Low Weights

Script for the Oberseminar: *Motivic Cohomology of Schemes*

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This script contains more material than can be covered in the talk; some proofs and details will be sketched or omitted during the presentation. This document is also an experiment in typesetting with Typst.

Conventions

Throughout, *category* means $(\infty, 1)$-category. All schemes are assumed qcqs unless otherwise stated. Since we work in the derived setting, Γ (and Γ_τ for a topology τ) always denotes the derived global section functor ($R\Gamma$ in the classical sense); similarly, H^n denotes the cohomology of the derived global sections. We write k for **connective algebraic K-theory** (the presheaf K^{cn} of $\mathbb{E}_\infty$-ring spectra) and K for **nonconnective algebraic K-theory** (Bass–Thomason–Trobaugh); on regular Noetherian schemes both agree with Weibel's homotopy K-theory KH.

1. Overview

In this talk, our goal is to define and study the first Chern class in $\mathbb{A}^1$-motivic cohomology.

Orientation of K-theory. The starting point is the *orientation* of algebraic K-theory. By [BEM25, Definition 4.4], there is a unique homotopy class of maps of presheaves of pointed spaces on qcqs schemes

$$1 - (-)^\vee : \Omega^\infty \mathcal{P}\mathrm{ic} \to \Omega^\infty \mathrm{k}$$

characterized by the property that, for each $m \geq 1$, the composition $(\mathbb{P}^m_{\mathbb{Z}}, \infty) \xrightarrow{[\mathcal{O}(1)]} \Omega^\infty \mathcal{P}\mathrm{ic} \xrightarrow{1-(-)^\vee} \Omega^\infty \mathrm{k}$ classifies $1 - [\mathcal{O}(-1)] \in \mathrm{K}_0(\mathbb{P}^m_{\mathbb{Z}})$. By adjunction, this yields a map $1 - (-)^\vee : \Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic} \to \mathrm{k}$ of presheaves of spectra.

From K-theory to motivic cohomology. The standard orientation of KGL, constructed from $1 - (-)^\vee$ in [BEM25, §4.1.1], induces compatible orientations of $s^0 \,\mathrm{KGL}_S$ and $f^* s^0 \,\mathrm{KGL}_{\mathbb{Z}}$ for any qcqs scheme $f : S \to \mathrm{Spec}(\mathbb{Z})$ (see [BEM25, Remark 3.48]). By the equivalence of the slice and Bott filtrations ([BEM25, §4.1.2]) and the σ^∞–ω^∞ adjunction, these orientations yield (see [BEM25, Remark 4.23]) a map

$$\Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A},\,\mathrm{cdh}}[2] \to \mathbb{Z}(1)^{\mathbb{A}}[2]$$

of presheaves of spectra on qcqs schemes. For any line bundle $\mathcal{L}$ on a qcqs scheme S, precomposing along $[\mathcal{L}] : \Sigma^\infty_+ S \to \Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic}$ defines the first Chern classes

$$c_1^{\mathbb{A},\,\mathrm{cdh}}(\mathcal{L}) \in H^2_{\mathbb{A},\,\mathrm{cdh}}(S, \mathbb{Z}(1)) \quad \text{and} \quad c_1^{\mathbb{A}}(\mathcal{L}) \in H^2_{\mathbb{A}}(S, \mathbb{Z}(1)).$$

Factoring through $\mathcal{P}\mathrm{ic}$. We will see in Lemma 3.3 that the map $\Sigma^\infty\Omega^\infty\mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A},\mathrm{cdh}}[2]$ factors through the counit $\Sigma^\infty\Omega^\infty\mathcal{P}\mathrm{ic} \to \mathcal{P}\mathrm{ic}$. This counit is the map from the free $\mathbb{E}_\infty$-group on the underlying space of $\mathcal{P}\mathrm{ic}$ back to $\mathcal{P}\mathrm{ic}$ itself: the left-hand side does not see the fact that line bundles may be tensored together, but the right-hand side and the map do. The factorization therefore says that the first Chern class depends only on the isomorphism class of $\mathcal{L}$ in $\mathrm{Pic}(S)$ and is additive under tensor product. In particular, we obtain maps of presheaves of spectra

$$c_1^{\mathbb{A},\mathrm{cdh}} : \mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A},\mathrm{cdh}}[2] \quad \text{and} \quad c_1^{\mathbb{A}} : \mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A}}[2].$$

Weight 0 and $\mathbb{Z}$-linearity. The construction of $c_1^{\mathbb{A}}$ requires understanding weight 0 first. Since $\mathbb{Z}(\star)^{\mathbb{A}}$ is a graded $\mathbb{E}_\infty$-algebra ([BEM25, Remark 4.22]), its weight-0 part $\mathbb{Z}(0)^{\mathbb{A}}$ serves as the base ring over which $\mathbb{Z}(1)^{\mathbb{A}}$ is a module. The identification $\mathbb{Z}(0)^{\mathrm{cdh}} \simeq \Gamma_{\mathrm{cdh}}(-,\mathbb{Z})$ — weight-0 motivic cohomology is the constant sheaf $\mathbb{Z}$ — underpins the $\mathbb{Z}$-linear structure within which c_1 is defined (Construction 3.1).

Difficulties and strategy. Over a general qcqs scheme, the direct construction faces two difficulties. First, the unstable slice computation $s_u^1\,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1]$ relies on the $\mathbb{A}^1$-invariance of K and the Zariski cohomological vanishing $H^n_{\mathrm{Zar}}(S,\mathbb{G}_m) = 0$ for $n > 1$, both of which require regularity. Second, the Picard sheaf of spectra satisfies $\mathcal{P}\mathrm{ic} \simeq (\tau_{\le 1}\Gamma_{\mathrm{Zar}}(-,\mathbb{G}_m))[1]$ in general; only on smooth schemes over a regular noetherian base does the truncation become redundant and the change of topology from Zariski to Nisnevich an equivalence, yielding $\mathcal{P}\mathrm{ic} \simeq \Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1]$ (Lemma 2.2).

Our strategy is therefore to construct $c_1^{\mathbb{A}}$ in stages, following [BEM25, §4.2.1]:

(1) In Section 2, we work over a regular noetherian scheme S and compute the first two stages of the unstable slice filtration of K-theory: $s_u^0\,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-,\mathbb{Z})$ and $s_u^1\,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1]$ (Lemma 2.1).

(2) In Section 3, we deduce that $\mathbb{Z}(\star)^{\mathbb{A}}$ is $\mathbb{Z}$-linear and define the first Chern class $c_1^{\mathbb{A}} : \mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A}}[2]$ on smooth schemes over a regular base (Construction 3.1). We also state the compatibility with the Atiyah–Hirzebruch filtration (Corollary 3.4).

(3) In Section 4, taking for granted that the first Chern class is an isomorphism over Dedekind domains [BEM25, Corollary 6.26] and the equivalence $\mathbb{Z}(j)^{\mathrm{cdh}} \simeq \mathbb{Z}(j)^{\mathbb{A}}$ for $j \le 1$, we deduce integral identifications of $\mathbb{A}^1$-motivic cohomology in low weights over arbitrary qcqs schemes (Corollary 4.4.1), and assemble them into a description of the second filtration quotient $\mathrm{KH}/\mathrm{Fil}^2_{\mathbb{A},\mathrm{cdh}}\,\mathrm{KH} \simeq L_{\mathrm{cdh}}\mathcal{P}\mathrm{ic}^\dagger$ (Corollary 4.5.2).

2. Unstable Slice Filtration

Throughout this section, let S be a regular noetherian scheme. We work in $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})$.

Recall that in $\mathsf{SH}(S)$, for an integer $j \in \mathbb{Z}$, one considers the full subcategory $\mathsf{SH}(S)(j) \subset \mathsf{SH}(S)$ generated under colimits by $\left\{\mathbb{T}_S^{\otimes j} \otimes M_{S(X)} \mid X \in \mathsf{Sm}_S\right\}$. The inclusion admits a right adjoint:

$$\begin{tikzcd} \mathsf{SH}(S)(j) \arrow[r, shift left=2, "\iota^j"] \arrow[r, phantom, "\perp"] & \mathsf{SH}(S) \arrow[l, shift left=2, "r^j"] \end{tikzcd}$$

Setting $\mathrm{Fil}^j := \iota^j \circ r^j$, for every motivic spectrum $E \in \mathsf{SH}(S)$ we obtain a filtration

$$\cdots \to \mathrm{Fil}^{j+1} E \to \mathrm{Fil}^{j} E \to \mathrm{Fil}^{j-1} E \to \cdots$$

known as the **slice filtration**. We define $s^j E$ as the cofiber $\mathrm{cofib}\Big(\mathrm{Fil}^{j+1}_{\text{slice}} E \to \mathrm{Fil}^{j}_{\text{slice}} E\Big)$; it is right orthogonal to $\mathsf{SH}(S)(j+1)$, i.e., the mapping spectrum $\mathrm{map}_{\mathsf{SH}(S)}(F, s^j E)$ vanishes for all $F \in \mathsf{SH}(S)(j+1)$.

