theorem A.4.7 (Mathew, [Mat16, Thm. 3.26]). *If $A \in \mathsf{CAlg}(C)$ is descendable, then the canonical functor*

$$C \xrightarrow{\sim} \lim_{\Delta} \mathrm{Mod}_{A^{\otimes(\bullet+1)}}(C)$$

is an equivalence.

Proof sketch. The adjunction $- \otimes A : \mathcal{C} \rightleftarrows \mathrm{Mod}_A(\mathcal{C}) : F$ satisfies the Barr–Beck–Lurie criterion. Conservativity of $- \otimes A$: if $X \otimes A \simeq 0$, the subcategory $\mathcal{Y} = \{Y \mid X \otimes Y \simeq 0\}$ contains A and is closed under finite (co)limits, retracts, and tensor products, so $\mathbb{1} \in \langle A \rangle \subset \mathcal{Y}$, giving $X \simeq 0$. Limit-exchange for split cosimplicial objects is similar, using Example A.4.5 and Theorem A.4.6. □

A.4.2. Faithfully Flat Maps Are Descendable

We specialize $\mathcal{C} = \mathrm{Mod}_R$, with $A = S$ for a connective ring map $R \to S$.

Definition A.4.8 (Flat / Faithfully flat). A connective R-module M is *flat* if $- \otimes_R M$ is t-exact, equivalently if $\pi_0 M$ is a flat $\pi_0 R$-module and the canonical map $\pi_n R \otimes_{\pi_0 R} \pi_0 M \to \pi_n M$ is an isomorphism for every n. A connective ring map $R \to S$ is *faithfully flat* if S is a flat R-module and $\pi_0 R \to \pi_0 S$ is faithfully flat in the classical sense (equivalently, $\mathrm{Spec}\, \pi_0 S \to \mathrm{Spec}\, \pi_0 R$ is surjective).

The technical engine is a flat-Ext vanishing lemma due to Lurie:

Lemma A.4.9 (Flat Ext vanishing, [Lur18, Lem. D.3.3.6]). *Let R be a connective ring and M a flat R-module such that $\pi_0 M$ is $\aleph_n$-presentable as a $\pi_0 R$-module. Then for any connective $N \in \mathrm{Mod}_R$ and any $k > n$,*

$$\mathrm{Ext}^k_R(M, N) = 0.$$

Theorem A.4.10 (Faithfully Flat Descent). *Let $R \to S$ be a faithfully flat connective ring map (with cardinality bound: $\pi_0 S$ is $\aleph_n$-presentable over $\pi_0 R$ for some n, e.g. $\aleph_\omega$, which holds for finitely or countably presented algebras). Then S is descendable as an R-algebra, and consequently*

$$\mathrm{Mod}_R \xrightarrow{\sim} \lim_{\Delta} \mathrm{Mod}_{S^{\otimes(\bullet+1)}}.$$

Proof. By Theorem A.4.7, descendability of S implies the equivalence; we show S is descendable.

Set $K := \mathrm{fib}(R \to S)$ and $C := \mathrm{cofib}(R \to S)$, related by $K \simeq C[-1]$. The structure map $\rho\colon K \to R$ assembles, for each $m \geq 1$, into the m-fold tensor power

$$\rho^{(m)}\colon K^{\otimes m} \xrightarrow{\rho \otimes \cdots \otimes \rho} R.$$

Claim: $\rho^{(m)}$ is null-homotopic for m sufficiently large.

By the shift $K \simeq C[-1]$,

$$\rho^{(m)} \in [K^{\otimes m}, R]_{\mathrm{Mod}_R} = [C^{\otimes m}[-m], R]_{\mathrm{Mod}_R} = \mathrm{Ext}^m_R(C^{\otimes m}, R).$$

Since S is flat, $C = \mathrm{cofib}(R \to S)$ is also flat (long exact sequence on π_*), and so is $C^{\otimes m}$. The cardinality assumption makes $\pi_0(C^{\otimes m})$ $\aleph_n$-presentable, so by Lemma A.4.9, $\mathrm{Ext}^m_R(C^{\otimes m}, R) = 0$ for $m > n$.

Concluding descendability: The factorization $\rho^{(m+1)}\colon K^{\otimes(m+1)} \xrightarrow{\mathrm{id} \otimes \rho^{(m)}} K \xrightarrow{\rho} R$ produces a cofiber sequence

$$K^{\otimes m} \otimes_R S \to \mathrm{cofib}(\rho^{(m+1)}) \to \mathrm{cofib}(\rho^{(m)}).$$

Inductively, $\mathrm{cofib}(\rho^{(m)}) \in \langle S \rangle$ for every m, since $K^{\otimes m} \otimes_R S \in \langle S \rangle$ and $\langle S \rangle$ is closed under cofibers. When $\rho^{(m)}$ is null-homotopic,

$$\mathrm{cofib}(\rho^{(m)}) \simeq R \oplus K^{\otimes m}[1],$$

exhibiting R as a retract of an object in $\langle S \rangle$. Thus $R \in \langle S \rangle$, i.e., S is descendable. □

A.4.3. Proof of the Main Theorem

Proof of Theorem A.3.1. Set $S := \prod_i R_{f_i}$ for a Zariski cover $(f_i)_{i\in I}$ — i.e., elements generating the unit ideal of R. (We may reduce to a finite I, since the unit ideal is generated by finitely many of the f_i.) The ring map $R \to S$ is faithfully flat: each R_{f_i} is flat over R, and $\mathrm{Spec}(S) = \coprod_i \mathrm{Spec}(R_{f_i}) \to \mathrm{Spec}(R)$ is surjective by the unit-ideal hypothesis. The cardinality bound of Theorem A.4.10 is then satisfied automatically.

Theorem A.4.10 gives an equivalence

$$\mathrm{Mod}_R \xrightarrow{\sim} \lim_{\Delta} \mathrm{Mod}_{S^{\otimes_R(\bullet+1)}}.$$

Tracing M through this equivalence: the unit of the adjunction sends M to the cosimplicial $S^{\otimes_R(\bullet+1)}$-module $[n] \mapsto M \otimes_R S^{\otimes_R(n+1)}$, and the equivalence reads

$$M \xrightarrow{\sim} \lim_{[n]\in\Delta} \left(M \otimes_R S^{\otimes_R(n+1)}\right).$$

This is exactly the Zariski sheaf condition for $\mathcal{F}_M$ along the cover $(\mathrm{Spec}\, R_{f_i})_i$: at level n, $S^{\otimes_R(n+1)} = \prod_{i_0,\dots,i_n} R_{f_{i_0}\cdots f_{i_n}}$, so the displayed limit is the equalizer (in the appropriate cosimplicial sense) corresponding to $\mathcal{F}_M$ on the cover. Hence $\mathcal{F}_M$ is a Zariski sheaf, and $L\mathcal{F}_M \simeq \mathcal{F}_M$. □

A.5. Naïve Čech as a Special Case

Theorem A.3.1 immediately clarifies the relationship between derived and classical Čech cohomology.

Derived Čech is sheaf cohomology, always. Let $\mathcal{F}$ be a sheaf on X and $\mathfrak{U} = (U_i \to X)_{i\in I}$ any cover. The sheaf condition along $\mathfrak{U}$ *is* the equivalence

$$\Gamma(X, \mathcal{F}) \xrightarrow{\sim} \lim_{[n]\in\Delta} \Gamma(U_n, \mathcal{F}).$$

No truncation, no spectral sequence, no acyclicity hypothesis. The right-hand side is the *derived Čech complex*.

Naïve Čech recovers derived Čech when intersections are acyclic. The classical naïve Čech complex is the cosimplicial *discrete abelian group* $[n] \mapsto H^0(U_n, \mathcal{F})$ — i.e., the degree-zero truncation of $\Gamma(U_n, \mathcal{F})$. By Corollary A.3.2, this truncation loses nothing precisely when each U_n is affine (so $\Gamma(U_n, \widetilde{M})$ is concentrated in degree zero). By the affine intersection property of separated schemes (Proposition 6.2.15), this is automatic for affine covers of separated schemes:

Corollary A.5.1 (Naïve Čech for Separated Schemes). *Let X be a separated scheme, $\mathfrak{U}$ an affine open cover, and $\mathcal{F}$ a quasi-coherent sheaf on X. Then naïve and derived Čech cohomology along $\mathfrak{U}$ coincide, and both equal sheaf cohomology:*

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^i(X, \mathcal{F}) \qquad \textit{for all } i \geq 0.$$

Proof. For X separated, the diagonal $X \to X \times X$ is a closed immersion, so any finite intersection of affine opens is affine; in particular each U_n is affine. Corollary A.3.2 then makes $\Gamma(U_n, \mathcal{F})$ concentrated in degree zero, where it agrees with $H^0(U_n, \mathcal{F})$. Hence the cosimplicial object $[n] \mapsto \Gamma(U_n, \mathcal{F})$ is discrete, and its limit is the Moore complex of the cosimplicial abelian group $[n] \mapsto H^0(U_n, \mathcal{F})$ — i.e., the naïve Čech complex. □

Cartan–Leray, reinterpreted. The classical Cartan–Leray spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(\mathcal{F})) \Longrightarrow H^{p+q}(X, \mathcal{F})$$

is precisely the spectral sequence associated to the Bousfield–Kan filtration on $\lim_{\Delta} \Gamma(U_\bullet, \mathcal{F})$. The Cartan–Leray hypothesis — vanishing of higher H^q on the cover — collapses the spectral sequence to its $E_2^{p,0}$ row, recovering naïve Čech.

