Galois cover

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This document serves as the script for the 6th talk in the seminar "Galois Theory in Algebra and Topology". It is based on [Szamuely, 2009], although we will cover some material beyond it.

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Contents

1 Motivation : Covering Spaces and Polynomial Equations

In earlier talks, we used the classical definition of covering spaces:

Definition 1.1. A continuous map $p\colon E \to B$ is called a *covering space* if, for every $b \in B$, there exists a neighbourhood V of b such that in the pullback square

$$\begin{tikzcd} E \times_B V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ V \arrow[r, hook] & B, \end{tikzcd}$$

the projection $E \times_B V \to V$ is, as a map over V, homeomorphic to a disjoint union of copies of V, that is,

$$E \times_B V \;\cong\; \coprod_{s \in S} V$$

as spaces over V, for some set S.

To motivate the concept of a Galois cover, we first need to introduce some related notions.

Definition 1.2. A continuous map $f\colon X \to Y$ is called *étale* if for every point $x \in X$, there exists open sets $U \subseteq X$ containing x and $V \subseteq Y$ containing $f(x)$ such that f induces a homeomorphism $U \to V$.

If f fails to be an étale map at some point $y \in Y$, then f is said to be *branched* at y, and $x = f(y)$ is called a *branch point*.

Thus, a branched covering is a finite-to-one map which is étale away from a finite branch locus. Equivalently, the deck transformation group acts freely over the étale locus, but may have fixed points over the branch points.

In fact, every covering map is an étale map.

Remark 1.3. A covering map is, by definition, an étale map with discrete fibres. Thus, if a continuous map $f\colon Y \to X$ fails to be étale at a point $y \in Y$, then f cannot be a covering map.

Indeed, if f is not étale at y, then no neighbourhood of y is mapped homeomorphically onto its image. Consequently, for every open neighbourhood V of $x = f(y)$, the preimage $f^{-1}(V)$ cannot be a disjoint union of open sets on each of which f restricts to a homeomorphism. This violates the defining condition for a covering map.

In particular, a *branched* (or *ramified*) map is a finite map that fails to be étale at finitely many points—the branch points. Over those points, sheets of the covering merge, and the deck transformation group has fixed points. Away from the branch locus, the map is étale and restricts to a genuine covering.

This provides the framework for studying how the roots of a polynomial vary continuously with its parameters.

1.1 A motivating example: the quadratic equation

Let

$$B = \{(b, c) \in \mathbb{C}^2 : b^2 - 4c \neq 0\},$$

the parameter space of monic quadratic polynomials with distinct roots. Define

$$X = \{(x, b, c) \in \mathbb{C} \times B : x^2 + bx + c = 0\}, \qquad p : X \to B,\ (x, b, c) \mapsto (b, c).$$

For fixed (b, c), the two solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

This suggests that $p : X \to B$ is a 2-sheeted cover, but the function $\sqrt{b^2 - 4c}$ is not single-valued on B. To obtain a single-valued square root, let

$$B' = \{(\Delta, b, c) \in \mathbb{C}^* \times B : \Delta^2 = b^2 - 4c\},$$

and $(\Delta, b, c) \mapsto (b, c)$ is a covering map. On B', the functions

$$s_{\pm}(\Delta, b, c) = \tfrac{1}{2}(-b \pm \Delta)$$

define two global sections of the pullback cover $B' \times_B X \to B'$. Hence

$$B' \times_B X \cong B' \times \{1, 2\},$$

so the quadratic equation becomes trivial after passing to the covering space B'.

1.2 The cubic equation

For the cubic equation $x^3 + px + q = 0$, define

$$\Delta(p, q) = \frac{q^2}{4} + \frac{p^3}{27}.$$

A root is given by the classical formula

$$x = C - \frac{p}{3C}, \qquad C = \sqrt[3]{\frac{q}{2} - \sqrt{\Delta(p, q)}}.$$

Let

$$B = \{(p, q) \in \mathbb{C}^2 : \Delta(p, q) \neq 0\}, \qquad X = \{(x, p, q) \in \mathbb{C} \times B : x^3 + px + q = 0\}.$$

To make the radicals single-valued, we perform two successive base changes:

$$B_1 = \{(y_1, p, q) : y_1^2 = \Delta(p, q)\},$$
$$B_2 = \{(y_2, y_1, p, q) : y_2^3 = q/2 - y_1\}.$$

The map $B_1 \to B$ is a covering map, hence étale. The second map $B_2 \to B_1$ is finite-to-one but *branched* at finitely many points; it fails to be étale exactly on the branch locus. Removing these branch points yields a genuine covering on which the three roots give global sections.

1.3 Motivation for Galois theory

These examples illustrate the key principle:

- the roots of a polynomial cannot be chosen continuously on the entire parameter space;
- but they *can* be chosen continuously after passing to an appropriate covering space;
- the minimal such covering encodes the monodromy action of the fundamental group on the roots.

This monodromy representation is the topological origin of the Galois group. Thus, covering spaces provide the geometric entry point to Galois theory.

1.4 Notation and convention

Throughout this talk, we adopt the following conventions:

- If $p\colon E \to B$ is a covering map and $b \in B$, we write E_b for the fiber $p^{-1}(b)$.

