A Brief Introduction To Monad And Higher Algebra

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Chapter 1

Introduction

1.1 Motivation: From Adjunctions To Monads

Consider the free-forgetful adjunction between the 1-categories of sets Set and 1-category of monoids Monoid. Let $F\colon \mathsf{Set} \to \mathsf{Monoid}$ be the free functor and $U\colon \mathsf{Monoid} \to \mathsf{Set}$ be the forgetful functor. We denote this adjunction by:

$$F\colon \mathsf{Set} \rightleftarrows \mathsf{Monoid}\colon U, \quad \text{with } F \dashv U.$$

The composition $T \coloneqq U \circ F$ is an endofunctor on Set. For any set X, the functor T maps it to the underlying set of the free monoid generated by X. This is the set of all finite lists (or words) with elements from X:

$$T(X) = \coprod_{n \geq 0} X^n,$$

where X^0 is a singleton set representing the empty word.

While the functor T itself "forgets" the monoid structure, the full adjunction carries more information. Specifically, it includes a unit of adjunction $\eta\colon \mathrm{id}_{\mathsf{Set}} \to U \circ F = T$ and a counit $\varepsilon\colon F \circ U \to \mathrm{id}_{\mathsf{Monoid}}$.

This raises a natural question: can we use the endofunctor T, along with the natural transformations derived from η and ε, to reconstruct the category of monoids? The answer is yes. To see how, we first examine the structure that T inherits from the adjunction.

The adjunction's unit and counit induce two natural transformations for T. They will induce two operations of T:

- The unit of the adjunction itself serves as a *unit operation* for T:

  $$\eta\colon \mathrm{id}_{\mathsf{Set}} \to T.$$

  For a set X, $\eta_X : X \to T(X)$ maps an element $x \in X$ to the singleton word (x).

- A *multiplication operation* $\mu\colon T^2 \to T$ can be constructed from the counit ε:

  $$\mu \coloneqq U\varepsilon F : T^2 \to T.$$

  For a set X, an element of $T^2(X)$ is a word of words of X, that is, a list of lists $((x_1, \cdots, x_n), (y_1, \cdots, y_m), \cdots)$ of elements from X. The map $\mu_X : T^2(X) \to T(X)$ acts by concatenation, or "flattening," the word of words into a single word

  $$((x_1, \cdots, x_n), (y_1, \cdots, y_m), \cdots) \to (x_1, \cdots, x_n, y_1, \cdots, y_m, \cdots).$$

These data guarantee T to be an algebra object in the endofunctor category $\mathsf{End}(\mathsf{Set})$ w.r.t. the monoidal structure provided by the composition. More precisely, the following diagrams commute, expressing the properties of unitality and associativity:

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T. \end{tikzcd}$$

In this case, we will say that (T, η, μ) is a *monad* over Set.

Now we can recover the algebraic structure. A *T-module* is a pair (M, a) consisting of a set M and a map $a\colon T(M) \to M$ (the "structure map") such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "\mathrm{id}_M"'] & T(M) \arrow[d, "a"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^2(M) \arrow[r, "T(a)"] \arrow[d, "\mu_M"'] & T(M) \arrow[d, "a"] \\ T(M) \arrow[r, "a"'] & M. \end{tikzcd}$$

Suppose we interpret the map a as the monoid's multiplication, where it takes a list of elements from M and multiplies them together. In that case, these two diagrams correspond precisely to the monoid axioms of identity and associativity, respectively.

Furthermore, a *morphism of T-modules* from (M, a) to (N, a') is a function $f\colon M \to N$ that respects the algebra structures, meaning the following diagram commutes:

$$\begin{tikzcd} T(M) \arrow[r, "a"] \arrow[d, "T(f)"'] & M \arrow[d, "f"] \\ T(N) \arrow[r, "a'"'] & N. \end{tikzcd}$$

A quick check reveals that this condition is equivalent to stating that f is a monoid homomorphism.

The category of T-modules and their morphisms, denoted Set^T. For this specific monad, we find a remarkable result: there is an equivalence of 1-categories:

$$\mathsf{Set}^T \simeq \mathsf{Monoid}.$$

This demonstrates that we have successfully recovered the category of monoids using only the data from the monad induced by the free-forgetful adjunction. This leads to a more general question: for an arbitrary adjunction

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G, \quad \text{with } F \dashv G.$$

when does this equivalence between $\mathcal{D}$ and the category of left T-modules of the induced monad hold?

The core structure that enabled this reconstruction is the monad (T, μ, η). The study of monads, their modules, and their relationship to adjunctions is a central theme in category theory.