A.6. Connection with the Locally Ringed Space Picture

The category Mod_X defined as a limit

$$\mathrm{Mod}_X = \lim_{(R,x:\ \mathrm{Spec}\,R \to X)} \mathrm{Mod}_R$$

agrees, when X is a scheme, with the classical category of quasi-coherent $\mathcal{O}_X$-modules on the locally ringed space $(|X|, \mathcal{O}_X)$ — see §Section 6.3. Under this identification, the derived global sections $\Gamma(X, \mathcal{F})$ defined here match the classical $\mathrm{R}\Gamma$ of an $\mathcal{O}_X$-module computed via injective resolutions. So all the classical formalism (injective resolutions, derived Hom, hypercohomology, etc.) gives an equivalent answer; the derived viewpoint adopted here is just a way to skip over the resolution machinery and work directly with the universal property.

Appendix B.

Schemes in Topos Theory

Note: Although this material did not appear in the algebraic geometry course, it was covered in the Higher Topos Theory course (notes available at [Hoy26b]). Due to its close connection with these lecture notes, it is recorded here.

B.0.1. Conventions

Throughout this appendix:

- Categories refer to $(\infty, 1)$-categories; classical categories are called 1-categories. We generally denote categories by $\mathcal{C}$.
- 2-Categories refer to $(\infty, 2)$-categories, generally denoted by $\mathbb{C}$.
- Topoi refer to $(\infty, 1)$-topoi. In a topos, coverings are effective epimorphisms.

In this appendix, we attempt to generalize the notion of scheme to arbitrary topoi, ultimately leading to Lurie's theory of fractured topoi.

Recall that in Theorem 6.3.4 there exists an adjunction:

$$\mathsf{LocRingSpace} \underset{\mathrm{Spec}}{\overset{\Gamma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{CAlg}^{\mathrm{op}}$$

where the left side is the category of locally ringed spaces, Γ is the global sections functor, and Spec is the prime spectrum functor, i.e., $\mathrm{Spec}(R) = (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. The spectrum functor Spec is fully faithful, with essential image consisting of affine locally ringed spaces.

The definition of schemes follows: let Aff denote the full subcategory of affine locally ringed spaces, then the category of schemes Sch can be identified with those objects in $\mathsf{LocRingSpace}$ that can be covered by objects from Aff along open immersions.

To replicate this construction in topos theory, we need to accomplish three tasks:

(i) Rewrite "locally ringed spaces" in the language of topos theory;

(ii) Construct the spectrum functor Spec in the topos-theoretic framework;

(iii) Define schemes as "objects covered by affine objects along admissible morphisms."

B.1. Topos-Theoretic Preliminaries

The constructions in this appendix use a small amount of topos-theoretic vocabulary. We recall the definitions and key results we need without proof; throughout we follow the conventions of the Higher Topos Theory notes [Hoy26b] (which themselves follow Lurie [Lur09; Lur18]).

B.1.1. Topoi

A colimit $X = \mathrm{colim}_i X_i$ in a presentable category $\mathcal{T}$ with pullbacks is **van Kampen** if the slice functor $\mathcal{T}^{\mathrm{op}} \to \mathsf{Cat}_\infty$, $X \mapsto \mathcal{T}_{/X}$, sends it to a limit. Equivalently, the colimit is both **universal** (preserved by every base change) and **effective** (every cartesian natural transformation $Y_\bullet \to X_\bullet$ is recovered from its colimit by pulling back along $X_i \to X$).

Definition B.1.1. A **topos** is a presentable category in which every small colimit is van Kampen.

B.1.2. Geometric and algebraic morphisms; points

A **geometric morphism** $\varphi\colon \mathcal{S} \to \mathcal{T}$ is a functor $\varphi_*\colon \mathcal{S} \to \mathcal{T}$ admitting a left exact left adjoint φ^*, called the **algebraic morphism** from $\mathcal{T}$ to $\mathcal{S}$. The resulting categories of topoi satisfy $\mathsf{Topos} \simeq \mathsf{Logos}^{\mathrm{op}}$, and both upgrade naturally to 2-categories $\mathbb{T}\mathsf{opos}$, $\mathbb{L}\mathsf{ogos}$ (with algebraic morphisms as 1-morphisms in $\mathbb{L}\mathsf{ogos}$, and $\mathrm{Hom}_{\mathbb{L}\mathsf{ogos}}(\mathcal{S},\mathcal{T}) \subset \mathsf{Fun}(\mathcal{S},\mathcal{T})$ on the nose). A **point** of $\mathcal{T}$ is a geometric morphism $\mathsf{An} \to \mathcal{T}$; equivalently, a left exact, colimit-preserving "stalk" functor $\mathcal{T} \to \mathsf{An}$. We write $\mathsf{Pt}(\mathcal{T}) = \mathrm{Hom}_{\mathbb{T}\mathsf{opos}}(\mathsf{An},\mathcal{T})$.

B.1.3. Étale geometric morphisms

Definition B.1.2. A geometric morphism $\varphi\colon \mathcal{T} \to \mathcal{S}$ is **étale** if there is an equivalence $\mathcal{T} \simeq \mathcal{S}_{/X}$ for some $X \in \mathcal{S}$ under which φ_* becomes the forgetful functor $\mathcal{S}_{/X} \to \mathcal{S}$.

We will use the following standard facts, all proved in [Hoy26b, Prop. 4.33–4.34]: étale morphisms are closed under composition and base change, satisfy left cancellation, and yield a 2-categorical equivalence $\mathcal{S} \xrightarrow{\sim} \mathsf{Topos}^{\text{ét}}_{/\mathcal{S}}$ via $X \mapsto \mathcal{S}_{/X}$. The prototypical examples are slice topoi: for any topological space X and open $U \subseteq X$, the inclusion induces an étale morphism $\mathsf{Shv}(U) \to \mathsf{Shv}(X)$; for any Zariski scheme $(\mathcal{T},\mathcal{O})$ and $U \in \mathcal{T}$, the map $(\mathcal{T}_{/U},\mathcal{O}|_U) \to (\mathcal{T},\mathcal{O})$ is étale.

B.1.4. Congruences

Let $\mathcal{T}$ be a topos and $\Sigma \subseteq \mathrm{Ar}(\mathcal{T})$.

- Σ is **saturated** if it contains the isomorphisms and is closed under composition and colimits in $\mathrm{Ar}(\mathcal{T})$.
- Σ is **strongly saturated** if it is saturated and satisfies the 2-out-of-3 property; equivalently ([Hoy26b, Prop. A.15]), Σ is the class of L-isomorphisms for some accessible localization $L\colon \mathcal{T} \to \mathcal{T}'$.
- Σ is a **modality** if it is the left class of a factorization system that is stable under base change.
- Σ is a **congruence** if it is strongly saturated and stable under base change; equivalently ([Hoy26b, Cor. 4.16]), Σ is the class of φ^*-isomorphisms for an algebraic morphism $\varphi^*\colon \mathcal{T} \to \mathcal{S}$. Congruences thus parametrize quotients of $\mathcal{T}$ inside Topos.

Inside the lattice of saturated classes one has $\mathsf{Cong}(\mathcal{T}) = \mathsf{SSat}(\mathcal{T}) \cap \mathsf{Mdl}(\mathcal{T})$, with both $\mathsf{SSat}, \mathsf{Mdl} \hookrightarrow \mathsf{Sat}(\mathcal{T})$.

Theorem B.1.3 (Anel–Biedermann–Finster–Joyal)**.** *For a small subset $\Sigma \subseteq \mathrm{Ar}(\mathcal{T})$, the congruence generated by Σ equals the modality generated by the closure of Σ under iterated diagonals.*

Definition B.1.4. For a modality Σ, set $D(\Sigma) \coloneqq \{u \in \Sigma \mid \Delta_u \in \Sigma\}$, and iterate: $D^{n+1} = D \circ D^n$, $D^\infty = \bigcap_n D^n$. The operation $\Sigma \mapsto D^\infty(\Sigma)$, called **décalage**, is right adjoint to the inclusion $\mathsf{Cong}(\mathcal{T}) \hookrightarrow \mathsf{Mdl}(\mathcal{T})$ ([Hoy26b, Prop. 5.30]).