1.5 Structure

- In Section 2, we discuss Galois coverings and the Galois correspondence, following [Szamuely, 2009].
- If time permits, Appendix A will sketch how covering space theory and the Galois correspondence extend to the setting of higher category theory.
- Appendix B outlines the theory of Galois categories and its generalisations. Due to time constraints, this part will not be covered in the talk.

2 Galois Cover

In the previous section, we saw that a covering map $p\colon E \to B$ encodes the idea that E locally looks like a disjoint union of copies of B. In many natural situations, the essential information of a covering is contained in the *symmetries* of its fibers: the ways in which the different sheets can be permuted while preserving the projection to the base. The goal of this section is to explain how these symmetries assemble into a "Galois group" and how the classical Galois correspondence reappears for covering spaces.

2.1 Automorphisms of coverings and Galois coverings

Definition 2.1. Let (E, p) be a covering space of B. An *automorphism* of (E, p) is a homeomorphism $f\colon E \to E$ such that

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

commutes. The group of all such automorphisms is denoted $\mathrm{Aut}(E|B)$.

For every $b \in B$, the fiber E_b is setwise stable under the induced left action of $\mathrm{Aut}(E|B)$.

Definition 2.2. Let B be connected and (E, p) a covering of B. We call (E, p) a *Galois covering* if:

- E is connected;
- the action of $\mathrm{Aut}(E|B)$ on each fiber E_b is transitive.

In this case, $\mathrm{Aut}(E|B)$ is called the *Galois group*.

Proposition 2.3. *Let (E, p) be a connected covering of B. Let $G = \mathrm{Aut}(E|B)$. The following are equivalent:*

1. *(E, p) is a Galois covering;*
2. *the map p factors as*
$$E \longrightarrow G\backslash E \xrightarrow{\overline{p}} B,$$
and $\overline{p}$ is a homeomorphism.

Proof sketch. The quotient $G\backslash E$ has points given by the G–orbits in E. The induced map $\overline{p}$ is injective if and only if each orbit is exactly a fiber, i.e., if and only if G acts transitively on fibers. Surjectivity and continuity are automatic. Thus the factorization identifies B with the orbit space precisely in the Galois case. ☐

2.2 Evenly acting groups and the basic criterion

A mild assumption of local connectedness is convenient, as it ensures a basis of connected neighborhoods, which simplifies lifting arguments.

Definition 2.4. A topological space X is *locally connected* if each point has a basis of connected open neighborhoods.

Definition 2.5. A left action of a group G on a space X is *even* if for each $x \in X$ there exists a neighborhood U of x such that

$$gU \cap g'U = \varnothing \qquad (g \neq g').$$

We now establish a central uniqueness lemma.

Lemma 2.6. *Let (E, p) be a connected covering of a locally connected space B. If $\phi \in \mathrm{Aut}(E|B)$ fixes a point of E, then $\phi = \mathrm{id}_E$.*

Proof sketch. Apply the more general Proposition 2.7 with $Z = E$, $f = \mathrm{id}_E$ and $g = \phi$. Since f and g agree at one point and satisfy $p \circ f = p \circ g$, and E is connected, they must agree everywhere. ☐

The lemma is a direct consequence of:

Proposition 2.7. *Let (E, p) be a covering of a locally connected space B. Let Z be connected and $f, g\colon Z \to E$ satisfy $p \circ f = p \circ g$. If $f(z_0) = g(z_0)$ for some $z_0 \in Z$, then $f = g$.*

Proof sketch. Look at the set $S = \{z \in Z\colon f(z) = g(z)\}$. Use local connectedness of B to show that near any point the fibers of p split uniquely, forcing S to be open. Closedness follows similarly. Since Z is connected and S is clopen and nonempty, $S = Z$. □

We now show that even actions give rise to Galois coverings.

Proposition 2.8. *If G acts evenly on a connected and locally connected space E, then the quotient $p_G\colon E \to G\backslash E$ is a Galois covering and*

$$\mathrm{Aut}(E|(G\backslash E)) \cong G.$$

Proof sketch. Evenness gives disjoint translates of a small open set, so p_G is a covering. The action of G gives a subgroup of $\mathrm{Aut}(E|(G\backslash E))$. Conversely, any automorphism ϕ must send a point e to a unique translate $g \cdot e$; by Lemma 2.6, $g^{-1}\phi = \mathrm{id}$, so $\phi = g$. Transitivity on fibers is evident since fibers are G–orbits. □

The converse holds as well:

Proposition 2.9. *Let (E, p) be a connected covering of a locally connected space B. Then the natural action of $G = \mathrm{Aut}(E|B)$ on E is even.*

Proof sketch. Choose a connected trivializing neighborhood V of $b = p(e)$ so that $p^{-1}(V)$ splits as disjoint copies $U_i \cong V$. Any two distinct automorphisms must send U_i into different U_j; otherwise they agree on a point and hence everywhere by Lemma 2.6. □

Finally, we record the following useful criterion.