1.2 Notation and Conventions

This note is rooted in higher categories and so we decided to drop "∞" from the notation, i.e. we refer to ∞-categories as categories and to (∞, n)-categories as n-categories. We refer the readers to [Mathew, 2016] for an excellent introduction to category theory.

Throughout this work, we adopt the following notations:

- The *simplex category* is denoted by $\mathbb{\Delta}$. The objects are finite, non-empty, linearly ordered sets together with order preserving maps. Every object of $\mathbb{\Delta}$ is isomorphic to $[n] \coloneqq \{0 < 1 < \cdots < n\}$, for some $n \geq 0$.
- We will say 'anima'(plural animae) instead of 'space' to refer to ∞-groupoids. And denote the category of animae by Ani.
- Cat is the category of categroies, $\mathsf{Cat}_{(1)} \subseteq \mathsf{Cat}$ is the subcategory spanned by 1-categories.
- We will say "static commutative rings" to mean the ordinary rings and denote the category of static commutative rings by $\mathsf{Ring}^{\mathrm{static}}$ and "commutative rings" to mean the commutative algebra in Sp and denote the category of commutative rings by CAlg.
- We also use "static" to indicate that a derived object lies in the heart, e.g. a static R-module is just an ordinary R-module (for a static commutative ring R).
- Throughout this note, we will use *cohomological index*.
- The term *initial functor* refers to a "right cofinal" functor in the sense of [Lurie, 2018a, Tag 02N0], while the term *final functor* refers to a "left cofinal" functor.

Chapter 2

Basic Concepts on Higher Algebra

In this chapter, we aim to employ the framework of algebraic patterns developed in [Chu and Haugseng, 2021] as a foundation for introducing the theory of higher algebra.

The chapter is organized as follows:

- In § 2.1, we introduce the definition of algebraic patterns and present several examples that will be used in subsequent chapters.
- In § 2.2, we discuss the algebra objects associated with algebraic patterns, which we refer to as "Segal objects." We then illustrate this notion through the examples introduced in § 2.1.
- In § 2.3, we introduce the notion of an operad over an algebraic pattern. Informally, for an algebraic pattern $\mathcal{P}$, a $\mathcal{P}$-operad $\mathcal{O}$ is itself an algebraic pattern in which every morphism can be expressed as a "$\mathcal{P}$-type operation." We explain this idea using the Comm-operad as a guiding example.
- In § 2.4, we define $\mathcal{O}$-monoidal categories and $\mathcal{O}$-algebras for a Cartesian pattern $\mathcal{O}$ (Definition 2.4.1).
- In § 2.5, we investigate the properties of nonunital $\mathbb{A}_n$-algebras. We show that for a monoidal $(m,1)$-category $\mathcal{C}$, there is an equivalence
$$\mathsf{Alg}^{\mathrm{nu}}(\mathcal{C}) \simeq \mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_n}(\mathcal{C})$$
whenever $n \geq m+2$.
- In § 2.6 and § 2.7, we construct the symmetric monoidal structure on Op and use it to show that the operad of little k-cubes $\mathbb{E}_k$ (Definition 2.7.2) encodes the same data as the pattern $\mathsf{E}_k \simeq \mathsf{Assoc}^{\times k}$ (Definition 2.1.12). As a consequence, for a symmetric monoidal $(m,1)$-category $\mathcal{C}$, we obtain an equivalence
$$\mathsf{Alg}_{\mathbb{E}_k}(\mathcal{C}) \simeq \mathsf{CAlg}(\mathcal{C})$$
whenever $k \geq m+1$.
- Finally, we introduce several symmetric monoidal structures that play an important role in later chapters.

For a comprehensive introduction to these topics, we refer to the excellent overview in [Cnossen, 2025] and [Haugseng, 2023].

2.1 Algebraic Pattern

Algebraic pattern is a blueprint for a notion of functors on a fixed category satisfying a Segal condition, suitable for formalizing homotopy-coherent algebra in the Cartesian setting.

Informally, algebraic pattern generalizes the active and inert morphisms in operads and chooses certain objects to control the Segal condition.

Definition 2.1.1. An *algebraic pattern* is a category $\mathcal{O}$ equipped with:

1. A collection of objects called *elementary objects*.
2. A factorization system $(\mathcal{O}^{\mathrm{inv}}, \mathcal{O}^{\mathrm{act}})$ where every morphism factors uniquely (up to equivalence) as an *inert* morphism followed by an *active* morphism.

We let $\mathcal{O}^{\mathrm{el}}$ denote the full subcategory of $\mathcal{O}$ spanned by the elementary objects and the inert morphisms between them. For any object $X \in \mathcal{O}$, we also write

$$\mathcal{O}^{\mathrm{el}}_{X/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{inv}}} \mathcal{O}^{\mathrm{inv}}_{X/}.$$

for the category of inter morphisms $X \to E$ with $E \in \mathcal{O}^{\mathrm{el}}$.