B.1.5. Lax slice 2-categories

For a 2-category $\mathbb{C}$ and $X \in \mathbb{C}$, the **lax slice** $\mathbb{C}_{//X}$ relaxes the strict slice by allowing comparison triangles to commute up to a *not-necessarily-invertible* 2-cell.

Definition B.1.5. The **(left-)lax slice** $\mathbb{C}_{//X}$ has

- objects: 1-morphisms $t : Y \to X$ in $\mathbb{C}$;
- 1-morphisms $(Y, t) \to (Z, u)$: pairs (f, ε) with $f : Y \to Z$ and a 2-cell $\varepsilon : u \circ f \to t$;
- 2-morphisms $(f, \varepsilon) \to (f', \varepsilon')$: pairs (ϕ, α) with $\phi : f \to f'$ a 2-cell and α an identification $\varepsilon \simeq \varepsilon' \circ (u \cdot \phi)$.

The strict slice $\mathbb{C}_{/X}$ embeds as the wide subcategory where ε is invertible ([GR17, §A.2.5]).

The crucial feature for our purposes is that right adjoints in $\mathbb{C}$ lift automatically into the lax slice: if $f \dashv g$ with counit $\varepsilon : fg \to \mathrm{id}_X$, then $(f, \mathrm{id}_f) \dashv (g, \varepsilon)$ in $\mathbb{C}_{//X}$ ([BH21, Lemma 8.10]). The counit ε is typically non-invertible, so (g, ε) does *not* live in the strict slice $\mathbb{C}_{/X}$ — only in the lax one. This is the reason lax slices appear in our treatment of structured topoi.

B.2. Ringed Topoi and Schemes

A scheme, classically, is a locally ringed space. To generalise schemes beyond spaces — to algebraic stacks, formal schemes, or derived-geometry objects — one replaces "space" by "topos" and works with *ringed topoi* instead. This section records the necessary definitions and explains how classical schemes embed into this framework as ringed topoi of a special form (those whose stalks are local rings and whose underlying topos is "spatial").

Definition B.2.1 (Ringed Topos)**.** A **ringed topos** is a pair $(\mathcal{T}, \mathcal{O})$, where $\mathcal{T}$ is a topos and $\mathcal{O}$ is a static ring sheaf on $\mathcal{T}$, i.e., a small-limit-preserving functor

$$\mathcal{O} : \mathcal{T}^{\mathrm{op}} \to \mathsf{CAlg}^{\heartsuit}.$$

***Remark* B.2.2.** We use the intrinsic notion of C-valued sheaves on a topos: for a topos $\mathcal{T}$ and a category C with small limits, a C-valued sheaf is a small-limit-preserving functor $\mathcal{T}^{\mathrm{op}} \to C$. When $\mathcal{T} \simeq \mathsf{Shv}_\tau(\mathcal{D})$ for a small site $\mathcal{D}$, this matches the classical notion of a C-valued τ-sheaf on $\mathcal{D}$.

Definition B.2.3 (Local Morphism)**.** Let $(f, f^\sharp) : (\mathcal{T}, \mathcal{O}_\mathcal{T}) \to (\mathcal{S}, \mathcal{O}_\mathcal{S})$ be a morphism of ringed topoi, where $f : \mathcal{T} \to \mathcal{S}$ is a geometric morphism and $f^\sharp : f^*\mathcal{O}_\mathcal{S} \to \mathcal{O}_\mathcal{T}$ is a morphism of ring sheaves on $\mathcal{T}$. We say $(f, f^\sharp)$ is **local** if $f^\sharp$ detects units: for each object U in $\mathcal{T}$ and the induced map $\varphi : (f^*\mathcal{O}_\mathcal{S})(U) \to \mathcal{O}_\mathcal{T}(U)$, the following square is a pullback:

$$\begin{array}{ccc} (f^*\mathcal{O}_\mathcal{S})(U)^\times & \xrightarrow{\varphi} & \mathcal{O}_\mathcal{T}(U)^\times \\ \downarrow & & \downarrow \\ (f^*\mathcal{O}_\mathcal{S})(U) & \xrightarrow{\varphi} & \mathcal{O}_\mathcal{T}(U) \end{array}$$

i.e., $\varphi^{-1}(\mathcal{O}_\mathcal{T}(U)^\times) = (f^*\mathcal{O}_\mathcal{S})(U)^\times$. We denote by $\mathsf{LocRingTopos}$ the category of ringed topoi and local morphisms between them.

***Remark* B.2.4** (Classical analogy). For locally ringed spaces, the corresponding condition is the familiar one: $(f, f^\sharp)$ is local iff for every point x, the induced stalk map

$$\mathcal{O}_{\mathcal{S},f(x)} \to \mathcal{O}_{\mathcal{T},x}$$

is a local ring homomorphism. Definition B.2.3 is the topos-theoretic counterpart, expressed via the unit-detecting square.

Construction B.2.5 (Zariski Spectrum). For a static ring R, define its **Zariski spectrum** $\mathrm{Spec}(R)_{\mathrm{Zar}}$ as the following ringed topos.

Consider the locale $\mathrm{Open}(\mathrm{Prim}(R))$ of open sets of the prime spectrum $\mathrm{Prim}(R)$, which is isomorphic to the locale Rad_R of radical ideals of R ordered by inclusion. Equipping it with the canonical topology gives the topos:

$$\mathcal{T}_R := \mathsf{Shv}(\mathrm{Prim}(R)).$$

The structure sheaf $\mathcal{O}_R \in \mathsf{CAlg}^\heartsuit(\mathcal{T}_R)$ sends the basic open $D(f) = \mathrm{Prim}(f)$ to the localization $R[f^{-1}]$; the assignment already satisfies Zariski descent, so it is a sheaf without further sheafification. Set $\mathrm{Spec}(R)_{\mathrm{Zar}} := (\mathcal{T}_R, \mathcal{O}_R)$. The resulting functor

$$\mathrm{Spec}(-)_{\mathrm{Zar}} \colon \mathsf{CAlg}^{\heartsuit,\mathrm{op}} \hookrightarrow \mathsf{LocRingTopos}$$

is right adjoint to global sections $\Gamma\colon \mathsf{LocRingTopos} \to \mathsf{CAlg}^{\heartsuit,\mathrm{op}}$, $(\mathcal{T}, \mathcal{O}) \mapsto \mathcal{O}(*_{\mathcal{T}})$, and fully faithful by Zariski descent.

***Remark* B.2.6.** Similarly, one can replace the Zariski topology with the étale topology to define the **étale spectrum** $\mathrm{Spec}(R)_{\text{ét}}$. In this case, $\mathcal{T}_R = \mathsf{Shv}_{\text{ét}}(\mathrm{Spec}(R))$, and the structure sheaf $\mathcal{O}_R$ is given by étale descent.

Definition B.2.7 (Zariski Scheme). A **Zariski scheme** is a ringed topos $(\mathcal{T}, \mathcal{O})$ such that there exists a covering family $\{U_i \to *_{\mathcal{T}}\}_{i \in I}$ in $\mathcal{T}$ such that for each $i \in I$, there is an equivalence of locally ringed topoi:

$$(\mathcal{T}_{/U_i}, \mathcal{O}|_{U_i}) \simeq \mathrm{Spec}(R_i)_{\mathrm{Zar}}$$

for some static ring R_i. Morphisms between Zariski schemes are local morphisms, and we denote their category by $\mathsf{Sch}_{\mathrm{Zar}}$.

***Remark* B.2.8.** Similarly, one can define **étale schemes** as ringed topoi locally equivalent to $\mathrm{Spec}(R)_{\text{ét}}$, denoting their category by $\mathsf{Sch}_{\text{ét}}$. Étale schemes are also called Deligne–Mumford ∞-stacks.

Example B.2.9.
- Classical schemes are precisely 0-localic Zariski schemes, i.e., the case where $\mathcal{T}$ is a 0-localic topos (a topos equivalent to the category of sheaves on some locale).
- Classical Deligne–Mumford stacks are precisely étale schemes with 1-localic topoi.
- If $(\mathcal{T}, \mathcal{O})$ is a Zariski scheme and $X \in \mathcal{T}$, then $(\mathcal{T}_{/X}, \mathcal{O}|_X)$ is still a Zariski scheme.

***Remark* B.2.10** (Relationship with the Functor-of-Points Definition). The topos-theoretic Definition B.2.7 agrees with the functor-of-points definition used in the main text. Concretely, given a classical scheme X, its small Zariski topos $\mathsf{Shv}(\mathrm{Prim}(X))$ together with the structure sheaf is a Zariski scheme in the sense above, and every 0-localic Zariski scheme arises in this way (Example B.2.9). The topos formulation has the additional virtue of accommodating higher-localic objects such as Deligne–Mumford stacks.

B.2.1. From Scheme Theory to Fractured Structures

Recast in topos-theoretic terms, classical scheme theory exhibits the following interconnected structures.