We have $\mathbb{Z}(0)^{\mathbb{A}} = \omega^\infty s^0 \,\mathrm{KGL}_S$ and $\mathbb{Z}(1)^{\mathbb{A}} = \omega^\infty s^1 \,\mathrm{KGL}_S[2]$. To define the first Chern class, we need the "slice filtration" of $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$. Since $\mathsf{SH}(S) = \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})[\mathbb{G}_m^{-1}]$, this filtration is also known as the **unstable slice filtration** (unstable under $- \otimes \mathbb{G}_m$).

For an integer $j \in \mathbb{Z}$, consider the full subcategory

$$\mathrm{Shv}_S(j) \subset \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$$

generated under colimits by $\Big\{\big(\Sigma^\infty \mathbb{T}^1_S\big)^{\otimes j} \otimes E \mid E \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})\Big\}$. By the same reasoning as for SH, we have an adjunction $\iota^j \dashv r^j$ and hence a functorial, exhaustive, multiplicative filtration for $E \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$, which is now $\mathbb{N}$-indexed. We denote this tower by

$$E \simeq \mathrm{Fil}^0_{\text{u-slice}} E \leftarrow \cdots \leftarrow \mathrm{Fil}^{j-1}_{\text{u-slice}} E \leftarrow \mathrm{Fil}^{j}_{\text{u-slice}} E \leftarrow \mathrm{Fil}^{j+1}_{\text{u-slice}} E \leftarrow \cdots$$

and write $s^j_u E$ for the corresponding graded pieces, which we call the **unstable slices** of E.

Now we compare the unstable slice filtration with the slice filtration on $\mathsf{SH}(S)$. Observing that

$$\sigma^\infty(\mathrm{Shv}_S(j)) \subset \mathsf{SH}(S)(j),$$

we obtain a Beck–Chevalley-style transformation $\iota^j \omega^\infty \Rightarrow \omega^\infty \iota^j$. This induces, for each $E \in \mathsf{SH}(S)$ and $j \geq 0$, natural maps

$$\mathrm{Fil}^j_{\text{u-slice}} \omega^\infty E \to \omega^\infty \,\mathrm{Fil}^j_{\text{slice}} E \quad \text{and} \quad s^j_u \omega^\infty E \to \omega^\infty s^j E.$$

In particular, we have natural maps $s^0_u \omega^\infty \,\mathrm{KGL} \to \omega^\infty s^0 \,\mathrm{KGL}_S = \mathbb{Z}(0)^{\mathbb{A}}$ and $s^1_u \omega^\infty \,\mathrm{KGL} \to \omega^\infty s^1 \,\mathrm{KGL}_S = \mathbb{Z}(1)^{\mathbb{A}}[2]$. It is well known that on a regular noetherian scheme S, we have $\mathrm{K} \simeq \mathrm{KH}$, hence $\omega^\infty \,\mathrm{KGL} \simeq \mathrm{K}$ on Sm_S. Everything thus reduces to showing that $\Gamma_{\mathrm{Nis}}(-, \mathbb{Z}) \simeq s^0_u \,\mathrm{K}$ and $\Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1] \simeq s^1_u \,\mathrm{K}$.

> **Lemma 2.1**: The following assertions hold in $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$:
> (1) The rank map $\mathrm{rk} : \mathrm{K} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$ induces an equivalence $s^0_u \,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$, so that $\mathrm{Fil}^1_{\text{u-slice}} \mathrm{K} \simeq \mathrm{K}^{\mathrm{rk}=0} := \mathrm{fib}(\mathrm{rk})$.
> (2) The determinant map $\det : \mathrm{K}^{\mathrm{rk}=0} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$ induces an equivalence $s^1_u \,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$.

To prove Lemma 2.1, we need some preparation.

Lemma 2.2: Let S be a regular noetherian scheme.

(1) There are natural isomorphisms

$$H^n_{\mathrm{Zar}}(S,\mathbb{Z}) = \begin{cases} \mathbb{Z}^{\pi_0(S)} & n = 0 \\ 0 & n > 0 \end{cases}$$

and

$$H^n_{\mathrm{Zar}}(S,\mathbb{G}_m) = \begin{cases} \mathcal{O}(S)^\times & n = 0 \\ \mathrm{Pic}(S) & n = 1. \\ 0 & n > 1 \end{cases}$$

(2) The change-of-topology maps

$$\Gamma_{\mathrm{Zar}}(S,\mathbb{Z}) \to \Gamma_{\mathrm{Nis}}(S,\mathbb{Z}) \quad \text{and} \quad \Gamma_{\mathrm{Zar}}(S,\mathbb{G}_m) \to \Gamma_{\mathrm{Nis}}(S,\mathbb{G}_m)$$

are equivalences.

(3) The canonical maps $\Gamma_{\mathrm{Nis}}(\mathbb{A}^1_S,\mathbb{Z}) \to \Gamma_{\mathrm{Nis}}(S,\mathbb{Z})$ and $\Gamma_{\mathrm{Nis}}(\mathbb{A}^1_S,\mathbb{G}_m) \to \Gamma_{\mathrm{Nis}}(S,\mathbb{G}_m)$ are equivalences.

Proof: (1). For the cohomology of $\mathbb{Z}$, the key point is the vanishing of higher cohomology of constant sheaves on irreducible schemes (Stacks, Tag 02UW). For $\mathbb{G}_m$, the H^0 and H^1 descriptions are standard. The vanishing of higher cohomology is classical: since S is regular (hence normal), the sheaf $\mathbb{G}_m$ admits a length-1 resolution by flasque sheaves

$$0 \to \mathbb{G}_m \to \bigoplus_{\eta \in S^{(0)}} i_{\eta *}\mathcal{O}_\eta^\times \to \bigoplus_{x \in S^{(1)}} i_{x*}\mathbb{Z} \to 0.$$

(2). The map $\Gamma_{\mathrm{Zar}}(-,F) \to \Gamma_{\mathrm{Nis}}(-,F)$ is a map of Zariski sheaves of spectra. To show it is an equivalence, it suffices to check on stalks. The stalk at $s \in S$ is

$$\Gamma_{\mathrm{Zar}}(\mathrm{Spec}(\mathcal{O}_{S,s}),F) \to \Gamma_{\mathrm{Nis}}(\mathrm{Spec}(\mathcal{O}_{S,s}),F)$$

where $\mathcal{O}_{S,s}$ is a regular noetherian local ring. We are thus reduced to showing: for every regular noetherian local ring R, the change-of-topology map $\Gamma_{\mathrm{Zar}}(\mathrm{Spec}(R),F) \to \Gamma_{\mathrm{Nis}}(\mathrm{Spec}(R),F)$ is an equivalence. We proceed by induction on $d = \dim R$.

Consider the cartesian square

$$\begin{tikzcd}
\mathrm{Spec}(R^h) \setminus \{\mathfrak{m}R^h\} \arrow[r] \arrow[d] & \mathrm{Spec}(R^h) \arrow[d] \\
\mathrm{Spec}(R) \setminus \{\mathfrak{m}\} \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

where R^h is the henselization. This is a cofiltered limit of Nisnevich distinguished squares, so applying $\Gamma_{\mathrm{Nis}}(-,F)$ yields a cartesian square. Now:

- On the two punctured spectra, which have dimension $< d$, the change-of-topology map is an equivalence by the inductive hypothesis.
- On $\mathrm{Spec}(R^h)$, the change-of-topology map is an equivalence because R^h is henselian: every Nisnevich cover of $\mathrm{Spec}(R^h)$ admits a section, so $H^n_{\mathrm{Nis}}(\mathrm{Spec}(R^h),F) = 0$ for $n > 0$, which matches $H^n_{\mathrm{Zar}}(\mathrm{Spec}(R^h),F) = 0$ by (1).

Since three of the four vertices agree, the change-of-topology map is an equivalence on $\operatorname{Spec}(R)$ if and only if $\Gamma_{\mathrm{Zar}}(-, F)$ also carries the above square to a cartesian square. We verify this case by case.