A morphism of algebraic patterns from $\mathcal{O}$ to $\mathcal{P}$ is a functor $f\colon \mathcal{O} \to \mathcal{P}$ that preserves inert and active morphisms and elementary objects.

We will use $\mathsf{AlgPatt}$ to denote the category of algebraic patterns.

Remark 2.1.2. A *factorization system* on a category $\mathcal{C}$ is a pair of subcategories $(\mathcal{L}, \mathcal{R})$ that contain all objects, such that for any morphism $f\colon X \to X'$, the anima of factorizations $X \xrightarrow{l} Y \xrightarrow{r} X'$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$ is contractible.

$$\left\{ \begin{tikzcd} & Y \arrow[dr, "r\in\mathcal{R}"] & \\ X \arrow[ur, "l\in\mathcal{L}"] \arrow[rr, "f"'] & & X' \end{tikzcd} \right\}$$

Proposition 2.1.3. $\mathsf{AlgPatt}$ *admits limits and filtered colimits, and the forgetful functor*

$$\mathsf{AlgPatt} \to \mathsf{Cat}$$

preserves these.

Proof. [Chu and Haugseng, 2021, Corollary 5.5] □

Definition 2.1.4. (Trivial Pattern) *Trivial Pattern* Triv is the final object of $\mathsf{AlgPatt}$. The underlying category of trivial pattern is the final category $*$.

Definition 2.1.5. (Empty Pattern) *Empty Pattern* $\varnothing$ is the initial object of $\mathsf{AlgPatt}$. The underlying category of empty pattern is the initial category $\varnothing$.

Definition 2.1.6. (Commutative Pattern) Consider the category of pointed finite sets Fin_* with $\langle n \rangle := (\{0, 1, \cdots, n\}, 0)$. We say a morphism $f\colon \langle n \rangle \to \langle m \rangle$ is:

- *Inert*, if f restricts to an isomorphism $\langle n \rangle \setminus f^{-1}(0) \to \langle m \rangle \setminus \{0\}$.
- *Active*, if $f^{-1}(0) = \{0\}$.

We make this an algebraic pattern by taking $\langle 1 \rangle$ to be the single elementary object and denote it by Comm. We refer to this pattern as *commutative pattern*.

Remark 2.1.7. In Definition 2.1.6, if we take $\langle 0\rangle$ and $\langle 1\rangle$ to be the elementary objects, we can get a new algebraic pattern $\mathsf{Fin}_*^\natural$.

Definition 2.1.8. (Associative Pattern) Consider the opposite of the simplex category, $\mathbb{\Delta}^{\mathrm{op}}$. A morphism $f\colon [n]\to[m]$ in $\mathbb{\Delta}^{\mathrm{op}}$ is:

- *Inert*, if its corresponding map $g\colon [m]\to[n]$ in $\mathbb{\Delta}$ is an interval inclusion (i.e., its image is a contiguous block of integers).
- *Active*, if its corresponding map $g\colon [m]\to[n]$ preserves endpoints (i.e., $g(0)=0$ and $g(m)=n$).

We choose the object $[1]$ as the unique elementary object. This algebraic pattern, denoted Assoc, is called the *associative pattern*.

Definition 2.1.9. ($\mathbb{A}_n$-Pattern) For $1\le n\le\infty$, let $\mathbb{A}_n$ denote the full subcategory of $\mathbb{\Delta}^{\mathrm{op}}$ spanned by the objects $[m]$ for $0\le m\le n$.

The category $\mathbb{A}_n$ can be endowed with an algebraic pattern structure inherited from Assoc, in which the inert and active morphisms are precisely those that are inert or active in Assoc, and the elementary object is given by $[1]$. We will refer to $\mathbb{A}_n$ as the *$\mathbb{A}_n$-pattern*.

Remark 2.1.10. One can find that the $\mathbb{A}_\infty$-pattern is the associative pattern.

Definition 2.1.11. (Nonunital $\mathbb{A}_n$-Pattern) Let $\mathbb{\Delta}_{\mathrm{inj}}\subseteq\mathbb{\Delta}$ denote the subcategory whose morphisms are strictly increasing map $[m]\hookrightarrow[n]$. For $0\le n\le\infty$, let $\mathbb{A}_n^{\mathrm{nu}}$ denote the full subcategory of $\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$ spanned by the objects $[m]$ for $1\le m\le n$ (so $\mathbb{A}_0^{\mathrm{nu}}\simeq\varnothing$).