- **Spec is fully faithful.** The functor $\mathrm{Spec}(-)_{\mathrm{Zar}} \colon \mathsf{CAlg}^{\heartsuit,\mathrm{op}} \hookrightarrow \mathsf{LocRingTopos}$ exhibits affine schemes as a full subcategory of locally ringed topoi (Construction B.2.5).
- **Schemes embed into the Zariski topos.** The functor

$$\mathsf{Sch}_{\mathrm{Zar}} \to \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{CAlg}^{\mathrm{op}}), \quad (\mathcal{T}, \mathcal{O}) \mapsto \mathrm{Hom}(\mathrm{Spec}(-)_{\mathrm{Zar}}, (\mathcal{T}, \mathcal{O}))$$

  is fully faithful.
- **Classifying topos for local rings.** The category $\mathcal{E} := \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{CAlg}^{\mathrm{fp,op}})$ is the classifying topos for local rings: algebraic morphisms $\mathcal{E} \to \mathcal{X}$ correspond bijectively to local ring sheaves on $\mathcal{X}$.
- **Étale slices.** A morphism of Zariski schemes is **étale** if the underlying geometric morphism is étale (Definition B.1.2); writing $\mathsf{Sch}^{\text{ét}}_{\mathrm{Zar}} := \mathsf{Sch}_{\mathrm{Zar}} \times_{\mathsf{Topos}} \mathsf{Topos}^{\text{ét}}$, this category is itself a topos, and

$$\mathsf{Sch}^{\text{ét}}_{\mathrm{Zar}} \simeq \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{CAlg}^{\mathrm{ad,op}}),$$

  where $\mathsf{CAlg}^{\mathrm{ad}} \subset \mathsf{CAlg}$ is the wide subcategory of Zariski localizations $R \to R[s^{-1}]$. (Caveat: this "étale" is the topos-theoretic notion; on affines it coincides with Zariski open immersion, not with the algebraic étale topology.)
- **Factorization system.** On any topos $\mathcal{T}$, every morphism $\alpha\colon A \to B$ of static ring sheaves factors uniquely as $A \to A[S^{-1}] \to B$, with the first arrow a localization and the second a local morphism.

These four facts can be repackaged as four pieces of structure on the classifying topos $\mathcal{E}$:

(i) **Corporeal subtopos.** Left Kan extension induces a non-fully-faithful functor:

$$j_! \colon \mathcal{E}^{\mathrm{corp}} := \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{CAlg}^{\mathrm{fp,ad,op}}) \hookrightarrow \mathcal{E},$$

whose image is also a topos. Objects in $\mathcal{E}^{\mathrm{corp}}$ are called **corporeal objects**.

(ii) **Admissibility structure.** The wide subcategory $\mathcal{E}^{\mathrm{ad}} \subset \mathcal{E}$ consists of representable étale morphisms.

(iii) **Factorization system.** The functor $\mathsf{Fun}_{\mathbb{T}\mathsf{opos}}(-, \mathcal{E}) \colon \mathsf{Topos}^{\mathrm{op}} \to \mathsf{Cat}$ lifts to $\mathsf{Topos}^{\mathrm{op}} \to \mathsf{Cat}^{\mathrm{fact}}$.

(iv) **Local morphisms.** The lax slice 2-category $\mathbb{T}\mathsf{opos}_{//\mathcal{E}}$ (for the definition and properties of lax slice 2-categories, see §Definition B.1.5) has a wide subcategory $(\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}}$ consisting of local morphisms.

These four structures are mutually equivalent — each determines the other three and each suffices to recover $\mathsf{Sch}_{\mathrm{Zar}}$. Lurie's theory of *fractured topoi* axiomatizes this phenomenon.

B.3. Fractured Topoi

The four equivalent structures on $\mathsf{Sch}_{\mathrm{Zar}}$ discussed above — Zariski admissibility, the étale site, infinitesimal thickenings, and the formal-affineness structure — share a common pattern: a topos equipped with a distinguished subcategory of "corporeal" objects whose behaviour controls the entire ambient category. Lurie axiomatised this pattern under the name *fractured topos*; the key example is precisely affine schemes sitting inside the big Zariski topos of schemes.

Definition B.3.1 (Fractured Topos). A **fractured topos** is a topos $\mathcal{E}$ equipped with a subcategory $j_! \colon \mathcal{E}^{\mathrm{corp}} \hookrightarrow \mathcal{E}$, satisfying the following conditions:

(i) The category $\mathcal{E}^{\mathrm{corp}}$ has pullbacks and $j_!$ preserves pullbacks.

(ii) $j_!$ has a right adjoint j^* that is conservative and preserves colimits.

(iii) For each morphism $U \to V$ in $\mathcal{E}^{\mathrm{corp}}$, the following square is a pullback:

$$\begin{array}{ccc} j_! j^* j_! U & \longrightarrow & j_! j^* j_! V \\ \downarrow & \lrcorner & \downarrow \\ j_! U & \longrightarrow & j_! V \end{array}$$

Objects of $\mathcal{E}^{\mathrm{corp}}$ are called **corporeal**. In all examples we will encounter, the following technical condition is also satisfied (and we tacitly assume it):

(4*) Retracts of corporeal objects in $\mathcal{E}$ are again corporeal.

Proposition B.3.2 (Basic Properties of Fractured Topoi). *For a fractured topos $(\mathcal{E}, \mathcal{E}^{\mathrm{corp}})$:*

(i) The category $\mathcal{E}^{\mathrm{corp}}$ forms a topos, and j^* is an algebraic morphism (i.e., j^* is left exact and preserves colimits).

(ii) For each $X \in \mathcal{E}^{\mathrm{corp}}$, the functor $j_! \colon \mathcal{E}^{\mathrm{corp}}_{/X} \to \mathcal{E}_{/j_!(X)}$ is fully faithful.

(iii) The topos $\mathcal{E}$ can be generated by corporeal objects along colimits.

B.3.1. Admissibility Structures and Geometric Sites

An equivalent — and in practice more convenient — repackaging of fractured structures uses *admissibility structures* on a category.

Definition B.3.3 (Admissibility Structure). Let C be a category with pullbacks. An **admissibility structure** on C is a wide subcategory $C^{\mathrm{ad}} \subseteq C$, whose morphisms are called **admissible morphisms**, satisfying the following conditions:

- C^{ad} is stable under base change in C.
- C^{ad} is left cancellative: given $f \colon X \to Y$ and $g \colon Y \to Z$, if g and $g \circ f$ are both admissible morphisms, then so is f.
- Admissible morphisms are closed under retracts.

Definition B.3.4 (Geometric Site). Let (C, C^{ad}) be a category with admissibility structure. A Grothendieck topology τ is said to be **compatible** with the admissibility structure if every τ-covering sieve contains a τ-covering sieve generated by admissible morphisms. In this case, the triple $(C, C^{\mathrm{ad}}, \tau)$ is called a **geometric site**.

Construction B.3.5. Let $(C, C^{\mathrm{ad}}, \tau)$ be a geometric site. Then C^{ad} inherits a topology from C, where covering sieves are defined as those that generate τ-covering sieves in C. The inclusion $j \colon C^{\mathrm{ad}} \hookrightarrow C$ induces a functor:

$$j_! \colon \mathsf{Shv}_\tau(C^{\mathrm{ad}}) \hookrightarrow \mathsf{Shv}_\tau(C),$$

which has a right adjoint $j^* \colon \mathsf{Shv}_\tau(C) \to \mathsf{Shv}_\tau(C^{\mathrm{ad}})$.

Theorem B.3.6. *The inclusion $j_! \colon \mathsf{Shv}_\tau(C^{\mathrm{ad}}) \hookrightarrow \mathsf{Shv}_\tau(C)$ from Construction B.3.5 exhibits a fractured topos.*

Example B.3.7.
- **Classical Zariski geometry.** Take $C = \mathsf{CAlg}^{\mathrm{fp,op}}$, τ the Zariski topology, and C^{ad} the wide subcategory of open immersions of the form $R \to R[s^{-1}]$.
- **Derived Zariski geometry.** Replace static rings in the above example with animated rings.

- **Spectral Zariski geometry.** Replace static rings with connective commutative ring spectra.
- **Classical étale geometry.** Take $C = \mathsf{CAlg}^{\mathrm{fp,op}}$, τ the étale topology, and C^{ad} étale morphisms.
- **Differential geometry.** Based on C^∞-rings.

B.3.2. Equivalence of Fractured and Admissibility Structures

The two notions — fractured structure and (geometric) admissibility structure — are equivalent.

Definition B.3.8. Let $(\mathcal{E}, \mathcal{E}^{\mathrm{corp}})$ be a fractured topos. A morphism $f : Y \to X$ in $\mathcal{E}$ is called **admissible** if its base change along every corporeal object $Z \in \mathcal{E}^{\mathrm{corp}}$ still lands in $\mathcal{E}^{\mathrm{corp}}$. This determines a wide subcategory $\mathcal{E}^{\mathrm{ad}} \subset \mathcal{E}$.