Proposition 2.10. *A connected covering $p\colon E \to B$ is Galois iff there exists one point $b \in B$ such that $\mathrm{Aut}(E|B)$ acts transitively on E_b.*

Proof sketch. The "only if" direction is immediate. For the converse, use that over a connected base all fibers are homeomorphic ([Szamuely, 2009, Cor. 2.1.4]); thus transitivity at one fiber forces transitivity at all fibers. □

2.3 The main theorem: Galois correspondence

Theorem 2.11. (Galois correspondence) *Let (E, p) be a Galois covering of a connected and locally connected space B, and let $G = \mathrm{Aut}(E|B)$. Then the assignment*

$$H \subseteq G \quad \longmapsto \quad \big(H\backslash E,\ p_H\colon H\backslash E \to B\big)$$

yields a bijection between subgroups $H \subseteq G$ and intermediate connected coverings

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & B. \end{tikzcd}$$

Moreover, $f\colon E \to Y$ is always a Galois covering *and the correspondence is order-reversing:*

$$H_1 \subseteq H_2 \quad \Longleftrightarrow \quad H_1\backslash E \to H_2\backslash E \textit{ is a Galois covering.}$$

Finally, the intermediate covering p_H is Galois if and only if $H \subseteq G$ is a normal subgroup.

Remark 2.12. This correspondence extends naturally to the setting of higher category theory: using the functor

$$\Pi_\infty\colon \mathsf{Top} \to \mathsf{An}$$

(constructed in Appendix A), classical covering spaces correspond to left fibrations with discrete fibers, and the above Galois theory becomes the statement that connected covering animae over B correspond to subgroups of $\pi_1(B, b)$.

Before proving the theorem, we establish a technical lemma.

Lemma 2.13. *In a commutative triangle*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & B \end{tikzcd}$$

with p and q coverings of a locally connected base B and Y connected, the map f is a covering.

Proof sketch. Trivialize both p and q over a small connected neighborhood $V \subset B$. Then $p^{-1}(V)$ and $q^{-1}(V)$ become disjoint unions of copies of V. By commutativity of the diagram, f sends each sheet of $p^{-1}(V)$ into exactly one sheet of $q^{-1}(V)$, homeomorphically. Thus f is locally a disjoint union of copies of V. $\square$

Proof of Theorem 2.11. We divide the proof into the following steps.

From subgroups H to intermediate coverings. Given $H \subseteq G$, consider

$$E \xrightarrow{p_H} H\backslash E \xrightarrow{\overline{p}_H} B.$$

Since $p = \overline{p}_H \circ p_H$ and p is a covering, local triviality over any sufficiently small $V \subset B$ implies

$$\overline{p}_H^{-1}(V) \;\cong\; \coprod_{[i]\in H\backslash E_b} V,$$

where $[i]$ ranges over H–orbits in the fiber E_b. Thus $\overline{p}_H$ is a covering.

From intermediate coverings to subgroups. Given a factorization

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & B \end{tikzcd}$$

with Y connected and q a covering, Lemma 2.13 implies f is a covering. Define

$$H \coloneqq \mathrm{Aut}(E|Y) = \{\phi \in G \mid f \circ \phi = f\}.$$

To show H acts transitively on the fibers of f, let $e_1, e_2 \in f^{-1}(y)$. Since p is Galois and $p(e_1) = p(e_2)$, there exists $\varphi \in G$ with $\varphi(e_2) = e_1$. One checks (using Proposition 2.7) that $f \circ \varphi = f$, so $\varphi \in H$. Thus, f is a Galois covering.

Showing the correspondences are inverse. These two constructions are inverse by definition: quotienting by H collapses exactly the fibers of f, and conversely $\mathrm{Aut}(E|H\backslash E) = H$.

Normality and Galois intermediate covers. If $H \subseteq G$ is a normal subgroup, then G/H acts naturally on $H\backslash E$ over B, and the quotient is $G\backslash E \cong B$, so p_H is Galois.

Conversely, if $q : Y \to B$ is a Galois cover, we show that $H \coloneqq \mathrm{Aut}(E|Y)$ is a normal subgroup of G. Recall that a subgroup H is normal in G if and only if it can be formed as the kernel of a group homomorphism $G \to K$. Here, we plan to define H as the kernel $\ker(G \to \mathrm{Aut}(Y|B))$. For this, we need to show that every $\phi \in G$ induces an automorphism of Y over B. In other words, we need an automorphism $\psi\colon Y \to Y$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[d, "f"'] & & E \arrow[d, "f"] \\ Y \arrow[rr, "\psi"] \arrow[dr, "q"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

For this, take a point $e \in E$ and let $b = p(e)$ in B. One can easily find that $f(e)$ and $f(\phi(e))$ are both in the fiber $q^{-1}(b)$, since $q(f(\phi(e))) = p(\phi(e)) = p(e) = q(f(e)) = b$. Since q is a Galois covering, $\mathrm{Aut}(Y|B)$ acts transitively on the fibers of q. Thus, there exists an automorphism $\psi \in \mathrm{Aut}(Y|B)$ with the property that $\psi(f(e)) = f(\phi(e))$. By Proposition 2.7, we have $\psi \circ f = f \circ \phi$. In fact, ψ is the unique element of $\mathrm{Aut}(Y|B)$ with this property. For if $\lambda \in \mathrm{Aut}(Y|B)$ also satisfies $\lambda(f(e)) = f(\phi(e))$, then $(\lambda^{-1} \circ \psi)(f(e)) = f(e)$. Since Y is connected, Lemma 2.6 (applied to the covering $q : Y \to B$) implies $\lambda^{-1} \circ \psi = \mathrm{id}$, so $\lambda = \psi$. □

The crucial link, which connects this automorphism group to the topology of the base space B, is the "monodromy action". This action describes how the fundamental group $\pi_1(B, b)$ permutes the points in the fiber E_b. In the Galois case, this action is transitive, and the automorphism group $\mathrm{Aut}(E|B)$ can be recovered from it.