The category $\mathbb{A}_n^{\mathrm{nu}}$ can be endowed with an algebraic pattern structure inherited from Assoc, in which the inert and active morphisms are precisely those that are inert or active in Assoc, and the elementary object is given by $[1]$. We will refer to $\mathbb{A}_n^{\mathrm{un}}$ as the *nonunital $\mathbb{A}_n$-pattern*.

Definition 2.1.12. (E_k-Pattern) Consider the k-copies of the opposite of simplex categories, $\mathbb{\Delta}^{k,\mathrm{op}} := (\mathbb{\Delta}^{\mathrm{op}})^{\times k}$, equipped with the factorization system where the inert and active maps are those that are inert or active in Assoc in each component. We choose the object $([1],\cdots,[1])$ to be the unique elementary object. This algebraic pattern, denoted E_k, is called the *E_k-pattern*.

Remark 2.1.13. In § 2.7, we will discuss the relation between the E_n-pattern and the $\mathbb{E}_k$-operad (Definition 2.7.2).

Next, we introduce patterns related to modules.

Definition 2.1.14. (Commutative Modules Pattern) The underlying category of *commutative modules pattern* CM is the category $\mathsf{Fin}_{*,\langle 1\rangle/}$. Its factorzation system is lifted from Comm along the canonical left fibration $\mathsf{Fin}_{*,\langle 1\rangle/}\to\mathsf{Fin}_*$. And the elementary objects are given by $\langle 1\rangle\to\{0\}\subseteq\langle 1\rangle$ and $\mathrm{id}_{\langle 1\rangle}$.

By construction, an object $\langle 1\rangle\to\langle n\rangle$ in $\mathsf{Fin}_{*,\langle 1\rangle/}$ can be regarded as a pair $(\langle n\rangle, i)$, where i is the image of $1\in\langle 1\rangle$. The morphism in CM, which is form $f\colon(\langle n\rangle,i)\to(\langle h\rangle,j)$, refers to the pointed map $\langle n\rangle\to\langle h\rangle$ with $f(i)=j$.

Definition 2.1.15. (Bimodule Pattern) The underlying category of *bimodule pattern* BM is the category $(\mathbb{\Delta}_{/[1]})^{\mathrm{op}}$. Its factorization system is lifted from Assoc along the canonical left fibration $(\mathbb{\Delta}_{/[1]})^{\mathrm{op}}\to\mathbb{\Delta}^{\mathrm{op}}$. And the elementary objects are given by $[1]\simeq\{0\}\to[1]$, $[1]\simeq\{1\}\to[1]$ and $\mathrm{id}_{[1]}$.

By construction, an object $[n]\to[1]$ in $\mathbb{\Delta}_{/[1]}$ can be viewed as an ordered sequence $(i_0,\dots,i_n)$ where $0\le i_0\le\cdots\le i_n\le 1$. The elementary objects correspond to the sequences $(0,0)$, $(0,1)$, and $(1,1)$.

Definition 2.1.16. (Left Module Pattern, Simplicial model version) The underlying category of *left module pattern* LM is the category $\mathbb{\Delta}^{\mathrm{op}} \times [1]$. Consider the functor $T \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$, sent $[n]$ to $[n] \star [0] \simeq [n+1]$. Then the functor induces a functor

$$(T \to \mathrm{id}) \colon \mathbb{\Delta}^{\mathrm{op}} \times [1] \to \mathbb{\Delta}^{\mathrm{op}}.$$

The algebraic pattern structure of LM is lifted from Assoc along $(T \to \mathrm{id})$. More precisely,

- $\mathbb{\Delta}^{\mathrm{op}} \times \{1\}$ is precisely Assoc.
- For each $[n] \in \mathbb{\Delta}$, the induced morphism $([n], 0) \to ([n], 1)$ is an inert morphism in LM.
- If $f \colon [h] \to [n]$ is an inert morphism in $\mathbb{\Delta}$ such that $f(h) = n$, then the corresponds morphism $([n], 0) \to ([h], 0)$ is an inert morphism in LM.
- If $g \colon [h] \to [n]$ is a morphism in $\mathbb{\Delta}$ such that $g(0) = 0$, then the corresponds morphism $([n], 0) \to ([h], 0)$ is an active morphism in LM.
- The elementary objects of LM are $([0], 0)$ (often denoted by $\mathfrak{m}$) and $([1], 1)$ (often denoted by $\mathfrak{a}$).

Remark 2.1.17. Saying 'left' versus 'right' is just a convention; the same algebraic pattern also encodes right actions.

2.2 Segal Object

The algebra represented by the algebraic pattern is called Segal object.