- When $d = 0$, R is a field, so $R = R^h$.
- When $d = 1$, R is a DVR. The punctured spectra are the spectra of the fraction fields of R and R^h respectively, and applying $\Gamma_{\mathrm{Zar}}(-, \mathbb{Z})$ and $\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)$ yields squares

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R^\times \arrow[r] \arrow[d] & \operatorname{Frac}(R)^\times \arrow[d] \\ R^{h\times} \arrow[r] & \operatorname{Frac}(R^h)^\times \end{tikzcd}$$

respectively. Both squares are cartesian in $\mathsf{D}(\mathbb{Z})$ (for the right square, the cokernels of the horizontal maps are both $\mathbb{Z}$, given by the discrete valuation), as desired.
- For $d \geq 2$: it suffices to show that the horizontal maps $\Gamma_{\mathrm{Zar}}(\operatorname{Spec}(R), F) \to \Gamma_{\mathrm{Zar}}(\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}, F)$ and $\Gamma_{\mathrm{Zar}}(\operatorname{Spec}(R^h), F) \to \Gamma_{\mathrm{Zar}}(\operatorname{Spec}(R^h) \setminus \{\mathfrak{m}R^h\}, F)$ are equivalences. For $F = \mathbb{Z}$: the punctured spectra are connected, so both sides are $\mathbb{Z}$ by (1). For $F = \mathbb{G}_m$: the normality of R gives $R^\times \simeq H^0_{\mathrm{Zar}}(\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}, \mathbb{G}_m)$ (as R is normal of dimension ≥ 2) and $\operatorname{Pic}(\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}) = 0$ by Grothendieck (Stacks, Tag 0F2H), and similarly for R^h in place of R.

(3). By part (2), we may replace Nis by Zar, and then use (1) to compute the cohomology. For $\mathbb{Z}$, note that the connected components of S and $\mathbb{A}^1_S$ are in bijection. For $\mathbb{G}_m$, we use that $A^\times \simeq A[T]^\times$ whenever A is a reduced ring, and $\operatorname{Pic}(A) \simeq \operatorname{Pic}(A[T])$ for A seminormal [Swa80, Tra70]. (In our setting A is regular, hence seminormal.) □

We now turn to the proof of Lemma 2.1. Our strategy is based on the following observation.

> **Observation 2.3**: Let $\mathrm{Shv}_S(j)^\perp$ denote the right orthogonal complement of $\mathrm{Shv}_S(j)$, i.e., the full subcategory of objects $G \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ with $\operatorname{map}(H, G) \simeq 0$ for all $H \in \mathrm{Shv}_S(j)$. Suppose we are given a fiber sequence
>
> $$A \to E \to B$$
>
> with $A \in \mathrm{Shv}_S(j)$ and $B \in \mathrm{Shv}_S(j)^\perp$. Then $A \simeq \mathrm{Fil}^j_{\text{u-slice}} E$ and $B \simeq E/\mathrm{Fil}^j_{\text{u-slice}} E$.
>
> Indeed, since $\mathrm{Fil}^j_{\text{u-slice}} E \to E$ is universal for maps from j-effective objects into E, the map $A \to E$ factors uniquely as $A \xrightarrow{\varphi} \mathrm{Fil}^j_{\text{u-slice}} E \to E$. We claim φ is an equivalence. Let $C := \operatorname{cofib}(\varphi)$; since $\mathrm{Shv}_S(j)$ is closed under colimits, $C \in \mathrm{Shv}_S(j)$. Applying the octahedral axiom to $A \to \mathrm{Fil}^j_{\text{u-slice}} E \to E$ produces a cofiber sequence
>
> $$C \to B \to E/\mathrm{Fil}^j_{\text{u-slice}} E.$$
>
> The first map is zero because $C \in \mathrm{Shv}_S(j)$ and $B \in \mathrm{Shv}_S(j)^\perp$, so the sequence splits: $E/\mathrm{Fil}^j_{\text{u-slice}} E \simeq B \oplus C[1]$. Now $E/\mathrm{Fil}^j_{\text{u-slice}} E \in \mathrm{Shv}_S(j)^\perp$ by definition of $\mathrm{Fil}^j_{\text{u-slice}}$, and the right orthogonal is closed under retracts, so $C[1] \in \mathrm{Shv}_S(j)^\perp$. Combined with $C[1] \in \mathrm{Shv}_S(j)$ this forces $C[1] \in \mathrm{Shv}_S(j) \cap \mathrm{Shv}_S(j)^\perp = 0$. Hence $C \simeq 0$, φ is an equivalence, and the identification $B \simeq E/\mathrm{Fil}^j_{\text{u-slice}} E$ follows from the fiber sequence.

Proof of Lemma 2.1: We apply Observation 2.3 in two steps.

(1) For the rank map $\mathrm{rk} : \mathrm{K} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$: the fiber sequence $\mathrm{K}^{\mathrm{rk}=0} \to \mathrm{K} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$ exhibits the 0-th unstable slice $s^0_u \,\mathrm{K} \simeq \mathrm{K} \,/\, \mathrm{Fil}^1_{\text{u-slice}} \,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$, provided we verify that $\mathrm{K}^{\mathrm{rk}=0} \in \mathrm{Shv}_S(1)$ and $\Gamma_{\mathrm{Nis}}(-, \mathbb{Z}) \in \mathrm{Shv}_S\,(1)^{\perp}$.

(2) For the determinant map $\det : \mathrm{K}^{\mathrm{rk}=0} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$: applying the same argument to $\mathrm{Fil}^1_{\text{u-slice}} \,\mathrm{K} = \mathrm{K}^{\mathrm{rk}=0}$ with $j = 2$ gives $s^1_u \,\mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$, provided we verify that $\mathrm{Fil}^2_{\text{u-slice}} \,\mathrm{K}^{\mathrm{rk}=0} \in \mathrm{Shv}_S(2)$ and $\Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1] \in \mathrm{Shv}_S\,(2)^{\perp}$.

Write $F := \mathrm{Fil}^2_{\text{u-slice}} \,\mathrm{K}^{\mathrm{rk}=0} = \mathrm{fib}\big(\mathrm{K}^{\mathrm{rk}=0} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]\big)$ for the second effective piece.

Right orthogonality. For $\Gamma_{\mathrm{Nis}}(-, \mathbb{Z}) \in \mathrm{Shv}_S\,(1)^{\perp}$, note that $\mathrm{Shv}_S(j)$ is generated under colimits and desuspensions by objects of the form $\Sigma^{\infty}_{+}\mathbb{P}^{j}_{X}/\Sigma^{\infty}_{+}\mathbb{P}^{j-1}_{X} \simeq \big(\Sigma^{\infty}\mathbb{T}^{1}_{S}\big)^{\otimes j} \otimes \Sigma^{\infty}_{+} X$. It suffices to show that $\Gamma_{\mathrm{Nis}}(X, \mathbb{Z}) \simeq \Gamma_{\mathrm{Nis}}(\mathbb{P}^1_X, \mathbb{Z})$ and $\Gamma_{\mathrm{Nis}}(\mathbb{P}^2_X, \mathbb{G}_m) \simeq \Gamma_{\mathrm{Nis}}(\mathbb{P}^1_X, \mathbb{G}_m)$ for all $X \in \mathsf{Sm}_S$. By Stacks, Tag 0BXJ and Lemma 2.2 (1), (2), we obtain the result.

Effectivity. It remains to show that $\mathrm{K}^{\mathrm{rk}=0} \in \mathrm{Shv}_S(1)$ and $F \in \mathrm{Shv}_S(2)$. For this we use the following key observation.[1]

> **Observation 2.4**: If $E \in \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ is Nisnevich-locally connective and $L_{\mathbb{A}^1,\mathrm{Nis}}\Sigma^{\infty}\Omega^{\infty}E \in \mathrm{Shv}_S(j)$, then $E \in \mathrm{Shv}_S(j)$.

Firstly, by Morel–Voevodsky's equivalence, we have $\Omega^{\infty}\,\mathrm{K} \simeq L_{\mathrm{Nis},\mathbb{A}^1}(\mathbb{Z} \times \mathrm{BGL})$, with the $\mathbb{Z}$ component corresponding to the rank, so $\Omega^{\infty}\,\mathrm{K}^{\mathrm{rk}=0} \simeq L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{BGL}$.

We claim that $\Omega^{\infty}F \simeq L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{BSL}$. Consider the fiber sequence of Nisnevich sheaves of pointed spaces

$$L_{\mathrm{Nis}}\,\mathrm{BSL} \to L_{\mathrm{Nis}}\,\mathrm{BGL} \overset{\det}{\to} L_{\mathrm{Nis}}\;\mathrm{B}\mathbb{G}_m.$$

Since $\mathrm{B}\mathbb{G}_m \simeq \Omega^{\infty}(\Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1])$ is already $\mathbb{A}^1$-invariant by Lemma 2.2 (3) and Nisnevich-locally connected, we may apply [Lur17, Lemma 5.5.6.17]: the fiber product $\mathrm{BSL} \simeq \mathrm{BGL} \times_{\mathrm{B}\mathbb{G}_m} *$ is taken over the pointed connected object $\mathrm{B}\mathbb{G}_m$, and HA 5.5.6.17 states that such fiber products preserve sifted colimits. Since $L_{\mathbb{A}^1}$ is computed as the geometric realization $|\{-\}(- \times \Delta^{\bullet}_S)|$, which is a sifted colimit, we obtain a fiber sequence

$$L_{\mathbb{A}^1}L_{\mathrm{Nis}}\,\mathrm{BSL} \to L_{\mathbb{A}^1}L_{\mathrm{Nis}}\,\mathrm{BGL} \to L_{\mathrm{Nis}}\;\mathrm{B}\mathbb{G}_m.$$

The middle term $L_{\mathbb{A}^1}L_{\mathrm{Nis}}\,\mathrm{BGL}$ is already a Nisnevich sheaf because $L_{\mathrm{Nis}}\,\mathrm{BGL}$ is $\mathbb{A}^1$-invariant on Sm_S: at stalks (regular local rings) it classifies vector bundles, which are $\mathbb{A}^1$-invariant by Quillen–Suslin / Lindel-type results. Hence $L_{\mathbb{A}^1}L_{\mathrm{Nis}}\,\mathrm{BGL} = L_{\mathrm{Nis}}\,\mathrm{BGL}$, and $\Omega^{\infty}F \simeq L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{BSL}$.