Definition B.3.9. An admissibility structure $\mathcal{E}^{\mathrm{ad}}$ on a topos $\mathcal{E}$ is called **local** if:

- $\mathcal{E}^{\mathrm{ad}}$ is a local class of morphisms, that is, it satisfies the following properties:
  - (i) It is closed under base change;
  - (ii) It is closed under composition;
  - (iii) Given a pullback square in $\mathcal{E}$ of the form

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle f'} \lrcorner & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

    in which $f' \in \mathcal{E}^{\mathrm{ad}}$ and g is an effective epimorphism, we also have $f \in \mathcal{E}^{\mathrm{ad}}$.

- For each $X \in \mathcal{E}$, the slice $\mathcal{E}^{\mathrm{ad}}_{/X} := (\mathcal{E}^{\mathrm{ad}})_{/X}$ is a presentable category, and the fully faithful inclusion $\mathcal{E}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{E}_{/X}$ preserves colimits (in particular, it has a right adjoint $\rho_X : \mathcal{E}_{/X} \to \mathcal{E}^{\mathrm{ad}}_{/X}$).

An object $X \in \mathcal{E}$ is called **corporeal** if ρ_X preserves colimits. A local admissibility structure is called **geometric** if $\mathcal{E}$ can be generated by corporeal objects along colimits.

Theorem B.3.10. *Let $\mathcal{E}$ be a topos. Then the assignments:*

$$(\mathcal{E}, \mathcal{E}^{\mathrm{corp}}) \mapsto (\mathcal{E}, \mathcal{E}^{\mathrm{ad}}) \quad \textit{(Definition B.3.8)}$$

and:

$$(\mathcal{E}, \mathcal{E}^{\mathrm{ad}}) \mapsto (\mathcal{E}, \mathcal{E}^{\mathrm{corp}}) \quad \textit{(Definition B.3.9)}$$

determine an isomorphism of posets:

$$\{\textit{fractured structures on } \mathcal{E}\} \simeq \{\textit{geometric admissibility structures on } \mathcal{E}\}.$$

B.3.3. Local Morphisms and Schemes

A third equivalent way to package the data of a fractured topos uses *local morphisms* in the lax slice 2-category.

Definition B.3.11. Let $(\mathcal{E}, \mathcal{E}^{\mathrm{corp}})$ be a fractured topos. Consider two geometric morphisms $f, g\colon \mathcal{T} \to \mathcal{E}$ and a natural transformation $\alpha\colon f^* \to g^*$. We say α is **local** if for each morphism $h\colon U \to V$ in $\mathcal{E}^{\mathrm{corp}}$, the following square is a pullback:

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha_U} & g^*U \\ \downarrow{\scriptstyle f^*h} & \lrcorner & \downarrow{\scriptstyle g^*h} \\ f^*V & \xrightarrow{\alpha_V} & g^*V \end{array}$$

This determines the right class of a factorization system on $\mathsf{Fun}_{\mathbb{T}\mathsf{opos}}(\mathcal{T}, \mathcal{E})$. Thus we obtain a wide subcategory $(\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}} \subseteq \mathbb{T}\mathsf{opos}_{//\mathcal{E}}$.

***Remark* B.3.12.** For $\mathcal{E} = \mathsf{Shv}_{\mathrm{Zar}}(\mathsf{CAlg}^{\mathrm{fp,op}})$, the lax slice $\mathbb{T}\mathsf{opos}_{//\mathcal{E}}$ recovers the category of ringed topoi, and $(\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}}$ the category of locally ringed topoi with local morphisms. The "detects units" condition of Definition B.2.3 is the unpacking of Definition B.3.11 in this case.

Local morphisms allow us to recover the notion of scheme intrinsically.

Theorem B.3.13. *Let $(\mathcal{E}, \mathcal{E}^{\mathrm{corp}})$ be a fractured topos. Consider the functor:*

$$\Gamma_X\colon (\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}} \to \mathsf{An}, \quad (f : \mathcal{T} \to \mathcal{E}) \mapsto \Gamma(f^*(X)).$$

(i) If X is corporeal, then Γ_X is representable, represented by $(\mathcal{E}^{\mathrm{ad}}_{/X}, \mathcal{E}^{\mathrm{ad}}_{/X} \hookrightarrow \mathcal{E}_{/X} \to \mathcal{E})$.

(ii) The resulting functor:

$$\mathrm{Re}\colon \{\textit{full subcategories of } \mathcal{E}^{\mathrm{corp}}\} \to (\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}}$$

is fully faithful.

(iii) A morphism f in $\mathcal{E}$ is admissible if and only if for every base change f' of f, if $\Gamma_{X'}$ is representable then $\Gamma_{Y'}$ is also representable and the morphism between representing objects is an étale morphism.

In particular, $(\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}}$ determines the fractured structure on $\mathcal{E}$.

Thus, for a general geometric site $\mathsf{G} = (\mathcal{C}, \mathcal{C}^{\mathrm{ad}}, \tau)$, we can define G-schemes:

Definition B.3.14 (G-Schemes)**.** Let $\mathsf{G} = (\mathcal{C}, \mathcal{C}^{\mathrm{ad}}, \tau)$ be a geometric site, and:

$$(\mathcal{E}, \mathcal{E}^{\mathrm{corp}}) = (\mathsf{Shv}_\tau(\mathcal{C}), \mathsf{Shv}_\tau(\mathcal{C}^{\mathrm{ad}}))$$

the corresponding fractured topos. When τ is the subcanonical topology, there exists a fully faithful spectrum functor:

$$\mathrm{Spec}_{\mathsf{G}}\colon \mathrm{Pro}(\mathcal{C}) \to (\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}}.$$

Define the category of **G-schemes** $\mathsf{Sch}_{\mathsf{G}} \subseteq (\mathbb{T}\mathsf{opos}_{//\mathcal{E}})^{\mathrm{loc}}$ as the full subcategory of objects locally in the image of $\mathrm{Spec}_{\mathsf{G}}$.

B.4. Six-Functor Formalisms and Fractured Topoi

Six-functor formalisms (see [Sch25]) provide a systematic source for constructing fractured topoi: every six-functor formalism canonically determines a topology, thereby giving a geometric site and a fractured topos.

Definition B.4.1 (D-Topology). Let (C, C^{ad}) be a category with admissibility structure, and:

$$\mathsf{D}\colon \mathsf{Span}(C, C^{\mathrm{ad}}) \to \mathsf{Pr}^{\mathrm{L}}$$

a presentable six-functor formalism. The **D-topology** τ_{D} on C is generated by the following covering families: a family $(X_i \to X)_i$ of morphisms in C^{ad} constitutes a τ_{D}-covering if and only if $\mathsf{D}_!$ satisfies descent along every base change of this family (i.e., universal !-descent).

Proposition B.4.2. *The triple $(C, C^{\mathrm{ad}}, \tau_{\mathsf{D}})$ forms a geometric site. In particular:*

$$(\mathsf{Shv}_{\tau_{\mathsf{D}}}(C), \mathsf{Shv}_{\tau_{\mathsf{D}}}(C^{\mathrm{ad}}))$$

is a fractured topos.

Theorem B.4.3 (Mann–Scholze). *The six-functor formalism* D *uniquely extends to a six-functor formalism on:*

$$(\mathsf{Shv}_{\tau_{\mathsf{D}}}(C), \mathsf{Shv}_{\tau_{\mathsf{D}}}(C)^{\mathrm{ad}})$$

such that both D^* *and* $\mathsf{D}_!$ *satisfy the sheaf condition.*

B.4.1. Fractured Localization

The fractured topos $(\mathsf{Shv}_{\tau_{\mathsf{D}}}(C), \mathsf{Shv}_{\tau_{\mathsf{D}}}(C^{\mathrm{ad}}))$ produced by a D-topology is generally too large: $\mathsf{D}_!$-isomorphisms — morphisms the six-functor formalism cannot tell apart — need not have been inverted by sheafification. Fractured localization remedies this by systematically inverting them while keeping the fractured structure intact.

General Fractured Localization

Definition B.4.4 (Fractured Localization). Let $(\mathcal{E}, j_!\colon \mathcal{E}^{\mathrm{corp}} \hookrightarrow \mathcal{E})$ be a fractured topos and $\Sigma \subseteq \mathrm{Ar}(\mathcal{E}^{\mathrm{corp}})$ a congruence. Define the **fractured localization** $(\mathcal{E}^+, (\mathcal{E}^+)^{\mathrm{corp}})$ as the following pullback:

$$\begin{array}{ccc} \mathcal{E}^+ & \longrightarrow & \mathcal{E} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle j^*} \\ \mathcal{E}^{\mathrm{corp}}[\Sigma^{-1}] & \longrightarrow & \mathcal{E}^{\mathrm{corp}} \end{array}$$

where:

- The bottom horizontal arrow is the fully faithful right adjoint to the localization functor $L_\Sigma\colon \mathcal{E}^{\mathrm{corp}} \to \mathcal{E}^{\mathrm{corp}}[\Sigma^{-1}]$ along the congruence Σ (i.e., the inclusion after sheafification);
- The right vertical arrow is the right adjoint j^* of the fractured topos;
- $\mathcal{E}^+$, as a full subcategory of $\mathcal{E}$, consists of those objects $X \in \mathcal{E}$ such that $j^*(X)$ lands in $\mathcal{E}^{\mathrm{corp}}[\Sigma^{-1}]$;
- Set $(\mathcal{E}^+)^{\mathrm{corp}} := \mathcal{E}^{\mathrm{corp}}[\Sigma^{-1}]$.