This entire picture—classifying coverings via the action of a "fundamental group" on a "fiber"—is a foundational concept that admits profound generalizations. The following appendices explore two such extensions:

- Appendix A lifts this theory to the setting of ∞-category theory. It replaces topological spaces with *animae* (∞-groupoids) and covering spaces with *covering animae*, providing a higher-categorical version of the Galois correspondence.
- Appendix B abstracts the algebraic structure of this correspondence. It introduces Grothendieck's *Galois categories*, an axiomatic framework that simultaneously captures the topological theory and its powerful analogue in algebraic geometry (the *étale fundamental group*).

Therefore, the classical theory from this section should be viewed as the first and most concrete illustration of a much broader categorical Galois theory.

A Covering Spaces in Higher Category Theory

In this appendix, we expand on Remark 2.12 and explain how classical covering space theory naturally extends to the setting of higher category theory. The central perspective is that covering spaces depend only on the fundamental ∞-groupoid of a topological space, and thus admit a clean formulation within the world of animae.

Classically, if B is a connected, locally path-connected topological space, then its covering spaces are classified by the functor category

$$\mathsf{Cov}(B) \;\simeq\; \mathsf{Fun}(\mathrm{B}\pi_1(B), \mathsf{Set}),$$

where $\pi_1(B)$ acts on the fiber of a covering. This correspondence plays a central role in the Galois theory of covering spaces discussed in Section 2.

Grothendieck's homotopy hypothesis asserts that topological spaces (up to weak homotopy equivalence) are equivalent to ∞-groupoids, or *animae*. From this perspective, covering spaces should be describable purely in terms of ∞-groupoids. This turns out to be true: a covering space corresponds to a left fibration whose fibers are discrete animae (i.e., sets). Using Lurie's straightening theorem, one obtains a conceptual equivalence

$$\mathsf{Cov}(B) \;\simeq\; \mathsf{Fun}(B, \mathsf{Set}),$$

where B is regarded as an anima. Passing to the homotopy categories then yields

$$h\mathsf{Cov}(B) \;\simeq\; \mathsf{Fun}(\Pi_1(|B|), \mathsf{Set}).$$

The goal of this appendix is to make the above perspective precise. We begin with our conventions.

Notation and conventions

Throughout this appendix:

- "category" means an $(\infty, 1)$-*category*.
- $\mathbb{\Delta}$ denotes the simplex category, whose objects are the ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are order-preserving maps.
- An denotes the category of animae (i.e., ∞-groupoids).
- Cat denotes the category of (possibly ∞-)categories.
- $\mathsf{sSet} = \mathsf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathsf{Set})$ is the category of simplicial sets.

We now review the passage from spaces to animae via the singular simplicial complex.

A.1 The singular simplicial complex

Construction A.1. (Singular simplicial complex) Let X be a topological space. Its *singular simplicial complex* $\mathrm{Sing}(X)$ is the simplicial set defined as follows.

- For $n \geq 0$, the standard topological n-simplex is

$$|\Delta^n| = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_i x_i = 1\}.$$

- An order-preserving map $\varphi : [n] \to [m]$ induces a continuous map

$$|\Delta^\varphi| : |\Delta^n| \to |\Delta^m|, \qquad y_i = \sum_{j \in \varphi^{-1}(i)} x_j.$$

  These assemble into a functor

$$|\Delta^\bullet| : \mathbb{\Delta} \to \mathsf{Top}.$$

- We define

$$\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, X).$$

  Equivalently,

$$\mathrm{Sing}(X) = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, X) : \mathbb{\Delta}^{\mathrm{op}} \to \mathsf{Set}.$$

A.2 Underlying anima

Definition A.2. (Underlying anima) The functor $\Pi_\infty : \mathsf{Top} \to \mathsf{An}$ is defined as the composite

$$\mathsf{Top} \xhookrightarrow{\text{Sing}} \mathsf{sSet} \hookrightarrow \mathsf{sAn} \xhookrightarrow{|-|} \mathsf{An},$$

where for a simplicial anima $X_\bullet$,

$$|X| = \operatorname{colim}_{[n]\in\Delta^{\text{op}}} X_n$$

denotes geometric realization. We call $\Pi_\infty(X)$ the *underlying anima* (or fundamental ∞-groupoid) of a topological space X.