Definition 2.2.1. Let $\mathcal{O}$ be an algebraic pattern. A functor $X \colon \mathcal{O} \to \mathcal{C}$ is called *Segal $\mathcal{O}$-object* of category $\mathcal{C}$ if for every $O \in \mathcal{O}$ the induced functor

$$\left(\mathcal{O}^{\mathrm{el}}_{O/}\right)^{\triangleleft} \to \mathcal{O} \xrightarrow{X} \mathcal{C}$$

is a limit diagram. If $\mathcal{C}$ has limit for diagrams indexed by $\mathcal{O}^{\mathrm{el}}_{O/}$ for all $O \in \mathcal{O}$ in which case we say that $\mathcal{C}$ is *$\mathcal{O}$-complete*, then this condition is equivalent to the canonical morphisms

$$X(O) \to \varprojlim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} X(E).$$

Now, we will provide some examples to explain how algebraic patterns work.

Example 2.2.2. (Segal $\mathsf{Trivial}$-Objects) Let $\mathcal{C}$ be a category. Then the Segal $\mathsf{Trivial}$-object in $\mathcal{C}$ is just an object in $\mathcal{C}$.

Example 2.2.3. (Segal Comm-Objects) Let $\mathcal{C}$ be a category with finite products, and let $X \colon \mathsf{Comm} \to \mathcal{C}$ be a functor. The Segal condition on X is

$$X(\langle n \rangle) \simeq \varprojlim_{(\langle n \rangle \to \langle 1 \rangle) \in \mathsf{Comm}^{\mathrm{inv}}} X(\langle 1 \rangle).$$

We can identify the category $\mathsf{Comm}^{\mathrm{el}}_{\langle n \rangle/}$ with the set of inert morphisms $\{\rho_i \colon i = 1, \ldots, n\}$, where $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ is given by

$$\rho_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Then, the Segal condition says that the canonical map

$$(\rho_i^*)_{i=1}^n \colon X(\langle n\rangle) \to \prod_{i=1}^n X(\langle 1\rangle)$$

is an equivalence. This means that for each non-basepoint element $i \in \langle n\rangle$ (where $i \neq 0$), we can specify a corresponding object $x_i \in X(\langle 1\rangle)$. Therefore, we can describe an object in $X(\langle n\rangle)$ as a sequence $(x_1, \dots, x_n)$.

Next, we will show how inert and active morphisms work:

- Let $f \colon \langle n\rangle \to \langle m\rangle$ be an inert morphism in Comm. Then f corresponds to the projection

$$X(f)\colon X(\langle n\rangle) \simeq X(\langle 1\rangle)^n \to X(\langle 1\rangle)^m \simeq X(\langle m\rangle), \quad (x_1, \dots, x_n) \mapsto (x_{f^{-1}(1)}, \dots, x_{f^{-1}(m)}).$$

- Let $g \colon \langle n\rangle \to \langle m\rangle$ be an active morphism in Comm, and let $I_j \coloneqq g^{-1}(j)$ for $j \in \{1, \dots, m\}$. Then g corresponds to the morphism

$$X(g)\colon X(\langle n\rangle) \simeq X(\langle 1\rangle)^n \to X(\langle 1\rangle)^m \simeq X(\langle m\rangle), \quad (x_1, \dots, x_n) \mapsto \left(\prod_{i_1\in I_1} x_{i_1}, \dots, \prod_{i_m \in I_m} x_{i_m}\right).$$

  In particular, the active morphism $s \colon \langle 2\rangle \to \langle 2\rangle$ that swaps 1 and 2 corresponds to the map $(x_1, x_2) \mapsto (x_2, x_1)$, which enforces the commutativity. Active morphisms represent the "commutative multiplication".

When we set $\mathcal{C} = \mathsf{Set}$, we find that Comm-Segal objects are precisely commutative monoids.

Example 2.2.4. (Segal Assoc-Objects) Let $\mathcal{C}$ be a category with finite products, and let $X \colon \mathsf{Assoc} \to \mathcal{C}$ be a simplicial object. The Segal condition on X is

$$X([n]) \simeq \varprojlim_{([n]\to[1])\in \mathsf{Assoc}^{\mathrm{inv}}} X([1]).$$

Now, let's analyze the limit above. An inert morphism $e_i \colon [n] \to [1]$ in Δ^{op} corresponds to an inclusion $[1] \hookrightarrow [n]$ in Δ with image $\{i-1, i\}$. Notice that $[n]$ is a linearly ordered set, and one can think of it as being cut into n pieces:

$$[n] = \left\{0 \xrightarrow{e_1} 1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} n\right\},$$

where the segment $i-1 \to i$ corresponds to the inert map e_i. The Segal condition says that the canonical map

$$(e_i^*)_{i=1}^n \colon X([n]) \to \prod_{i=1}^n X([1])$$

is an equivalence. This means we can associate each arrow $i-1 \to i$ in $[n]$ with a corresponding object $x_i \in X([1])$. Therefore, we can describe an object in $X([n])$ as a sequence $(x_1, \dots, x_n)$.