By Observation 2.4, it suffices to show that $L_{\mathbb{A}^1,\mathrm{Nis}}\Sigma^{\infty}\,\mathrm{BGL} \in \mathrm{Shv}_S(1)$ and $L_{\mathbb{A}^1,\mathrm{Nis}}\Sigma^{\infty}\,\mathrm{BSL} \in \mathrm{Shv}_S(2)$. We use the Schubert cell decomposition: since $\mathrm{BGL} = \mathrm{colim}_{n,m}\,\mathrm{Gr}(n, n+m)$ and $\mathrm{BSL} = \mathrm{colim}_{n,m}\,\mathrm{SGr}(n, n+m)$, and $\mathrm{Shv}_S(j)$ is closed under colimits, it suffices to check each finite (special linear) Grassmannian.

By [Wen10, Proposition 3.7], for a split connected reductive group G with standard parabolic P, the projective homogeneous variety $X = G/P$ admits a Schubert stratification $X = X_n \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ with strata $X_i \setminus X_{i-1} \simeq \mathbb{A}^{n_i}$. Write $d_i = \dim X - n_i$ for the codimension. The open inclusion $X \setminus X_i \hookrightarrow X \setminus X_{i-1}$ has closed complement $X_i \setminus X_{i-1} \simeq \mathbb{A}^{n_i}$ with trivial normal bundle of rank d_i. By motivic purity in $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$,

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[1] See also [BE21, Lemma 5.2] for detecting stable effectivity in motivic homotopy theory.

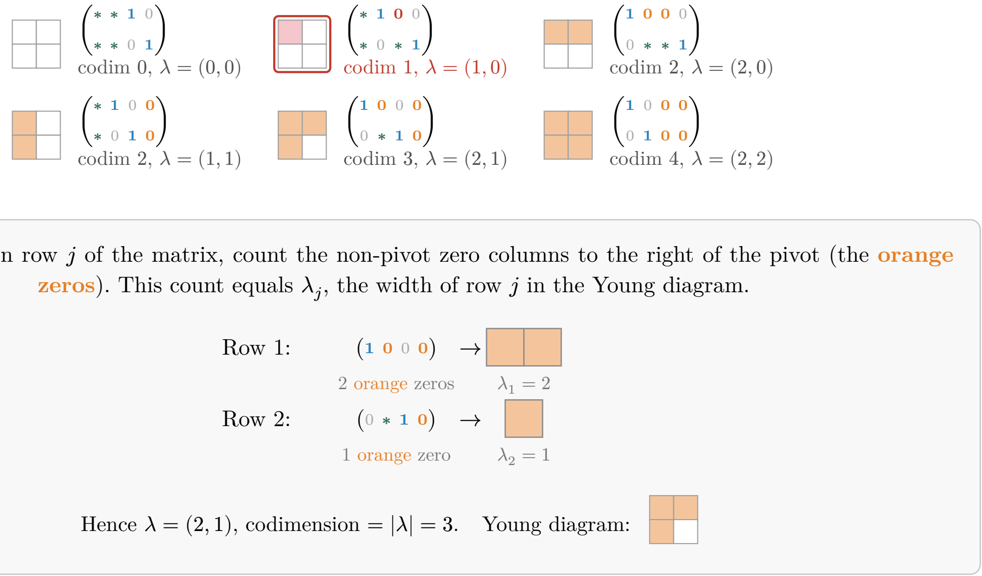
$$\mathrm{cofib}(\Sigma^\infty(X \setminus X_i) \to \Sigma^\infty(X \setminus X_{i-1})) \simeq \left(\Sigma^\infty \mathbb{T}_S^1\right)^{\otimes d_i}.$$

Equivalently, in $\mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ there is a fiber sequence

$$\Sigma^\infty\left(\mathbb{A}^{d_i} \setminus \{0\}\right) \to \Sigma^\infty(X \setminus X_i) \to \Sigma^\infty(X \setminus X_{i-1}),$$

where the leftmost term is identified via the purity equivalence $\Sigma^\infty\left(\mathbb{A}^{d_i} \setminus \{0\}\right) \simeq \left(\Sigma^\infty \mathbb{T}_S^1\right)^{\otimes d_i}[-1]$. Iterating along the stratification, $L_{\mathrm{Nis},\mathbb{A}^1}\Sigma^\infty X$ is an iterated extension of copies of $\left(\Sigma^\infty \mathbb{T}_S^1\right)^{\otimes d_i}$, so $L_{\mathrm{Nis},\mathbb{A}^1}\Sigma^\infty X \in \mathrm{Shv}_S(\min\{d_i \mid d_i > 0\})$.

For $\mathrm{Gr}(k, N)$: the cells are indexed by Young diagrams λ in a $k \times (N-k)$ box, with codimension $|\lambda|$; the minimum positive codimension is 1 (the partition $(1, 0, \cdots, 0)$), so $\Sigma^\infty \mathrm{BGL} \in \mathrm{Shv}_S(1)$.



Reading rule: In row j of the matrix, count the non-pivot zero columns to the right of the pivot (the **orange zeros**). This count equals λ_j, the width of row j in the Young diagram.

Hence $\lambda = (2, 1)$, codimension $= |\lambda| = 3$. Young diagram:

Figure 1: Schubert cells of $\mathrm{Gr}(2, 4)$. Each cell $C_\lambda \simeq \mathbb{A}^{4-|\lambda|}$ is parametrized by a 2×4 matrix in echelon form. 1 marks pivots, $*$ marks free parameters, 0 marks *Young diagram zeros* (non-pivot zeros to the right of the pivot), and 0 marks structural zeros. The number of orange zeros in row j equals λ_j. The cell $\lambda = (1, 0)$ (red) has nontrivial determinant and does not lift to $\mathrm{SGr}(2, 4)$.

For $\mathrm{SGr}(k, N)$: the codimension-1 cell of $\mathrm{Gr}(k, N)$ (the partition $(1, 0, \cdots, 0)$) has nontrivial determinant and does not lift to SGr, so the minimum positive codimension is 2. Hence $\Sigma^\infty \mathrm{BSL} \in \mathrm{Shv}_S(2)$. ☐

Finally, we prove Observation 2.4.

Proof of Observation 2.4: Any spectrum A satisfies $A \simeq \mathrm{colim}_n(\Sigma^\infty\Omega^\infty(A[n]))[-n]$ (from the $\Sigma^\infty\Omega^\infty$ comonad resolution). For Nisnevich-locally connective E, the delooping $\Omega^\infty(E[n])$ is Nisnevich-locally equivalent to the iterated bar construction $\mathrm{B}^n(\Omega^\infty E)$. Since B^n is built from products and colimits, and $\Sigma^\infty(A_1 \times A_2) \simeq \Sigma^\infty A_1 \oplus \Sigma^\infty A_2 \oplus (\Sigma^\infty A_1 \otimes \Sigma^\infty A_2)$, and $\mathrm{Shv}_S(j)$ is closed under colimits, sums, tensors, and desuspensions, the result follows. ☐

3. First Chern Class

We now define the first Chern class in $\mathbb{A}^1$-motivic cohomology, at least over regular noetherian bases. Recall from Section 2 the natural comparison maps $s^j_u \omega^\infty E \to \omega^\infty s^j E$ for $E \in \mathsf{SH}(S)$. Applied to $E = \mathrm{KGL}_S$, noting that $\omega^\infty \mathrm{KGL} = \mathrm{KH} \simeq \mathrm{K}$ on Sm_S for regular S, we obtain maps

$$s^0_u \mathrm{K} \to \omega^\infty s^0 \mathrm{KGL}_S = \mathbb{Z}(0)^{\mathbb{A}} \quad \text{and} \quad s^1_u \mathrm{K} \to \omega^\infty s^1 \mathrm{KGL}_S = \mathbb{Z}(1)^{\mathbb{A}}[2].$$

By Lemma 2.1, $s^0_u \mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$ and $s^1_u \mathrm{K} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$, so these become maps from sheaves we understand.