Proposition B.4.5. *Under the above setup:*

(i) The inclusion $\mathcal{E}^+ \hookrightarrow \mathcal{E}$ is a left exact localization, i.e., $\mathcal{E}^+$ is a topos.

(ii) The composition of the left adjoint $j_!^\Sigma$ and right adjoint $j^{*,\Sigma}$:

$$j_!^\Sigma\colon \mathcal{E}^{\mathrm{corp}}[\Sigma^{-1}] \hookrightarrow \mathcal{E}^+$$

forms a fractured topos.

(iii) The admissibility structure on $\mathcal{E}^+$ is inherited from $\mathcal{E}$: a morphism in $\mathcal{E}^+$ is admissible if and only if it is admissible in $\mathcal{E}$.

Fractured Localization for Six-Functor Formalisms We apply this to a presentable six-functor formalism $\mathsf{D}\colon \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\mathrm{ad}}) \to \mathsf{Pr}^{\mathrm{L}}$. By Proposition B.4.2 and Theorem B.4.3, the D-topology gives a fractured topos $(\mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}), \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}}))$ to which D extends.

Construction B.4.6 ([Hoy26b, Construction 7.29]). Consider the restriction to the corporeal category:

$$\mathsf{D}_!|_{\mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})^{\mathrm{op}}} \colon \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})^{\mathrm{op}} \to \mathsf{Cat}.$$

Define:

$$\Sigma_{\mathsf{D}} := \{f \mid \text{all base changes and all iterated diagonals of } f \text{ are } \mathsf{D}_!\text{-isomorphisms}\}.$$

More precisely, Σ_{D} is the maximal congruence contained in the class of $\mathsf{D}_!$-isomorphisms in $\mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})$. In the language of décalage (Definition B.1.4), $\Sigma_{\mathsf{D}} = D^{\infty}(\Sigma_0)$, where Σ_0 is the modality consisting of all universal $\mathsf{D}_!$-isomorphisms.

Remark **B.4.7.** By definition of the D-topology, every $\mathsf{D}_!$-isomorphism is an effective epimorphism (its base changes generate D-covers). Hence $\Sigma_{\mathsf{D}} \subseteq \mathsf{Conn}_{\infty}$, the class of ∞-connected morphisms.

Definition B.4.8. Define $\mathsf{Shv}^{+}_{\tau_{\mathsf{D}}}(\mathcal{C})$ as the following pullback:

$$\begin{array}{ccc}
\mathsf{Shv}^{+}_{\tau_{\mathsf{D}}}(\mathcal{C}) & \longrightarrow & \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}) \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle j^*} \\
\mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})[\Sigma_{\mathsf{D}}^{-1}] & \longrightarrow & \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})
\end{array}$$

Then:

$$(\mathsf{Shv}^{+}_{\tau_{\mathsf{D}}}(\mathcal{C}),\ \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})[\Sigma_{\mathsf{D}}^{-1}])$$

is a fractured topos.

Lemma B.4.9. *The six-functor formalism further descends to:*

$$\mathsf{D}\colon \mathsf{Span}\big(\mathsf{Shv}^{+}_{\tau_{\mathsf{D}}}(\mathcal{C}),\ \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})[\Sigma_{\mathsf{D}}^{-1}]\big) \to \mathsf{Pr}^{\mathrm{L}}.$$

Remark **B.4.10.** Lemma B.4.9 relies on a general property of six-functor formalisms: if all base changes of $f \in \mathsf{Shv}_{\tau_{\mathsf{D}}}(\mathcal{C}^{\mathrm{ad}})$ are $\mathsf{D}_!$-isomorphisms, then f is also a D^*-isomorphism. This follows from self-duality in the span category — an admissible $Y \to X$ makes $\mathsf{D}(Y)$ a self-dual $\mathsf{D}(X)$-module, and the duality swaps f^* with $f_!$ — and similarly, universal !-descent implies universal $*$-descent.

B.5. Analytic Stacks

As a flagship example of the construction above, we briefly describe Clausen–Scholze's theory of analytic stacks.

Definition B.5.1 (Condensed Anima). **Condensed animae** are hypercomplete sheaves on the category CHaus of compact Hausdorff spaces (equipped with the topology where surjective families are coverings). Equivalently, they are hypercomplete sheaves on the category ProFin of profinite sets. In practice, the light condensed version is often used:

$$\mathsf{CondAn} := \mathsf{Shv}(\{\text{metrizable profinite sets}\})^{\mathrm{hyp}}.$$

Definition B.5.2 (Analytic Ring). An **analytic ring** is a pair $A = (A^{\triangleright}, \mathrm{Mod}_A)$, where $A^{\triangleright}$ is a condensed ring and $\iota\colon \mathrm{Mod}_A \hookrightarrow \mathrm{Mod}_{A^{\triangleright}}$ is a reflective subcategory (with left adjoint L), satisfying:

- Mod_A is closed under colimits.
- Mod_A is closed under internal Hom $\underline{\mathrm{Hom}}(M,-)$ (for each $M \in \mathrm{Mod}_{A^\triangleright}$).
- ιL preserves connective objects (with respect to the canonical t-structure).
- $A^\triangleright \in \mathrm{Mod}_A$.

Morphisms of analytic rings are ring homomorphisms such that the pushforward functor preserves complete modules. We denote the category of analytic rings by AnRing.

Definition B.5.3. A morphism $f: A \to B$ of analytic rings is called:

- **Proper** if f_* induces an equivalence:

$$\mathrm{Mod}_B \xrightarrow{\sim} \mathrm{Mod}_{B^\triangleright} \times_{\mathrm{Mod}_{A^\triangleright}} \mathrm{Mod}_A.$$

- An **open immersion** if f^* has a left adjoint $f_!$ satisfying the projection formula.
- **!-able** if f factors as an open map followed by a proper map.

We use $\mathsf{AnRing}^! \subset \mathsf{AnRing}$ to denote the wide subcategory spanned by !-able morphisms.

Proposition B.5.4. *There exists a presentable six-functor formalism:*

$$\mathsf{D}\colon \mathsf{Span}(\mathsf{AnRing}^{\mathrm{op}}, \mathsf{AnRing}^{!,\mathrm{op}}) \to \mathsf{Pr}^L, \quad A \mapsto \mathrm{Mod}_A.$$

Definition B.5.5 (Analytic Stacks)**.** The category of **analytic stacks** is defined as:

$$\mathsf{AnStk} := \mathsf{Shv}^+_{\tau_{\mathsf{D}}}(\mathsf{AnRing}^{\mathrm{op}}),$$

i.e., the topos obtained via the D-topology and fractured localization.

***Remark* B.5.6.** The definition above incorporates the corrections to the original Clausen–Scholze IHES formulation. The six-functor formalism automatically extends to analytic stacks (Theorem B.4.3 together with Lemma B.4.9).

B.6. Gestalten

The theory of **Gestalten**, recently developed by Scholze–Stefanich [Sch26], packages algebraic geometry, condensed mathematics, and higher category theory into a single framework. Its core idea revisits the most basic duality in algebraic geometry — "affine schemes are commutative rings, opposite-wise".

B.6.1. Presentable Symmetric Monoidal n-Categories

The starting point of the construction is the following observation: for $C \in \mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}_\kappa)$ (where κ is a regular cardinal), the functor $A \mapsto \mathrm{Mod}_A(C)$ yields a fully faithful embedding:

$$\mathsf{CAlg}(C) \hookrightarrow \mathsf{CAlg}(\mathrm{Mod}_C(\mathsf{Pr}^{\mathrm{L}}_\kappa)),$$

whose right adjoint is $\mathcal{D} \mapsto \mathrm{End}_{\mathcal{D}}(\mathbb{1})$. Equivalently, operations at the commutative algebra level (colimits, tensor products) can be faithfully lifted to the categorical level.

Taking C to be $\mathsf{Pr}^{\mathrm{L}}_\kappa$ itself and iterating, we can define presentable n-categories:

Definition B.6.1 (Presentable n-Categories). Define inductively:

$$0\mathsf{Pr} := \mathsf{Ani}, \qquad (n+1)\mathsf{Pr} := \mathrm{Mod}_{n\mathsf{Pr}}(\mathsf{Pr}^{\mathrm{L}}_{\aleph_1}).$$

From here on, we fix $\kappa = \aleph_1$ and omit the subscript. Each $n\mathsf{Pr}$ is a symmetric monoidal $\aleph_1$-presentable category.