A.3 Homotopy hypothesis

Fact A.3. (Grothendieck's homotopy hypothesis) The functor

$$\Pi_\infty : \mathsf{Top} \to \mathsf{An}$$

inverts precisely the weak homotopy equivalences. Equivalently,

$$\mathsf{Top}[\text{weak homotopy equivalences}^{-1}] \;\simeq\; \mathsf{An}.$$

Informally: *animae are topological spaces up to weak homotopy equivalence.*

A.4 Left fibrations

We now introduce the higher-categorical analogue of a covering map. Recall that, in ordinary category theory, a functor $p : E \to B$ is a discrete opfibration if every arrow in B admits a unique lift in E. A left fibration is the $(\infty, 1)$-categorical version of this idea.

Definition A.4. (Left fibration) A functor of categories $p : \mathcal{E} \to \mathcal{B}$ is a *left fibration* if it is right orthogonal to the inclusion $\{0\} \hookrightarrow [1]$, i.e., for every commutative square

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ {[1]} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

there exists a filler, unique up to a contractible space of choices. Equivalently, the induced square

$$\begin{tikzcd} \mathcal{E}[1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{B}[1] \arrow[d] \\ \mathcal{E}[0] \arrow[r] & \mathcal{B}[0] \end{tikzcd}$$

is a pullback. Here, $\mathcal{E}[n] = \operatorname{Hom}_{\mathsf{Cat}}([n], \mathcal{E})$. We denote by $\mathsf{LFib}(\mathcal{B})$ the full subcategory of $\mathsf{Cat}_{/\mathcal{B}}$ spanned by left fibrations over $\mathcal{B}$.

Intuitively, a left fibration is a functor for which morphisms in $\mathcal{B}$ admit contractibly unique lifts in $\mathcal{E}$. The key consequence is that a left fibration behaves like a "family of animae parametrised by $\mathcal{B}$," with functoriality along arrows in $\mathcal{B}$.

A.5 Straightening and unstraightening

Let $p : \mathcal{E} \to \mathcal{B}$ be a left fibration. Consider the pullback diagram

$$\begin{tikzcd} \mathcal{E}[0] \arrow[d] & \mathcal{E}[1] \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \mathcal{E}[0] \arrow[d] \\ \mathcal{B}[0] & \mathcal{B}[1] \arrow[l, "s"'] \arrow[r, "t"] & \mathcal{B}[0] \end{tikzcd}$$

By definition of left fibration, the left-hand square is a pullback. For a morphism $f\colon b \to b'$ in $\mathcal{B}$, the fiber $\mathcal{E}[1]_f$ fits into a zigzag of equivalences

$$\mathcal{E}_b \xleftarrow{\sim} \mathcal{E}[1]_f \to \mathcal{E}_{b'}.$$

Inverting the first equivalence (from the source fiber) yields a transport functor

$$f_!\colon \mathcal{E}_b \longrightarrow \mathcal{E}_{b'}.$$

The assignment $f \mapsto f_!$ is functorial, and one obtains:

Theorem A.5. (Lurie) *For any category $\mathcal{B}$, there is an equivalence of categories*

$$\mathrm{St}_{\mathcal{B}}\colon \mathsf{LFib}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{B}, \mathsf{An}),$$

known as the straightening equivalence.

This equivalence identifies left fibrations over $\mathcal{B}$ with functors from $\mathcal{B}$ into animae; thus, each left fibration encodes a "local system of animae" on $\mathcal{B}$.

A.6 Covering animae

We now specialize this framework to the ∞-categorical analogue of a covering space.

Definition A.6. (Covering anima) Let B be an anima. A *covering anima* over B is a left fibration $p\colon E \to B$ such that for every $b \in B$, the fiber E_b is a set (i.e., a static anima). We write $\mathsf{Cov}(B) \subseteq \mathsf{LFib}(B)$ for the full subcategory spanned by covering animae.

The following result shows that classical covering spaces are examples.

Proposition A.7. ([Lurie, 2018, Tag 021J]) *Let B be a topological space and $p\colon E \to B$ a classical covering map. Then the induced map* $\mathrm{Sing}(p)\colon \mathrm{Sing}(E) \to \mathrm{Sing}(B)$ *is a covering anima.*

Warning A.8. The converse need not hold: a covering anima need not arise from a classical covering space.

A.7 The fundamental theorem of covering animae

Combining Definition A.6 with Theorem A.5, we obtain:

Theorem A.9. *Let B be an anima. Then there is an equivalence*

$$\mathsf{Cov}(B) \;\simeq\; \mathsf{Fun}(B, \mathsf{Set}).$$

Passing to homotopy categories (cf. [Lurie, 2018, Tag 004J]), one obtains an equivalence

$$h\mathsf{Cov}(B) \;\simeq\; \mathsf{Fun}(\Pi_1(|B|), \mathsf{Set}),$$

where $\Pi_1(|B|)$ is the fundamental groupoid of the geometric realization $|B|$.

In other words, covering animae over B are exactly set-valued functors on the anima B. When B is a space, this reduces to ordinary π_1-actions.

A.8 Galois correspondence for animae

We now obtain the higher-categorical Galois correspondence.