Construction 3.1: Let S be a regular noetherian scheme. The composite

$$\Gamma_{\mathrm{Nis}}(-, \mathbb{Z}) \simeq s^0_u(\mathrm{K}) \to \omega^\infty s^0 \mathrm{KGL}_S = \mathbb{Z}(0)^{\mathbb{A}}$$

defines a map of $\mathbb{E}_\infty$-algebras, endowing $\mathbb{Z}(\star)^{\mathbb{A}}$ with a $\mathbb{Z}$-linear structure. The composite

$$\Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1] \simeq s^1_u(\mathrm{K}) \to \omega^\infty s^1 \mathrm{KGL}_S = \mathbb{Z}(1)^{\mathbb{A}}[2]$$

defines the **first Chern class**

$$c_1^{\mathbb{A}} : \mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A}}[2]$$

on smooth S-schemes, where we identify $\mathcal{P}\mathrm{ic} \simeq \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$ on Sm_S by Lemma 2.2.

The weight-0 map is a map of $\mathbb{E}_\infty$-algebras because $s^0 \mathrm{KGL}_S \simeq \mathrm{H}\mathbb{Z}_S$ is an $\mathbb{E}_\infty$-ring spectrum and the rank map $\mathrm{K} \to \Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$ respects the multiplicative structure. The weight-1 map $c_1^{\mathbb{A}}$ is then a map of modules over this base ring.

Remark 3.2: For a line bundle $\mathcal{L}$ on $Y \in \mathsf{Sm}_S$, the first Chern class $c_1^{\mathbb{A}}(\mathcal{L}) \in H^2_{\mathbb{A}}(Y, \mathbb{Z}(1))$ is the image of the class $[\mathcal{L}] \in \mathrm{Pic}(Y) = H^1_{\mathrm{Nis}}(Y, \mathbb{G}_m) = \pi_0(\Gamma_{\mathrm{Nis}}(Y, \mathbb{G}_m)[1])$ under the map induced by $c_1^{\mathbb{A}} : \mathcal{P}\mathrm{ic} \to \mathbb{Z}(1)^{\mathbb{A}}[2]$. By construction, it depends only on the isomorphism class of $\mathcal{L}$ in $\mathrm{Pic}(Y)$.

We next explain how the orientation of K-theory interacts with its unstable slices.

Lemma 3.3: Let S be a regular noetherian scheme.

(1) The mapping anima $\mathrm{Map}(\Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic}, \Gamma_{\mathrm{Nis}}(-, \mathbb{Z}))$ in presheaves of spectra on Sm_S is contractible. In particular, the map $1 - (-)^\vee : \Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic} \to \mathrm{K}$ factors uniquely through $\mathrm{K}^{\mathrm{rk}=0}$.

(2) The resulting diagram of presheaves of spectra on Sm_S

$$\begin{tikzcd} \Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic} \arrow[r, "1 - (-)^\vee"] \arrow[d] & \mathrm{K}^{\mathrm{rk}=0} \arrow[d, "\det"'] \\ \mathcal{P}\mathrm{ic} \arrow[r] & \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1] \end{tikzcd}$$

commutes up to homotopy, where the left vertical map is the canonical counit and the bottom map is $\mathcal{P}\mathrm{ic} \simeq (\tau_{\leq 1} \Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m))[1] \to \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1]$ (an equivalence on Sm_S by Lemma 2.2).

Proof:

(1) Since $\Sigma^\infty \Omega^\infty \mathcal{P}\mathrm{ic}$ is Nisnevich-locally connective and $\Gamma_{\mathrm{Nis}}(-, \mathbb{Z})$ is coconnective, the mapping anima is contractible.

(2) Over regular S, Lemma 2.2 identifies Zariski and Nisnevich cohomology of $\mathbb{G}_m$ on smooth S-schemes and removes the $\tau^{\leq 1}$. The claim then reduces to showing that the composition

$$\Sigma^\infty\Omega^\infty\mathscr{P}\mathrm{ic} \xrightarrow{1-(-)^\vee} \mathrm{K}^{\mathrm{rk}=0} \xrightarrow{\det} \Gamma_{\mathrm{Zar}}(-,\mathbb{G}_m)[1] \xrightarrow{\simeq} \mathscr{P}\mathrm{ic}$$

is homotopic to the counit $\Sigma^\infty\Omega^\infty\mathscr{P}\mathrm{ic} \to \mathscr{P}\mathrm{ic}$. Via the Σ^∞–Ω^∞ adjunction, this amounts to showing that

$$\Omega^\infty\mathscr{P}\mathrm{ic} \xrightarrow{1-(-)^\vee} \Omega^\infty\,\mathrm{K}^{\mathrm{rk}=0} \xrightarrow{\det} \Omega^\infty(\Gamma_{\mathrm{Zar}}(-,\mathbb{G}_m)[1]) \xrightarrow{\simeq} \Omega^\infty\mathscr{P}\mathrm{ic}$$

is the identity. By the uniqueness of the orientation, it suffices to verify that for each $m \geq 1$ the composition

$$(\mathbb{P}^m_S,\infty) \xrightarrow{[\mathcal{O}(1)]} \Omega^\infty\mathscr{P}\mathrm{ic} \xrightarrow{1-(-)^\vee} \Omega^\infty\,\mathrm{K}^{\mathrm{rk}=0} \xrightarrow{\det} \Omega^\infty(\Gamma_{\mathrm{Zar}}(-,\mathbb{G}_m)[1]) \xrightarrow{\simeq} \Omega^\infty\mathscr{P}\mathrm{ic}$$

classifies $\mathcal{O}(1)$. The first two maps compose to the class $1-[\mathcal{O}(-1)] \in \mathrm{K}_0^{\mathrm{rk}=0}(\mathbb{P}^m_S)$. It remains to check that $\det : \mathrm{K}_0^{\mathrm{rk}=0}(\mathbb{P}^m_S) \to \mathrm{Pic}(\mathbb{P}^m_S)$ sends $1-[\mathcal{O}(-1)]$ to $\mathcal{O}(1)$. We prove a more general statement.

Recall Morel–Voevodsky's equivalence $\Omega^\infty\,\mathrm{K} \simeq L_{\mathrm{Nis},\mathbb{A}^1}(\mathbb{Z}\times\mathrm{BGL})$, characterized by the property that for a smooth S-scheme Y and a rank-d vector bundle $\mathcal{V}$ on Y, the composition

$$Y \xrightarrow{[\mathcal{V}]} L_{\mathrm{Nis}}\,\mathrm{BGL}_d \to L_{\mathrm{Nis},\mathbb{A}^1}(\mathbb{Z}\times\mathrm{BGL}) \xrightarrow{\simeq} \Omega^\infty\,\mathrm{K}$$

classifies $[\mathcal{V}] \in \mathrm{K}_0(Y)$. Here the middle map is given by d on the first factor and by $\mathrm{GL}_d \to \mathrm{GL}$ on the second.

Restricting to rank-0 parts gives $L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{BGL} \simeq \Omega^\infty\,\mathrm{K}^{\mathrm{rk}=0}$, with the composition

$$S \xrightarrow{[\mathcal{V}]} L_{\mathrm{Nis}}\,\mathrm{BGL}_d \to L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{BGL} \simeq \Omega^\infty\,\mathrm{K}^{\mathrm{rk}=0}$$

classifying $[\mathcal{V}]-d \in \mathrm{K}_0^{\mathrm{rk}=0}(Y)$. Specializing to $d=1$ (so $\mathcal{V}=\mathcal{L}$ is a line bundle), we arrange this as the top row of the diagram

$$\begin{tikzcd}
S \arrow[r, "{[\mathcal{L}]}"] & L_{\mathrm{Nis}}\,\mathrm{B}\mathbb{G}_m \arrow[r] & L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{BGL} \arrow[r, "\sim"] \arrow[d, "L_{\mathrm{Nis},\mathbb{A}^1}\,\mathrm{B}(\det)"'] & \Omega^\infty\,\mathrm{K}^{\mathrm{rk}=0} \arrow[d, "\det"'] \\
 & & L_{\mathrm{Nis}}\,\mathrm{B}\mathbb{G}_m \arrow[r, "\sim"] & \Omega^\infty(\Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1])
\end{tikzcd}$$

The square commutes by definition of det (induced by $\det : \mathrm{K}_1(-) \to H^0_{\mathrm{Nis}}(-,\mathbb{G}_m)$). The unwritten diagonal $L_{\mathrm{Nis}}\,\mathrm{B}\mathbb{G}_m \to L_{\mathrm{Nis}}\,\mathrm{B}\mathbb{G}_m$ is the identity, since the determinant of $\mathrm{GL}_1 \hookrightarrow \mathrm{GL}$ is the identity on $\mathbb{G}_m$. Since the top row classifies $[\mathcal{L}]-1$, we conclude that $\det([\mathcal{L}]-1)=\mathcal{L}$, and therefore $\det(1-[\mathcal{L}^\vee]) = \det([\mathcal{L}^\vee]-1)^{-1} = (\mathcal{L}^\vee)^{-1} = \mathcal{L}$, as required. □

The functoriality of the Beck–Chevalley transformation $\mathrm{Fil}^j_{\text{u-slice}}\,\omega^\infty \to \omega^\infty\,\mathrm{Fil}^j_{\mathrm{slice}}$ across the cofiber projection $\mathrm{Fil}^j_{\mathrm{slice}} \to s^j$, applied to KGL_S at $j=1$, yields:

> **Corollary 3.4**: Let S be a regular noetherian scheme. The following diagram of presheaves of spectra on Sm_S commutes:
>
> $$\begin{tikzcd} \mathrm{K}^{\mathrm{rk}=0} \arrow[r] \arrow[d, "\det"'] & \mathrm{Fil}^1_{\mathbb{A}} \mathrm{KH} \arrow[d, "\mathrm{edge}"'] \\ \Gamma_{\mathrm{Nis}}(-, \mathbb{G}_m)[1] \arrow[r, "c_1^{\mathbb{A}}[2]"] & \mathbb{Z}(1)^{\mathbb{A}}[2] \end{tikzcd}$$
>
> where the right vertical map is the edge map of the Atiyah–Hirzebruch filtration.