This yields a sequence of fully faithful embeddings:

$$\mathsf{CAlg}(0\mathsf{Pr}) \hookrightarrow \mathsf{CAlg}(1\mathsf{Pr}) \hookrightarrow \mathsf{CAlg}(2\mathsf{Pr}) \hookrightarrow \cdots,$$

where the transition functors are $A \mapsto \mathrm{Mod}_A(n\mathsf{Pr})$, with right adjoints $\mathcal{D} \mapsto \mathrm{End}_{\mathcal{D}}(\mathbb{1})$.

***Remark* B.6.2.** $n\mathsf{Pr}$ is enriched over $n-1\mathsf{Pr}$: for $\mathcal{C}, \mathcal{D} \in n\mathsf{Pr}$, the internal Hom object $\mathsf{Hom}_{n\mathsf{Pr}}(\mathcal{C}, \mathcal{D}) \in n-1\mathsf{Pr}$ is determined by the universal property $\mathsf{Hom}_{n\mathsf{Pr}}(\mathcal{C}, \mathcal{D}) \otimes \mathcal{C} \to \mathcal{D}$. This endows $n\mathsf{Pr}$ with an n-categorical structure. In particular, $2\mathsf{Pr}$ is a "presentable symmetric monoidal 2-category."

B.6.2. Stefanich Rings

Definition B.6.3 (Stefanich Ring). The category of **Stefanich rings** is defined as the colimit in $1\mathsf{Pr}$ of the above chain of fully faithful embeddings:

$$\mathsf{StRing} := \mathrm{colim}_{1\mathsf{Pr}}\left(\mathsf{CAlg}(\mathsf{Ani}) \hookrightarrow \mathsf{CAlg}(1\mathsf{Pr}) \hookrightarrow \mathsf{CAlg}(2\mathsf{Pr}) \hookrightarrow \cdots\right).$$

Equivalently, taking right adjoints yields a limit in $\widehat{\mathsf{Cat}}$:

$$\mathsf{StRing} \simeq \lim_{\widehat{\mathsf{Cat}}}\left(\mathsf{CAlg}(\mathsf{Ani}) \xleftarrow{\mathrm{End}_{(-)}(\mathbb{1})} \mathsf{CAlg}(1\mathsf{Pr}) \xleftarrow{\mathrm{End}_{(-)}(\mathbb{1})} \cdots\right).$$

A Stefanich ring $A = (A_0, A_1, A_2, \ldots)$ consists of $A_n \in \mathsf{CAlg}(n\mathsf{Pr})$ together with equivalences $A_n \simeq \mathrm{End}_{A_{n+1}}(\mathbb{1})$. The unit object is $(*, \mathsf{Ani}, 1\mathsf{Pr}, 2\mathsf{Pr}, \ldots)$.

***Remark* B.6.4** (Categorical Spectrum Analogy). Stefanich rings can be viewed as "categorical spectra": in classical spectrum theory, a spectrum is a sequence of spaces $(X_0, X_1, \ldots)$ equipped with equivalences $X_n \simeq \Omega X_{n+1}$, where Ω denotes taking loop spaces; a Stefanich ring is a sequence of symmetric monoidal categories equipped with equivalences $A_n \simeq \mathrm{End}_{A_{n+1}}(\mathbb{1})$, where $\mathrm{End}_{(-)}(\mathbb{1})$ plays the role of "taking loop spaces" at the categorical level.

Slice Categories and the $\mathrm{Spec}_n \dashv \Gamma_n$ Adjunction The slice category of Stefanich rings admits a limit-colimit description similar to the global category. For each $n \geq 0$, there exists an adjoint pair:

$$\mathrm{Mod}_{A_n}(1\mathsf{Pr}) \underset{G_n}{\overset{F_n}{\rightleftarrows}} A_{n+1} \quad (F_n \dashv G_n)$$

where $G_n(X) = \underline{\mathsf{Hom}}_{A_{n+1}}(\mathbb{1}, X)$. By the definition of Stefanich rings, $G_n(\mathbb{1}_{A_{n+1}}) \simeq A_n$.

Proposition B.6.5. *Let A be a Stefanich ring. Then there is an equivalence:*

$$\mathsf{StRing}_{A/} \simeq \mathrm{colim}_{1\mathsf{Pr}}\left(\mathsf{CAlg}(A_1) \hookrightarrow \mathsf{CAlg}(A_2) \hookrightarrow \cdots\right),$$

where the transition functors are $R \mapsto F_n(\mathrm{Mod}_R(A_n))$. For each $n \geq 0$, the structure map yields a fully faithful embedding:

$$\mathrm{Spec}_n^{\mathrm{op}} : \mathsf{CAlg}(A_{n+1}) \hookrightarrow \mathsf{StRing}_{A/},$$

whose right adjoint $\Gamma_n^{\mathrm{op}} : \mathsf{StRing}_{A/} \to \mathsf{CAlg}(A_{n+1})$, $B \mapsto (B/A)_n$ extracts the n-th layer data.

***Remark* B.6.6.** The notation is motivated by geometry: at the level of Gestalten $\mathsf{Gest} = \mathsf{StRing}^{\mathrm{op}}$, $\mathrm{Spec}_n^{\mathrm{op}}$ is the "relative spectrum functor at the n-th categorical level," and Γ_n^{op} is the "global sections functor at the n-th level." This precisely generalizes the classical $\mathrm{Spec} \dashv \Gamma$ adjunction.

Tensor Products and Spectrification Limits of Stefanich rings are computed levelwise. Colimits (in particular, tensor products) are more subtle: taking colimits levelwise yields a sequence $(D_n)_n$ equipped with maps $D_n \to \mathrm{End}_{D_{n+1}}(\mathbb{1})$, but these maps are generally not equivalences—one needs to perform **spectrification** to make them equivalences. Spectrification is realized by iteratively replacing $D'_n := \mathrm{End}_{D_{n+1}}(\mathbb{1})$ and taking a transfinite colimit; since $D \mapsto \mathrm{End}_D(\mathbb{1})$ preserves $\aleph_1$-filtered colimits, the process stabilizes after $\aleph_1$ steps.

This mechanism is analogous to the process in classical stable homotopy theory of spectrifying prespectra $(X_n, X_n \to \Omega X_{n+1})$ to obtain genuine spectra $(X_n \simeq \Omega X_{n+1})$.

B.6.3. Properties of Morphisms

Morphisms between Stefanich rings carry a family of properties graded by categorical level — categorifications of classical notions like affineness, properness, smoothness, etc. We introduce them in order from basic to composite.

n-Affine Morphisms n-Affineness asserts: starting from the n-th layer, the entire Stefanich ring is uniquely determined by a single commutative algebra datum $(B/A)_n \in \mathsf{CAlg}(A_{n+1})$ through iteratively taking module categories—in other words, **spectrification above the n-th layer is unnecessary**.

Definition B.6.7 (n-Affine)**.** Let A be a Stefanich ring and $n \geq 0$. An A-algebra $B \in \mathsf{StRing}_{A/}$ is called **n-affine** if the counit $\mathrm{Spec}_n^{\mathrm{op}}(\Gamma_n^{\mathrm{op}}(B)) \xrightarrow{\sim} B$ is an equivalence. Equivalently, for all $m \geq n$, the canonical functor $\mathrm{Mod}_{(B/A)_m}(A_{m+1}) \to B_{m+1}$ is an equivalence.

***Remark* B.6.8** (Levels of Affineness)**.** Classical affine schemes are exactly the 0-affine Stefanich rings: B is determined by the ring $(B/A)_0 \in \mathsf{CAlg}(A_1)$. Gaitsgory's 1-affineness (Remark 6.2.17, $\mathsf{ShvCat}(\mathcal{Y}) \simeq \mathsf{QCoh}(\mathcal{Y})$-mod) is the case $n = 1$: B is determined by $(B/A)_1 \in \mathsf{CAlg}(A_2)$. More generally, for n-affine B everything from the n-th layer up is generated by purely algebraic data — at sufficient categorical depth, geometry collapses back to algebra.

Proposition B.6.9. *n-Affine morphisms are closed under small colimits, base change, composition, and diagonals.*

n-Prim and n-Proper Morphisms n-Primness (n-prim) is a categorified version of "the pushforward functor f_* has an internal right adjoint within the category." Its motivation comes from primness in six-functor theory (see the article on six-functor formalisms): there, $\mathbb{1}_X$ being f-prim means that f_* can be obtained by twisting $f_!$, i.e., $f_!(\delta_f \otimes -) \simeq f_*(-)$.