Corollary A.10. (Galois correspondence, anima version) *Let B be a connected anima (i.e., $\pi_0(B) = *$), and let $b \in B$. Then there is a bijection*

$$\{\textit{connected coverings of } B\} \xleftrightarrow{1:1} \{H \subseteq \pi_1(B, b)\}.$$

Proof sketch. Write $G = \pi_1(B, b)$. Since B is connected, the fundamental groupoid $\Pi_1(|B|)$ is equivalent to the one-object groupoid with endomorphism group G. Hence

$$\mathsf{Fun}(\Pi_1(|B|), \mathsf{Set}) \;\simeq\; \mathsf{Fun}(G, \mathsf{Set}) = G\text{-}\mathsf{Set}.$$

A G-set is connected if and only if it consists of a single G-orbit. Such orbits are in bijection with coset spaces G/H for subgroups $H \subseteq G$. $\square$

B Axiomatic Galois theory

In this appendix, we introduce the notion of a *Galois category*. Our exposition follows [Mathew, 2016, §5.1].

B.1 Étale fundamental group

To motivate the general definition, we briefly recall the étale fundamental group, originally introduced by Grothendieck in [Grothendieck and Raynaud, 2004].

The notion of an étale morphism is the algebro–geometric analogue of the topological étale map (Definition 1.2).

Definition B.1. (Étale morphism)

- Let $\varphi\colon R \to A$ be a morphism of commutative rings. We say that φ is *étale* if it is finitely presented, flat, and the module of relative Kähler differentials $\Omega_{A/R}$ vanishes.
- Let $f\colon X \to S$ be a morphism of schemes. We say that f is *étale* if for every point $x \in X$, there exist affine opens $U = \operatorname{Spec} A \subset X$ containing x and $V = \operatorname{Spec} R \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale.

This definition allows us to define the algebraic analogue of a finite covering space.

Definition B.2. (Finite étale covering) Let $f\colon X \to S$ be a morphism of schemes. We say that f is a *finite covering morphism* (or *finite étale cover*) if it is both finite and étale.

We write

$$\mathsf{Cov}_S \subset \mathsf{Sch}_S$$

for the full subcategory spanned by finite étale covers of S.

Example B.3. The basic example of a finite étale cover is the fold map $\coprod_S X \to X$. If X is connected, then a morphism $Y \to X$ is finite étale if and only if, after base change along some finitely presented, faithfully flat morphism $X' \to X$, the pullback

$$\begin{tikzcd} X' \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

becomes isomorphic to $\coprod_S X' \to X'$ for a finite set S. When X is not connected, the degree may vary from component to component, reflecting that "locally" here means *locally in the flat topology*.

This shows that finite étale covers play the expected algebro–geometric role of classical covering spaces.

We next examine the simplest case.

Example B.4. (Finite étale covers of a point) Let k be an algebraically closed field and $X = \operatorname{Spec} k$. Then any finite étale $Y \to X$ is a finite disjoint union of points. Hence

$$\mathsf{Cov}_X \simeq \mathsf{FinSet},$$

via $Y \mapsto \pi_0(Y)$.

Now assume X is connected and fix a geometric point $\overline{x} \to X$. Grothendieck's key insight is that the étale fundamental group of X can be extracted *purely formally* from the category Cov_X together with the fiber functor

$$\mathrm{Fib}_{\overline{x}}\colon \mathsf{Cov}_X \longrightarrow \mathsf{FinSet}, \qquad (Y \to X) \longmapsto Y_{\overline{x}}.$$

This perspective motivates Grothendieck's axiomatic definition of Galois categories.

Definition B.5. (Étale fundamental group) Let X be a connected scheme and $\overline{x} \to X$ a geometric point. The *étale fundamental group* of $(X, \overline{x})$ is the group of invertible natural transformations of the fiber functor:

$$\pi_1^{\text{ét}}(X, \overline{x}) \coloneqq \operatorname{Aut}(\mathrm{Fib}_{\overline{x}}).$$

This mirrors the topological description

$$\pi_1(B,b) \;\cong\; \mathrm{Aut}(E_b)$$

for a universal covering $E \to B$.

A crucial difference is that $\pi_1^{\text{ét}}(X,\overline{x})$ is naturally a *profinite group*. Indeed, finite étale covers of X form a cofiltered diagram

$$\{Y_i \to X\}_{i\in I},$$

each equipped with a finite Galois group $\mathrm{Aut}_X(Y_i)$. The transition maps assemble these into a cofiltered diagram of finite groups, and one obtains a canonical identification

$$\pi_1^{\text{ét}}(X,\overline{x}) \;\simeq\; \varprojlim_{i\in I} \mathrm{Aut}_X(Y_i).$$

Thus the étale fundamental group is the profinite group collecting all finite Galois covers of X into a single inverse limit.

Theorem B.6. (Grothendieck) *The fiber functor induces an equivalence*

$$\mathsf{Cov}_X \xrightarrow{\;\sim\;} \mathsf{FinSet}_{\pi_1^{\text{ét}}(X,\overline{x})}.$$

In particular, connected finite étale covers of X correspond bijectively to open subgroups of $\pi_1^{\text{ét}}(X,\overline{x})$.

This is the algebro–geometric analogue of monodromy for topological covering spaces. The abstract framework capturing these formal properties is the notion of a *Galois category*, introduced next.