4. Low Weights over Arbitrary Schemes

We extend the results of the previous sections from smooth schemes over regular noetherian bases to arbitrary qcqs schemes.

4.1. From regular bases to qcqs schemes

The constructions of Section 3 were carried out on smooth schemes over a regular noetherian base S. To extend them to arbitrary qcqs schemes, we cdh-locally left Kan extend from smooth $\mathbb{Z}$ -schemes.

We first recall the motivic filtrations on homotopy K-theory. For a qcqs scheme S with structure map $f : S \to \mathrm{Spec}(\mathbb{Z})$, the slice filtration on KGL induces two natural, multiplicative, exhaustive $\mathbb{N}$-indexed filtrations on $\mathrm{KH}(S)$ ([BEM25, Theorem 4.39(1)]):

$$\mathrm{Fil}^j_{\mathbb{A},\,\mathrm{cdh}} \mathrm{KH}(S) \coloneqq \mathrm{map}_{\mathsf{SH}(S)}\left(\mathbb{1}_S, f^* \mathrm{Fil}^j_{\mathrm{slice}} \mathrm{KGL}_{\mathbb{Z}}\right) \quad \text{and} \quad \mathrm{Fil}^j_{\mathbb{A}} \mathrm{KH}(S) \coloneqq \mathrm{map}_{\mathsf{SH}(S)}\left(\mathbb{1}_S, \mathrm{Fil}^j_{\mathrm{slice}} \mathrm{KGL}_S\right),$$

called the **$\mathbb{A}^1$-cdh-motivic filtration** and the **$\mathbb{A}^1$-motivic filtration** respectively. Their graded pieces recover the $\mathbb{A}^1$-motivic cohomology theories:

$$\mathrm{gr}^j_{\mathbb{A},\,\mathrm{cdh}} \mathrm{KH}(S) \simeq \mathbb{Z}(j)^{\mathbb{A},\,\mathrm{cdh}}(S)[2j] \quad \text{and} \quad \mathrm{gr}^j_{\mathbb{A}} \mathrm{KH}(S) \simeq \mathbb{Z}(j)^{\mathbb{A}}(S)[2j].$$

There is a natural multiplicative map $\mathrm{Fil}^\star_{\mathbb{A},\,\mathrm{cdh}} \mathrm{KH}(S) \to \mathrm{Fil}^\star_{\mathbb{A}} \mathrm{KH}(S)$.

To carry out the extension we use the rigidity / left Kan extension framework of [BEM25, §2.3], whose relevant ingredients we restate.

> **Proposition 4.1.1 [BEM25, Proposition 2.23]**: Let $\mathcal{C}$ be a presentable category. The following diagram of $\mathcal{C}$-valued functor categories commutes:
>
> $$\begin{tikzcd} \mathsf{Fun}(\mathsf{Sch}_{\mathbb{Z}}^{\mathrm{qcqs,op}}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathsf{CAlg}_{\mathbb{Z}}, \mathcal{C}) \arrow[r, "="] & \mathsf{Fun}(\mathsf{CAlg}_{\mathbb{Z}}, \mathcal{C}) \arrow[r] & \mathsf{Fun}\left(\mathsf{CAlg}_{\mathbb{Z}}^{?\mathrm{loc}}, \mathcal{C}\right) \\ \mathsf{Fun}(\mathsf{Sm}_{\mathbb{Z}}^{\mathrm{op}}, \mathcal{C}) \arrow[r] \arrow[u, "L_{\mathbb{Z}}^{\mathrm{sm}}"] & \mathsf{Fun}(\mathsf{SmAlg}_{\mathbb{Z}}, \mathcal{C}) \arrow[r, "\simeq"] \arrow[u] & \mathsf{Fun}^{\mathrm{fin}}(\mathsf{IndSmAlg}_{\mathbb{Z}}, \mathcal{C}) \arrow[r] \arrow[u] & \mathsf{Fun}\left(\mathsf{EssSmAlg}_{\mathbb{Z}}^{?\mathrm{loc}}, \mathcal{C}\right) \arrow[u] \end{tikzcd}$$
>
> Here
>
> - the vertical arrows are all left Kan extensions along the indicated fully faithful inclusions,
> - the horizontal arrow labelled $\simeq$ is left Kan extension,
> - the remaining horizontal arrows are restrictions, and $?\mathrm{loc} \in \{\mathrm{loc}, \mathrm{hloc}, \mathrm{shloc}\}$.

Combining these with $L_{\mathrm{cdh}}\,\mathrm{k} \simeq \mathrm{KH}$ ([KST18, Theorem 6.3] or [BEM25, Theorem 7.12(1)]) and cdh-locally left Kan extending Construction 3.1 and Corollary 3.4 yields:

Theorem 4.1.2 [BEM25, Theorem 4.39(3)(4)] : For any qcqs scheme S:

(1) *Weight zero and $\mathbb{Z}$-linear structure.* There is a natural map of presheaves of $\mathbb{E}_\infty$-algebras

$$\Gamma_{\mathrm{cdh}}(-,\mathbb{Z}) \to \mathbb{Z}(0)^{\mathbb{A},\,\mathrm{cdh}}$$

on qcqs schemes, fitting into a commutative diagram

$$\begin{tikzcd}
\Gamma_{\mathrm{cdh}}(-,\mathbb{Z}) \arrow[rr] & & \mathbb{Z}(0)^{\mathbb{A},\,\mathrm{cdh}} \\
& \mathrm{KH} \arrow[ul, "\mathrm{rk}"] \arrow[ur, "\mathrm{edge}"'] &
\end{tikzcd}$$

(2) *Weight one and first Chern class.* There is a natural map

$$c_1^{\mathbb{A},\,\mathrm{cdh}} : \Gamma_{\mathrm{cdh}}(-,\mathbb{G}_m)[-1] \to \mathbb{Z}(1)^{\mathbb{A},\,\mathrm{cdh}},$$

fitting into a commutative diagram

$$\begin{tikzcd}
\mathrm{KH}^{\mathrm{rk}=0} \arrow[r] \arrow[d, "\mathrm{det}"'] & \mathrm{Fil}^1_{\mathbb{A},\,\mathrm{cdh}}\,\mathrm{KH} \arrow[d, "\mathrm{edge}"] \\
\Gamma_{\mathrm{cdh}}(-,\mathbb{G}_m)[1] \arrow[r, "{c_1^{\mathbb{A},\,\mathrm{cdh}}[2]}"] & \mathbb{Z}(1)^{\mathbb{A},\,\mathrm{cdh}}[2]
\end{tikzcd}$$

of presheaves of spectra on qcqs schemes.

Proof: **Three LKE properties on local Z-algebras.** The three functors $\mathsf{CAlg}^{\mathrm{loc}}_{\mathbb{Z}} \to \mathsf{D}(\mathbb{Z})$,

$$A \mapsto \mathbb{Z}, \quad A \mapsto A^\times, \quad A \mapsto \mathrm{k}(A),$$

are each left Kan extended from $\mathsf{EssSmAlg}^{\mathrm{loc}}_{\mathbb{Z}}$:

- $A \mapsto \mathbb{Z}$: the LKE diagram is constantly $\mathbb{Z}$ over a contractible category.
- $A \mapsto A^\times$: $\mathbb{G}_{m,\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$ is a smooth affine $\mathbb{Z}$-scheme; restricted to $\mathsf{EssSmAlg}^{\mathrm{loc}}_{\mathbb{Z}}$, the corepresentable functor $A^\times = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[t,t^{-1}], A)$ is LKE by the standard cofinality argument.
- $A \mapsto \mathrm{k}(A)$: Bhatt–Lurie (Talk 1; [Elm+20, Example A.0.6]).