Next, we will show how inert and active morphisms work:

- Let $f \colon [n] \to [m]$ be an inert morphism in Assoc, and let $f^{\mathrm{op}} \colon [m] \to [n]$ be the corresponding morphism in Δ. In the Segal object X, the morphism $X(f)$ corresponds to a projection:

$$X([n]) \simeq X([1])^n \to X([1])^m \simeq X([m]), \quad (x_1, \dots, x_n) \mapsto (x_{f^{\mathrm{op}}(1)}, \dots, x_{f^{\mathrm{op}}(m)}).$$

- Let $g \colon [n] \to [m]$ be an active morphism in Assoc, and let $I_j \coloneqq \{(g^{\mathrm{op}})^{-1}(j-1)+1, \cdots, (g^{\mathrm{op}})^{-1}(j)\}$ for $j \in \{1, \dots, m\}$. In the Segal object X, the morphism $X(g)$ corresponds to:

$$X([n]) \simeq X([1])^n \to X([1])^m \simeq X([m]), \quad (x_1, \dots, x_n) \mapsto \left(\prod_{i_1\in I_1} x_{i_1}, \dots, \prod_{i_m \in I_m} x_{i_m}\right),$$

  which represents "multiplication".

When we set $\mathcal{C} = \mathsf{Set}$, we find that Assoc-Segal objects are precisely monoids.

Remark 2.2.5. Note that:

- the morphism $[0] \to [1]$ in Assoc corresponds to the morphism $1\colon * \to X([1])$, which is the unit of the associative algebra.
- the degenerate morphism $\mathrm{s}^i\colon [n] \to [n-1]$ will corresponds to the morphism $X([n-1]) \to X([n])$, $(x_1, \cdots, x_{n-1}) \mapsto (x_1, \cdots, \underset{i\text{-th}}{1}, \cdots, x_{n-1})$.

Therefore, if we remove these data, we will get the Segal $\mathbb{A}_\infty^{\mathrm{nu}}$-object (Definition 2.1.11).

Example 2.2.6. (Segal CM-Objects) Let $\mathcal{C}$ be a category with finite products, and $X\colon \mathsf{CM} \to \mathcal{C}$ be a functor. The Segal condition on X is

$$X(\langle n\rangle, i) \simeq \varprojlim_{(\langle h\rangle, j) \in \mathsf{CM}^{\mathrm{el}}_{(\langle n\rangle, i)/}} X(\langle h\rangle, j).$$

Now, let's analyze the limit above. When $i = 0$, the Segal condition is equivalent to saying that

$$X(\langle n\rangle, 0) \simeq \prod_{j=1}^{n} X(\langle 1\rangle, 0).$$

When $i \neq 0$, the Segal condition is equivalent to saying that

$$X(\langle n\rangle, i) \simeq \prod_{j=1}^{i-1} X(\langle 1\rangle, 0) \times X(\langle 1\rangle, 1) \times \prod_{j=i+1}^{n} X(\langle 1\rangle, 0).$$

We will refer to $X(\langle 1\rangle, 0)$ as A and $X(\langle 1\rangle, 1)$ as M. Next, we will show how inert and active morphisms work:

- Let $f\colon (\langle n\rangle, i) \to (\langle h\rangle, j)$ be an inert morphism in CM.
  - If $i = 0$, we have $j = 0$ by construction of CM, in this case, f corresponds to the projection
  $$\begin{gathered} A^n \to A^h \\ (a_1, \cdots, a_n) \mapsto (a_{f^{-1}(1)}, \cdots, a_{f^{-1}(h)}). \end{gathered}$$
  - If $i \neq 0$ with $j = 0$, then f corresponds to the projection
  $$\begin{gathered} A^{i-1} \times M \times A^{n-i} \to A^h \\ (a_1, \cdots, m, \cdots, a_n) \mapsto (a_{f^{-1}(1)}, \cdots, a_{f^{-1}(h)}). \end{gathered}$$
  - If $i \neq 0$ with $j \neq 0$, then f corresponds to the projection
  $$\begin{gathered} A^{i-1} \times M \times A^{n-i} \to A^{j-1} \times M \times A^{h-j} \\ (a_1, \cdots, m, \cdots, a_n) \mapsto (a_{f^{-1}(1)}, \cdots, m, \cdots, a_{f^{-1}(h)}). \end{gathered}$$
- Let $g\colon (\langle n\rangle, i) \to (\langle h\rangle, j)$ be an active morphism in CM.
  - If $i = 0$, then we have $j = 0$ by construction of CM, in this case, let $I_j \coloneqq g^{-1}(j)$ for $j \in \{1, \ldots, m\}$, we have g corresponds to the morphism
  $$\begin{gathered} A^n \to A^h \\ (a_1, \cdots, a_n) \mapsto \left(\prod_{i_1 \in I_1} a_{i_1}, \cdots, \prod_{i_h \in I_h} a_{i_h}\right). \end{gathered}$$