Definition B.6.10 (n-Prim)**.** Let $f : A \to B$ be a morphism of Stefanich rings and $n \geq 0$. We say f is **n-prim** if for all $m \geq n+1$, the functor $f^*_{m-1} : \mathbb{1} \to (B/A)_m$ in A_{m+1} has a right adjoint $f_{m-1,*} : (B/A)_m \to \mathbb{1}$. In other words, the external right adjoint $f_{m-1,*} : B_m \to A_m$ can be refined to an internal right adjoint in A_{m+1}.

Proposition B.6.11 ([Sch26, Proposition 6.10])**.**

(i) Morphisms that are n-affine and $\aleph_1$-presentable are n-prim.

(ii) n-Prim morphisms are closed under countable colimits, base change, and composition.

(iii) Base change preserves the construction of $f_{m,*}$: the formation of $f_{m,*}$ commutes with base change.

*However, n-primness is **not** closed under diagonals.*

Since n-primness is not closed under diagonals, we need a stronger concept:

Definition B.6.12 (n-Proper)**.** We say $f\colon A \to B$ is **n-proper** if f and all its iterated diagonals $\Delta_f, \Delta_{\Delta_f}, \ldots$ are all n-prim.

Proposition B.6.13. *n-Proper morphisms are closed under countable colimits, base change, and composition. Moreover, morphisms that are n-affine and $\aleph_1$-presentable are n-proper.*

***Remark* B.6.14.** 0-Properness is closely related to Gaitsgory–Rozenblyum's notion of **rigidity**: an $\aleph_1$-presentable 1-affine Stefanich A-algebra B is 0-proper if and only if $(B/A)_1 \in \mathsf{CAlg}(A_2)$ is rigid in A_2.

n-Suave and n-Étale Morphisms n-Suaveness is the left-adjoint counterpart of n-primness: it requires the pushforward functor to have a *left* (not right) adjoint. Its motivation comes from suaveness in six-functor theory: $\mathbb{1}_X$ being f-suave means that $f^!$ can be obtained by twisting f^*, i.e., $\omega_f \otimes f^*(-) \simeq f^!(-)$.

Definition B.6.15 (n-Suave)**.** Let $f\colon A \to B$ be a morphism of Stefanich rings and $n \geq 0$. We say f is **n-suave** if for all $m \geq n+1$, the functor $f^*_{m-1}\colon \mathbb{1} \to (B/A)_m$ in A_{m+1} has a left adjoint $f_{m-1,\sharp}\colon (B/A)_m \to \mathbb{1}$, and moreover: for $m \geq n$, setting:

$$(B/A)^!_m := f_{m,\sharp}(\mathbb{1}) \in A_{m+1},$$

for all $m \geq n+1$, the counit map $(B/A)^!_m \to \mathbb{1}$ has a right adjoint in A_{m+1}.

***Remark* B.6.16.** $(B/A)^!_m$ is naturally a cocommutative coalgebra in A_{m+1}, whose dual is the algebra $(B/A)_m$. The notation suggests "!-sheaves": for an fpqc stack X (which is 1-suave over $\mathrm{Spec}(\mathbb{S})$), $(X/\mathbb{S})^!_1$ is precisely the category of !-sheaves (obtained by taking limits along !-pullbacks). For $m \geq 2$, ambidexterity gives $(B/A)^!_m = (B/A)_m$—the two differ only at the lowest level.

Proposition B.6.17. *n-Suave morphisms are closed under base change and composition, and the construction of $f_{m,\sharp}$ commutes with base change.*

Analogous to the relationship between n-properness and n-primness, n-étaleness is defined by requiring all iterated diagonals to be suave:

Definition B.6.18 (n-Étale)**.** We say $f\colon A \to B$ is **n-étale** if f and all its iterated diagonals are n-suave.

Proposition B.6.19. *n-Étale morphisms are closed under base change and composition.*

Ambidexterity These four classes of morphisms are linked through ambidexterity, which connects adjacent levels.

Proposition B.6.20 (Ambidexterity)**.** *Let $f\colon A \to B$ be a morphism of Stefanich rings and $n \geq 0$.*

(i) If f and Δ_f are both n-prim, then f is $(n+1)$-suave. In particular, n-proper $\Rightarrow$ $(n+1)$-étale.

(ii) If f and Δ_f are both n-suave, then f is $(n+1)$-prim. In particular, n-étale $\Rightarrow$ $(n+1)$-proper.

Under either of the above assumptions, for $m \geq n+2$ we have $f_{m,\sharp} \simeq f_{m,*}$, and this functor preserves dualizable objects.

***Remark* B.6.21.** Morally, ambidexterity says that at sufficiently high categorical levels left and right adjoints automatically coincide — a phenomenon foreshadowed by Hopkins–Lurie's theory of ambidexterity, here generalized across all categorical levels. For fpqc stacks, for example, the morphism is étale at level 1 (left adjoint exists) and proper at level 2 (right adjoint exists); from level 2 onward the two adjoints agree.

The above can be summarized in the following implication diagram:

$$\cdots \Rightarrow n\text{-proper} \Rightarrow (n+1)\text{-étale} \Rightarrow (n+2)\text{-proper} \Rightarrow \cdots$$

$$\cdots \Rightarrow n\text{-étale} \Rightarrow (n+1)\text{-proper} \Rightarrow (n+2)\text{-étale} \Rightarrow \cdots$$

Properness and étaleness thus implicate each other at alternating levels: once the chain begins at some finite stage, all higher properties follow automatically.

B.6.4. The Category of Gestalten

Definition B.6.22 (Gestalten). The **category of Gestalten** is defined as $\mathsf{Gest} := \mathsf{StRing}^{\mathrm{op}}$.

Theorem B.6.23 (Scholze–Stefanich). *Let A be a Stefanich ring and $n \geq 0$. Then the opposite category of n-étale Stefanich algebras over A:*

$$(\mathsf{StRing}_{A/}^{n\text{-ét}})^{\mathrm{op}}$$

is a countable topos (i.e., its $\aleph_1$-ind-completion is a topos).

***Remark* B.6.24.** This theorem has two limitations. The first is "countable"—this is a universe size constraint, because StRing and its opposite cannot both be presentable. The second is that it fixes the level of n-étaleness—but due to ambidexterity, all algebraic stacks are 1-étale, and n-étaleness increases by at most 1 or 2 under reasonable operations, so in practice fixing $n = 10$ suffices for everything.

Six-Functor Formalisms and Gestalten Six-functor formalisms supply a systematic source of non-affine Gestalten.

Construction B.6.25. Let $\mathsf{D}\colon \mathsf{Span}(C, C^{\mathrm{ad}}) \to 1\mathsf{Pr}$ be a presentable six-functor formalism. For $X \in C$, the kernel 2-category:

$$\mathbb{K}_{\mathsf{D},X} \in \mathsf{CAlg}(2\mathsf{Pr})$$

(see the article on kernel 2-categories) yields a "lifted six-functor formalism":

$$\mathbb{K}_{\mathsf{D}}\colon \mathsf{Span}(C, C^{\mathrm{ad}}) \to 2\mathsf{Pr}.$$

Iterating this construction, we define a Stefanich ring:

$$A_{\mathsf{D},X} = (\mathsf{D}(X), \mathbb{K}_{\mathsf{D},X}, \mathbb{K}_{\mathbb{K}_{\mathsf{D}},X}, \ldots),$$

thereby yielding a functor $[-]_{\mathsf{D}}\colon C \to \mathsf{Gest}$.

***Remark* B.6.26.** The successive layers of $A_{\mathsf{D},X}$ encode D-sheaves at increasing categorical depth:

- $\mathsf{D}(X)$ — the category of sheaves (0-sheaf data),
- $\mathbb{K}_{\mathsf{D},X}$ — the 2-category of kernel transformations between sheaves (1-sheaf data),
- $\mathbb{K}_{\mathbb{K}_{\mathsf{D}},X}$ — kernel transformations of kernel transformations (2-sheaf data),
- and so on.

The equivalences $\mathrm{End}_{\mathbb{K}_{\mathsf{D},X}}(\mathbb{1}) \simeq \mathsf{D}(X)$ glue adjacent layers together.

When the Künneth formula holds (i.e., $\mathsf{D}(X) \otimes_{\mathsf{D}(*)} \mathsf{D}(Y) \xrightarrow{\sim} \mathsf{D}(X \times Y)$), we have $\mathbb{K}_{\mathsf{D}} \simeq \mathrm{Mod}_{\mathsf{D}(*)}(1\mathsf{Pr})$, in which case $A_{\mathsf{D},X}$ is a 1-affine Gestalt.

Connection with Fractured Topoi

Proposition B.6.27. *The functor* $[-]_{\mathsf{D}} \colon \mathcal{C}^{\mathrm{ad}} \to \mathsf{Gest}$ *preserves pullbacks, and a family of morphisms is a* D-*covering if and only if its image is an effective epimorphism in Gestalten. Therefore, sheafification is the left Kan extension of* D *along* $\mathsf{D} \hookrightarrow \mathsf{Gestalten}$*, in accordance with the universal property of the latter.*

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