By Proposition 4.1.1, the corresponding presheaves on $\mathsf{Sch}^{\mathrm{qcqs}}_{\mathbb{Z}}$, namely $\Gamma_{\mathrm{Nis}}(-,\mathbb{Z})$, $\Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)$, and k, are left Kan extended from $\mathsf{Sm}_{\mathbb{Z}}$.

Extending the natural transformations. Applied to $S = \mathrm{Spec}(\mathbb{Z})$ (regular noetherian), Construction 3.1 and Corollary 3.4 furnish, on $\mathsf{Sm}_{\mathbb{Z}}$:

$$\mathrm{rk} : \mathrm{k} \to \Gamma_{\mathrm{Nis}}(-,\mathbb{Z}), \quad \mathrm{det} : \mathrm{k}^{\mathrm{rk}\,=0} \to \Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1],$$

together with the commutative diagram of Corollary 3.4, and the maps $\Gamma_{\mathrm{Nis}}(-,\mathbb{Z}) \to \mathbb{Z}(0)^{\mathbb{A}}$ and $c_1^{\mathbb{A}}[2] : \Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1] \to \mathbb{Z}(1)^{\mathbb{A}}[2]$.

The universal property of $L^{\mathrm{sm}}_{\mathbb{Z}}$ reads, for any presheaf G on $\mathsf{Sch}^{\mathrm{qcqs}}$,

$$\mathrm{Map}(L^{\mathrm{sm}}_{\mathbb{Z}} F, G) \simeq \mathrm{Map}\big(F|_{\mathsf{Sm}_{\mathbb{Z}}}, G|_{\mathsf{Sm}_{\mathbb{Z}}}\big).$$

Each source presheaf — k, $\mathrm{k}^{\mathrm{rk}\,=0}$ (fiber of an LKE map between LKE functors, still LKE since LKE is a filtered colimit), $\Gamma_{\mathrm{Nis}}(-,\mathbb{Z})$, $\Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1]$ — is LKE from $\mathsf{Sm}_{\mathbb{Z}}$ by the previous paragraph. Therefore all maps and commutative diagrams extend uniquely from $\mathsf{Sm}_{\mathbb{Z}}$ to $\mathsf{Sch}^{\mathrm{qcqs}}$.

cdh sheafification. Apply L_{cdh}. Kerz–Strunk–Tamme ([KST18, Theorem 6.3]; also [BEM25, Theorem 7.12(1)]) gives $L_{\mathrm{cdh}}\,\mathrm{k} \simeq \mathrm{KH}$, and by definition $L_{\mathrm{cdh}}\Gamma_{\mathrm{Nis}}(-,\mathbb{Z}) = \Gamma_{\mathrm{cdh}}(-,\mathbb{Z})$, $L_{\mathrm{cdh}}\Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m) = \Gamma_{\mathrm{cdh}}(-,\mathbb{G}_m)$. The targets $\mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}}$ are cdh sheaves whose restriction to $\mathsf{Sm}_{\mathbb{Z}}$ recovers $\omega^{\infty} s^{j}\,\mathrm{KGL}_{\mathbb{Z}}[-2j] = \mathbb{Z}(j)^{\mathbb{A}}|_{\mathsf{Sm}_{\mathbb{Z}}}$ by smooth base change of slices. Therefore cdh-sheafifying the extended maps produces the diagrams (1) and (2). ☐

4.2. Rational low weights

> **Corollary 4.2.1 [BEM25, Corollary 4.60(1)]** : The maps of Theorem 4.1.2 are rational equivalences.

Proof: For each $j \geq 0$, the presheaf $\mathbb{Q}(j)^{\mathbb{A},\mathrm{cdh}}$ is a direct summand of $\mathrm{KH}_{\mathbb{Q}}$ by [BEM25, Theorem 4.47(2)]. Since $\mathrm{KH}_{\mathbb{Q}}$ is the cdh sheafification of the left Kan extension of its restriction to smooth $\mathbb{Z}$-schemes (see the proof of [BEM25, Theorem 7.12(1)]), the same is true of $\mathbb{Q}(j)^{\mathbb{A},\mathrm{cdh}}$. The result then follows by left Kan extending and cdh sheafifying the rational equivalences over regular noetherian schemes ([BEM25, Proposition 4.56]). ☐

4.3. Assumed results

To pass from rational to integral equivalences, we need deeper inputs from the body of [BEM25].

> **Theorem 4.3.1 Assumed results**:
>
> (1) *(Low weights for cdh-motivic cohomology.)* For any qcqs scheme S:
>
> $$\mathbb{Z}(0)^{\mathrm{cdh}}(S) \simeq \Gamma_{\mathrm{cdh}}(S,\mathbb{Z}) \quad \text{and} \quad c_1^{\mathrm{cdh}} : \Gamma_{\mathrm{cdh}}(S,\mathbb{G}_m)[-1] \xrightarrow{\sim} \mathbb{Z}(1)^{\mathrm{cdh}}(S)$$
>
> ([BEM25, Theorem 7.12(5)(6)]).
>
> (2) *($c_1^{\mathbb{A}}$ is an isomorphism over Dedekind domains.)* For any scheme X smooth over a field or a mixed characteristic Dedekind domain:
>
> $$\Gamma_{\mathrm{Nis}}(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}(0)^{\mathbb{A}}(X) \quad \text{and} \quad c_1^{\mathbb{A}} : \Gamma_{\mathrm{Nis}}(X,\mathbb{G}_m)[-1] \xrightarrow{\sim} \mathbb{Z}(1)^{\mathbb{A}}(X)$$
>
> ([BEM25, Corollary 6.26]).
>
> (3) *(Comparison in low weights.)* For any qcqs scheme S and $j \leq 1$:
>
> $$\mathbb{Z}(j)^{\mathrm{cdh}}(S) \simeq \mathbb{Z}(j)^{\mathbb{A}}(S).$$

4.4. Integral low weights

> **Corollary 4.4.1 [BEM25, Corollary 9.12(1)]** : For any qcqs scheme S:
>
> $$\mathbb{Z}(0)^{\mathbb{A}}(S) \simeq \Gamma_{\mathrm{cdh}}(S,\mathbb{Z}) \quad \text{and} \quad \mathbb{Z}(1)^{\mathbb{A}}(S) \simeq \Gamma_{\mathrm{cdh}}(S,\mathbb{G}_m)[-1].$$

Proof: Since $\mathbb{Z}(j)^{\mathbb{A}}$ is $\mathbb{A}^1$-invariant by construction, Theorem 4.3.1 (3) implies that $\mathbb{Z}(j)^{\mathrm{cdh}}$ is also $\mathbb{A}^1$-invariant for $j \leq 1$. Both maps in the comparison chain $\mathbb{Z}(j)^{\mathrm{cdh}} \to \mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}} \to \mathbb{Z}(j)^{\mathbb{A}}$ are then equivalences, and the result follows from Theorem 4.3.1 (1). ☐

4.5. The second filtration quotient

We now describe the filtration quotients $\mathrm{KH}/\mathrm{Fil}^{j}_{\mathbb{A},\mathrm{cdh}}\,\mathrm{KH}$ for $j = 1, 2$ as single recognizable objects.

Definition 4.5.1: For a qcqs scheme S, the **derived Picard anima** is

$$\mathcal{P}\mathrm{ic}^{\dagger}(S) := \mathrm{Pic}(\mathsf{D}(S)),$$

the Picard anima of the quasi-coherent module category $\mathsf{D}(S)$ of S — i.e., the anima of $\otimes$-invertible objects in $\mathsf{D}(S)$. Locally on S, every such invertible has the form $\mathcal{L}[n]$ for a line bundle $\mathcal{L}$ and an integer n; the cohomological grading n is locally constant on S, defining a **rank** map of presheaves of spectra on $\mathsf{Sch}^{\mathrm{qcqs}}$,

$$\mathrm{rk} : \mathcal{P}\mathrm{ic}^{\dagger} \to \Gamma(-, \mathbb{Z}),$$

whose fiber over 0 is $\mathcal{P}\mathrm{ic}$:

$$\mathcal{P}\mathrm{ic} \to \mathcal{P}\mathrm{ic}^{\dagger} \xrightarrow{\mathrm{rk}} \Gamma(-, \mathbb{Z}).$$

This exhibits $\mathcal{P}\mathrm{ic}^{\dagger}$ as an extension of $\Gamma(-, \mathbb{Z})$ by $\mathcal{P}\mathrm{ic}$ in presheaves of spectra on qcqs schemes.