Definition B.7. (Grothendieck [Grothendieck and Raynaud, 2004, Exp. V, Sec. 4]) A *classical Galois category* is a category $\mathcal{C}$ equipped with a fiber functor $\mathrm{Fib}\colon \mathcal{C} \to \mathsf{FinSet}$ satisfying:

1. $\mathcal{C}$ has finite limits, and Fib preserves them.
2. $\mathcal{C}$ has finite coproducts, and Fib preserves them.
3. $\mathcal{C}$ admits quotients by finite group actions, and Fib preserves such quotients.
4. Fib is conservative and preserves effective epimorphisms.
5. Every morphism $X \to Y$ factors as an effective epimorphism $X \to Y'$ followed by a monomorphism $Y' \hookrightarrow Y$ which is a summand inclusion.

Given such a pair $(\mathcal{C}, \mathrm{Fib})$, Grothendieck's Galois theory shows that the category is determined by the automorphism group of Fib.

Definition B.8. The *Galois group* of a classical Galois category $(\mathcal{C}, \mathrm{Fib})$ is

$$\pi_1(\mathcal{C}) \coloneqq \mathrm{Aut}(\mathrm{Fib}).$$

This group is naturally profinite, and the functor Fib lifts to

$$\mathcal{C} \longrightarrow \mathsf{FinSet}_{\pi_1(\mathcal{C})}.$$

Proposition B.9. (Grothendieck [Grothendieck and Raynaud, 2004, Exp. V, Thm. 4.1]) *Let $(\mathcal{C}, \mathrm{Fib})$ be a classical Galois category. Then the lift*

$$\mathcal{C} \longrightarrow \mathsf{FinSet}_{\pi_1(\mathcal{C})}$$

is an equivalence.

In particular, if X is a connected scheme with geometric point $\overline{x} \to X$, one may show that Cov_X with the fiber functor $\mathrm{Fib}_{\overline{x}}$ is a classical Galois category. Its fundamental group is precisely the étale fundamental group—a deep and useful invariant of X. For varieties over an algebraically closed field of characteristic 0, this group is the profinite completion of the topological fundamental group of the associated complex analytic space.

B.2 Toward non-connected setting

We now present an exposition of Galois theory appropriate for the non-connected setting. Namely, to a type of category which we will simply call a Galois category, we will attach a *profinite groupoid*: that is, a pro-object in the $(2,1)$-category $\mathsf{Gpd}_{\mathrm{fin}}$. The advantage of this approach, which relies heavily on descent theory, is that it does not require the *a priori* assumption of a fiber functor, since we might not have one.

Definition B.10. Let $\mathcal{C}$ be a category admitting finite limits. Given a map $f\colon y \to x$ in $\mathcal{C}$, we have an adjunction

$$\begin{tikzcd} \mathcal{C}_{/y} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{/x}. \arrow[l, shift left, "f^*"] \end{tikzcd}$$

We say f is an *effective descent morphism* if the above adjunction is monadic.

Consider the bar construction in $\mathcal{C}$

$$\mathrm{B}_\bullet(f) \coloneqq \Big(\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} y \times_x y \rightrightarrows y \Big)$$

which is a simplicial object in $\mathcal{C}_{/x}$, so one can also regard it as an augmented simplicial object with $\mathrm{B}_{-1}(f) \coloneqq x$. Applying the pullback functor everywhere, we get a cosimplicial category

$$\mathcal{C}_{/\,\mathrm{B}_\bullet(f)} = \Big(\cdots \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} \mathcal{C}_{y\times_x y} \leftleftarrows \mathcal{C}_{/y} \Big)$$

One can also regard it as an augmented cosimplicial object with $\mathcal{C}_{/\,\mathrm{B}_{-1}(f)} \coloneqq \mathcal{C}_{/x}$. Thus, saying f is an effective descent morphism is equivalent to saying that

$$\mathcal{C}_{/x} \simeq \mathrm{tot}\Big(\cdots \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} \mathcal{C}_{y\times_x y} \leftleftarrows \mathcal{C}_{/y} \Big)$$

where tot means the totalization, i.e. the limit of the cosimplicial diagram.

By the Barr–Beck theorem, one can easily obtain:

Proposition B.11. *Let $\mathcal{C}$ be a category with finite limits, then $f\colon y \to x$ is an effective descent morphism if and only if f^* is conservative and preserves f^*-split reflexive coequalizers.*

Remark B.12. One can also note that if $\mathcal{C}$ is furthermore an $(\infty,1)$-category, then f is an effective descent morphism if and only if f^* is conservative and preserves f^*-split cosimplicial objects.

We can now define the Galois category.

Definition B.13. A *Galois category* is a category $\mathcal{C}$ such that:

1. $\mathcal{C}$ is extensive and admits finite limits.
2. Given an object $x \in \mathcal{C}$, there is an effective descent morphism $x' \to *$ and a decomposition $x' = x'_1 \sqcup \cdots \sqcup x'_n$ such that each map $x \times x'_i \to x'_i$ decomposes as the fold map $x \times x'_i \simeq \coprod_{S_i} x'_i \to x'_i$ for a finite set S_i.