– If $i \neq 0$, then by the description of the active morphism, we know that $j \neq 0$. Let $I_j := g^{-1}(j)$ for $j \in \{1, \ldots, m\}$, we have g corresponds to the morphism

$$\begin{aligned} A^{i-1} \times M \times A^{n-i} &\to A^{j-1} \times M \times A^{h-j} \\ (a_1, \cdots, a_n) &\mapsto \left(\prod_{i_1 \in I_1} a_{i_1}, \cdots, \left(\prod_{i_j \in I_j} a_{i_j} \right) \cdot m, \cdots, \prod_{i_h \in I_h} a_{i_h} \right). \end{aligned}$$

which represents "action".

Example 2.2.7. (Segal LM-Objects) Let $\mathcal{C}$ be a category with finite products, and $X \colon \mathsf{LM} \to \mathcal{C}$ be a functor. The Segal condition on X is

$$X([n], i) \simeq \varprojlim_{([h], j) \in \mathsf{LM}^{\mathrm{el}}_{([n], i)/}} X([h], j).$$

Now, let's analyze the limit above. Consider the canonical projection $p \colon ([n], 0) \to ([0], 0)$ in LM corresponds to an inclusion $[0] \simeq \{n\} \hookrightarrow [n]$.

When $i = 1$, the Segal condition is equivalent to saying that

$$((e_i, 1)^*)_{i=1}^n : X([n], 1) \simeq \prod_{i=1}^n X([1], 1).$$

When $i = 0$, the Segal condition is equivalent to saying that

$$((e_i, 0)^*)_{i=1}^n \times p \colon X([n], 0) \to \left(\prod_{i=1}^n X([1], 1) \right) \times X([0], 0).$$

We will refer to $X([1], 1)$ as A and $X([0], 0)$ as M.

That is, the Segal LM-objects in $\mathcal{C}$ consists of those natural transformations $M_\bullet \Rightarrow A_\bullet$ of simplicial objects $M_\bullet, A_\bullet \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that $A_\bullet$ is a Segal Assoc-object in $\mathcal{C}$ and for all $n \geq 0$, we have

$$M_n = A^n \times M.$$

Now we will show how inert and active morphisms work:

- $\{1\} \times \mathbb{\Delta}^{\mathrm{op}}$ consistent with Assoc.
- For each $[n] \in \mathbb{\Delta}$, the induced morphism $([n], 0) \to ([n], 1)$ corresponds to the projection

$$\begin{aligned} M_n = A^n \times M &\to A^n \\ (a_1, \cdots, a_n, m) &\mapsto (a_1, \cdots, a_n). \end{aligned}$$

- Let $f \colon ([n], 0) \to ([h], 0)$ be an inert morphism in LM, denote its image under $(T \to \mathrm{id})$ as $\tilde{f} \colon [n+1] \to [h+1]$. Then f corresponds to the projection

$$\begin{aligned} M_n = A^n \times M &\to A^h \times M = M_h \\ (a_1, \cdots, a_n, m) &\mapsto \left(a_{\tilde{f}^{\mathrm{op}}(1)}, \cdots, a_{\tilde{f}^{\mathrm{op}}(h)}, m \right). \end{aligned}$$

- Let $g \colon ([n], 0) \to ([h], 0)$ be an active morphism in LM, denote its image under $(T \to \mathrm{id})$ as $\tilde{g} \colon [n+1] \to [h+1]$. let $I_j := \{(\tilde{g}^{\mathrm{op}})^{-1}(j-1)+1, \cdots, (\tilde{g}^{\mathrm{op}})^{-1}(j)\}$ for $j \in \{1, \ldots, h+1\}$. Then g corresponds to the morphism

$$\begin{aligned} M_n = A^n \times M &\to A^h \times M = M_h \\ (a_1, \cdots, a_n, m) &\mapsto \left(\prod_{i_1 \in I_1} a_{i_1}, \cdots, \prod_{i_h \in I_h} a_{i_h}, \left(\prod_{i_{h+1} \in I_{h+1}} a_{i_{h+1}} \right) \cdot m \right) \end{aligned}$$

which represents "left action".