The determinant

$$\det : \mathrm{K} \to \mathcal{P}\mathrm{ic}^{\dagger}$$

of presheaves of spectra on $\mathsf{Sch}^{\mathrm{qcqs}}$ sends a perfect complex to its Knudsen–Mumford determinant (top exterior power on connective resolutions); it is a map of $\mathbb{E}_{\infty}$-ring spectra. The rank map $\mathrm{rk} : \mathrm{K} \to \Gamma(-, \mathbb{Z})$ factors as

$$\mathrm{K} \xrightarrow{\det} \mathcal{P}\mathrm{ic}^{\dagger} \xrightarrow{\mathrm{rk}} \Gamma(-, \mathbb{Z}),$$

and the restriction $\det|_{\mathrm{K}^{\mathrm{rk}=0}} : \mathrm{K}^{\mathrm{rk}=0} \to \mathcal{P}\mathrm{ic}$ recovers the classical determinant on rank-0 K-theory. It is a structural property of the $\mathbb{A}^1$-motivic filtration on K-theory that

$$\mathrm{K} \,/\, \mathrm{Fil}^1_{\mathbb{A}}\,\mathrm{K} \simeq \Gamma(-, \mathbb{Z}) \quad \text{and} \quad \mathrm{K} \,/\, \mathrm{Fil}^2_{\mathbb{A}}\,\mathrm{K} \simeq \mathcal{P}\mathrm{ic}^{\dagger}$$

in general (before cdh sheafification); the following corollary is the cdh-sheafified statement.

Corollary 4.5.2: For any qcqs scheme S, there are natural equivalences of presheaves of spectra on qcqs schemes:

$$\mathrm{Fil}^1_{\mathbb{A},\,\mathrm{cdh}}\,\mathrm{KH} \simeq L_{\mathrm{cdh}}\,\mathrm{fib}(\mathrm{rk} : \mathrm{K} \to \mathbb{Z}[0]),$$
$$\mathrm{Fil}^2_{\mathbb{A},\,\mathrm{cdh}}\,\mathrm{KH} \simeq L_{\mathrm{cdh}}\,\mathrm{fib}\big(\det : \mathrm{K} \to \mathcal{P}\mathrm{ic}^{\dagger}\big).$$

Equivalently,

$$\mathrm{KH} \,/\, \mathrm{Fil}^1_{\mathbb{A},\,\mathrm{cdh}}\,\mathrm{KH} \simeq \Gamma_{\mathrm{cdh}}(-, \mathbb{Z}) \quad \text{and} \quad \mathrm{KH} \,/\, \mathrm{Fil}^2_{\mathbb{A},\,\mathrm{cdh}}\,\mathrm{KH} \simeq L_{\mathrm{cdh}}\mathcal{P}\mathrm{ic}^{\dagger}.$$

Proof: Write Fil^j for $\mathrm{Fil}^j_{\mathbb{A},\,\mathrm{cdh}}\,\mathrm{KH}$ throughout.By [BEM25, Theorem 4.39(1)], $\mathrm{Fil}^0 \,/\, \mathrm{Fil}^1 \simeq \mathbb{Z}(0)^{\mathbb{A},\,\mathrm{cdh}}$ and $\mathrm{Fil}^1 \,/\, \mathrm{Fil}^2 \simeq \mathbb{Z}(1)^{\mathbb{A},\,\mathrm{cdh}}[2]$. By Corollary 4.4.1 (and the comparison $\mathbb{Z}(j)^{\mathbb{A},\,\mathrm{cdh}} \simeq \mathbb{Z}(j)^{\mathbb{A}}$ for $j \leq 1$ from its proof), these become

$$\mathrm{Fil}^0 \,/\, \mathrm{Fil}^1 \simeq \Gamma_{\mathrm{cdh}}(-, \mathbb{Z}) \quad \text{and} \quad \mathrm{Fil}^1 \,/\, \mathrm{Fil}^2 \simeq \Gamma_{\mathrm{cdh}}(-, \mathbb{G}_m)[1] \simeq L_{\mathrm{cdh}}\mathcal{P}\mathrm{ic}.$$

The Fil^1 formula. The first equivalence gives

$$\mathrm{Fil}^1 \simeq \mathrm{fib}(\mathrm{KH} \to \Gamma_{\mathrm{cdh}}(-, \mathbb{Z})).$$

Since the rank map at the level of K is $\mathrm{rk} : \mathrm{K} \to \mathbb{Z}[0]$ (the constant presheaf $\mathbb{Z}$ in degree 0), and L_{cdh} is left exact, we have

$$L_{\mathrm{cdh}}\,\mathrm{fib}(\mathrm{rk}:\mathrm{K}\to\mathbb{Z}[0])\simeq\mathrm{fib}(L_{\mathrm{cdh}}\,\mathrm{K}\to L_{\mathrm{cdh}}\mathbb{Z}[0])\simeq\mathrm{fib}(\mathrm{KH}\to\Gamma_{\mathrm{cdh}}(-,\mathbb{Z}))\simeq\mathrm{Fil}^1,$$

using $L_{\mathrm{cdh}}\,\mathrm{K}\simeq\mathrm{KH}$ (Kerz–Strunk–Tamme) and $L_{\mathrm{cdh}}\mathbb{Z}[0]\simeq\Gamma_{\mathrm{cdh}}(-,\mathbb{Z})$ (definition of cdh cohomology of the constant sheaf).

The $\mathbf{Fil^2}$ formula. Recall that the rank map $\mathrm{rk}:\mathrm{KH}\to\Gamma_{\mathrm{cdh}}(-,\mathbb{Z})$ factors through the determinant $\det:\mathrm{KH}\to L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}^{\dagger}$, since $\mathrm{rk}:\mathscr{P}\mathrm{ic}^{\dagger}\to\Gamma(-,\mathbb{Z})$ is the second map in the fiber sequence $\mathscr{P}\mathrm{ic}\to\mathscr{P}\mathrm{ic}^{\dagger}\to\Gamma(-,\mathbb{Z})$. Concretely, det restricts to $\det|_{\mathrm{KH}^{\mathrm{rk}=0}}:\mathrm{Fil}^1\to L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}$.

Consider the diagram of presheaves of spectra

$$\begin{tikzcd}
\mathrm{Fil}^2 \arrow[r] \arrow[d] & \mathrm{Fil}^1 \arrow[r] \arrow[d, "\det"'] & \mathrm{Fil}^0=\mathrm{KH} \arrow[d, "\det"'] \\
0 \arrow[r] \arrow[d] & L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic} \arrow[r] \arrow[d] & L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}^{\dagger} \arrow[d, "\mathrm{rk}"'] \\
0 \arrow[r] & 0 \arrow[r] & \Gamma_{\mathrm{cdh}}(-,\mathbb{Z})
\end{tikzcd}$$

whose commutativity follows from $\mathrm{rk}=\mathrm{rk}\circ\det$ together with $\det|_{\mathrm{Fil}^1}$ landing in $L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}$. We assemble the cartesian-ness of the top rectangle in four steps:

(1) The **bottom-right square** is cartesian by the definition of $\mathscr{P}\mathrm{ic}^{\dagger}$ as the extension $\mathscr{P}\mathrm{ic}\to\mathscr{P}\mathrm{ic}^{\dagger}\to\Gamma(-,\mathbb{Z})$ (applied after L_{cdh}).
(2) The **right rectangle** (columns 1–2, rows 0–2) is cartesian because $\mathrm{Fil}^1\simeq\mathrm{fib}(\mathrm{rk}:\mathrm{KH}\to\Gamma_{\mathrm{cdh}}(-,\mathbb{Z}))$ by the Fil^1 formula above.
(3) By pasting (1) and (2), the **top-right square** is cartesian: $\mathrm{Fil}^1\simeq\mathrm{KH}\times_{L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}^{\dagger}}L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}$.
(4) The **top-left square** is cartesian because $\mathrm{Fil}^2\simeq\mathrm{fib}(\mathrm{Fil}^1\to L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic})$, equivalently $\mathrm{Fil}^1/\mathrm{Fil}^2\simeq L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}$ (Corollary 4.4.1). Pasting with (3) gives that the top rectangle is cartesian.

The cartesian-ness of the top rectangle gives

$$\mathrm{Fil}^2\simeq\mathrm{fib}\big(\det:\mathrm{KH}\to L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}^{\dagger}\big)\simeq L_{\mathrm{cdh}}\,\mathrm{fib}\big(\det:\mathrm{K}\to\mathscr{P}\mathrm{ic}^{\dagger}\big),$$

using left exactness of L_{cdh} again. Equivalently $\mathrm{KH}/\mathrm{Fil}^2\simeq L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}^{\dagger}$. □

Remark 4.5.3: Writing $\mathrm{Fil}^j_{\mathbb{A},\mathrm{cdh}}\,\mathrm{KH}\simeq L_{\mathrm{cdh}}\,\mathrm{fib}(\ldots:\mathrm{K}\to\ldots)$ rather than $\mathrm{fib}\big(\ldots:\mathrm{K}^{\mathrm{rk}=0}\to\mathscr{P}\mathrm{ic}\big)$ has the advantage that the filtration **quotients** $\Gamma_{\mathrm{cdh}}(-,\mathbb{Z})$ and $L_{\mathrm{cdh}}\mathscr{P}\mathrm{ic}^{\dagger}$ are immediately visible. The two formulations agree because L_{cdh} is left exact and in particular preserves fibers.

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