We use $\mathsf{GalCat} \subseteq \mathsf{Cat}$ to denote the subcategory spanned by the Galois categories with the functor between them (which are required to preserve coproducts, effective descent morphisms, and finite limits). It is a $(2,1)$-category. We will use $\mathrm{Hom}_{\mathsf{GalCat}}(\mathcal{C},\mathcal{D})$ to denote the groupoid of functors beteen Galois categories $\mathcal{C}$ and $\mathcal{D}$.

In other words, we might say that an object $x \in \mathcal{C}$ is in *elementary form* if $x \simeq \coprod_S *$ for some finite set S. More generally, if there exists a decomposition $* \simeq *_1 \sqcup \cdots \sqcup *_n$, such that, as an object of $\mathcal{C} \simeq \prod_i \mathcal{C}_{/*_i}$, each $y \times_* *_i \to *_i$ is in elementary form.

In fact, in a Galois category $\mathcal{C}$, we have the following result.

Proposition B.14. *Given a map $f\colon y \to x$ in the Galois category $\mathcal{C}$, the following are equivalent:*

1. *f is an effective descent morphism.*

2. *f is an effective epimorphism.*

3. *For every $x' \to x$ with x nonempty, the pullback $y \times_x x'$ is nonempty.*

We now introduce the Galois correspondence for this new definition. This is equivalent to providing an alternative description of the $(2,1)$-category GalCat in terms of groupoids.

Example B.15. Let G be a profinite group. Then the category FinSet_G of finite G-sets is a Galois category. To see this, it suffices to show the second axiom of Definition B.13 holds. Given any finite G-set T, we have an effective descent morphism $G \to *$ such that $T \times G$, as a G-set, is a disjoint union of copies of G.

This Galois category enjoys a convenient universal property.

Definition B.16. Let $\mathcal{C}$ be a Galois category and let G be a finite group. A *G-torsor* in $\mathcal{C}$ consists of an object $x \in \mathcal{C}$ with a G-action such that there exists an effective descent morphism $y \to *$ such that $y \times x \in \mathcal{C}_{/y}$, as an object with a G-action, is given by

$$y \times x \simeq \coprod_G y,$$

where G acts on the latter by permuting the summands. For instance, x could be $\coprod_G *$. The collection of G-torsors forms a full subcategory $\mathsf{Tors}_G(\mathcal{C}) \subseteq \mathrm{Hom}_{\mathsf{Cat}}(\mathrm{B}G, \mathcal{C})$.

The Galois category FinSet_G has a natural example of a G-torsor: namely, G itself.

Proposition B.17. *Let $\mathcal{C}$ be a Galois category, then there is an equivalence*

$$\mathrm{Hom}_{\mathsf{GalCat}}(\mathsf{FinSet}_G, \mathcal{C}) \simeq \mathsf{Tors}_G(\mathcal{C}).$$

More generally, we can build Galois categories from finite groupoids. This will be very important from a 2-categorical point of view.

Definition B.18. We say a groupoid $\mathcal{G}$ is *finite* if $\mathcal{G}$ has finitely many isomorphism classes of objects and for every $x \in \mathcal{G}$, we have $\mathrm{Aut}_{\mathcal{G}}(x)$ is a finite group. The collection of all finite groupoids, functors and natural transformations is naturally organized into a $(2,1)$-category $\mathsf{Gpd}_{\mathrm{fin}}$.

In other words, a finite groupoid is a 1-truncated homotopy type such that π_0 is finite, as is π_1 with any choice of basepoint.

Given a finite groupoid $\mathcal{G}$, the category $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{G}, \mathsf{FinSet})$ forms a Galois category. This is a generalization of FinSet_G and if we interpret $\mathcal{G}$ as a 1-truncated homotopy type, then this is precisely the category of finite *covering spaces* of $\mathcal{G}$.

It follows that we get a functor of $(2,1)$-categories

$$\mathsf{Gpd}_{\mathrm{fin}}^{\mathrm{op}} \to \mathsf{GalCat}, \quad \mathcal{G} \mapsto \mathrm{Hom}_{\mathsf{Cat}}(\mathcal{G}, \mathsf{FinSet}).$$

In order to proceed further, we need a basic formal property of GalCat.

Proposition B.19. *The $(2,1)$-category GalCat admits filtered colimits and the inclusion $\mathsf{GalCat} \hookrightarrow \mathsf{Cat}$ preserves these colimits.*

It follows that we get a natural functor

$$\mathsf{Pro}(\mathsf{Gpd}_{\mathrm{fin}})^{\mathrm{op}} \simeq \mathsf{Ind}(\mathsf{Gpd}_{\mathrm{fin}}^{\mathrm{op}}) \to \mathsf{GalCat}.$$

We give this object a name:

Definition B.20. A *profinite groupoid* is an object of $\mathsf{Pro}(\mathsf{Gpd}_{\mathrm{fin}})$.

Thus, we can state the main Galois correspondence:

Theorem B.21. *The functor*

$$\mathsf{Pro}(\mathsf{Gpd}_{\mathrm{fin}})^{\mathrm{op}} \to \mathsf{GalCat}$$

is an equivalence of $(2,1)$-categories.

Definition B.22. Given a Galois category $\mathcal{C}$, we define the *Galois groupoid* $\Pi_{\leq 1}\mathcal{C}$ of $\mathcal{C}$ as the associated profinite groupoid of $\mathcal{C}$.

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