2.3 Operad Over An Algebraic Pattern

In this section, we introduce *operads*, which are mathematical structures designed to encode the abstract properties of algebraic operations. Building on the notion of algebraic patterns, operads allow us to describe entire algebraic theories, such as the theory of homotopy-coherence algebras.

For more heuristics of operads, we refer to the excellent overview in [Cnossen, 2025, Chapter 10].

An operad $\mathcal{O}$ over an algebraic pattern $\mathcal{P}$ can be regarded as a category of "$\mathcal{P}$-type" operation, where the "$\mathcal{P}$-type" means Segal condition of $\mathcal{P}$.

Definition 2.3.1. (Operad) Let $\mathcal{P}$ be an algebraic pattern. An $\mathcal{P}$*-operad* is a functor $p\colon \mathcal{O} \to \mathcal{P}$ with the algebraic pattern structure lifted from $\mathcal{P}$ such that:

1. $\mathcal{O}$ has p-coCartesian lifts of inert morphisms in $\mathcal{P}$.
2. For $P \in \mathcal{P}$, let $\mathcal{O}_P$ denote the fiber of P. For $X \in \mathcal{O}_P$, if $\xi\colon \left(\mathcal{P}^{\mathrm{el}}_{P/}\right)^{\triangleleft} \to \mathcal{C}$ is a diagram of coCartesian morphisms over the object of $\mathcal{P}^{\mathrm{el}}_{P/}$, then for $Y \in \mathcal{O}_{P'}$, the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{O}}(Y,X) \arrow[r] \arrow[d] & \varprojlim\limits_{\alpha\colon P\to O\in\mathcal{P}^{\mathrm{el}}_{P/}} \mathrm{Hom}_{\mathcal{O}}(Y,\xi(\alpha)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{P}}(P',P) \arrow[r] & \varprojlim\limits_{\alpha\colon P\to O\in\mathcal{P}^{\mathrm{el}}_{P/}} \mathrm{Hom}_{\mathcal{P}}(P',O) \end{tikzcd}$$

   is Cartesian.
3. The functor $\mathcal{O}_P \to \varprojlim\limits_{O\in\mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}_O$ is an equivalence.

We refer to $\mathsf{Op}(\mathcal{P})$ as the category of $\mathcal{P}$-operads and functors over $\mathcal{P}$ that preserve inert coCartesian morphisms.

Remark 2.3.2. When we consider the category(without algebraic pattern structure) of an $\mathcal{P}$-operad $\mathcal{O}$, we will denote it as $\mathcal{O}^{\otimes}$.

Example 2.3.3. (Operad) Comm-operad is precisely the operad in the sense of [Lurie, 2017, Definition 2.1.1.10]. Unless specified otherwise, we will use the term "operad" to mean a Comm-operad and denote its category by Op.

Example 2.3.4. (Generalized Operad) By Remark 2.1.7, one can consider $\mathsf{Fin}^{\natural}_{*}$-operad, which is precisely the generalized operad in the sense of [Lurie, 2017, Definition 2.3.2.1]. Unless specified otherwise, we will use the term "generalized operad" to mean a $\mathsf{Fin}^{\natural}_{*}$-operad and denote its category by $\mathsf{Op}^{\mathrm{gen}}$.

Remark 2.3.5. Consider the inclusion $\mathsf{Op} \subseteq \mathsf{Op}^{\mathrm{gen}}$. By [Lurie, 2017, Proposition 2.1.4.6 and Remark 2.3.2.4] and [Lurie, 2009, Propositon A.3.7.6], one can find that both Op and $\mathsf{Op}^{\mathrm{gen}}$ are presentable. And $\mathsf{Op} \hookrightarrow \mathsf{Op}^{\mathrm{gen}}$ preserves limits. Therefore, it admits a left adjoint

$$\mathrm{L}_{\mathrm{gen}}\colon \mathsf{Op}^{\mathrm{gen}} \to \mathsf{Op}.$$

Example 2.3.6. (Planar Operad) Assoc-operad is the planar operad in the sense of [Lurie, 2017, Definition 4.1.3.2]. Unless specified otherwise, we will use the term "planar operad" to mean a Assoc-operad and denote its category by $\mathsf{Op}^{\mathrm{ns}}$.

If $f\colon \mathcal{O} \to \mathcal{P}$ is a functor between algebraic patterns that preserves the factorization system and elementary objects, and moreover, the induced functor

$$\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$$

is initial, that is, let $F\colon \mathcal{P}^{\mathrm{el}}_{f(X)/} \to \mathcal{C}$, then $\varprojlim F \simeq \varprojlim F \circ f$. Then the pullback functor

$$f^*\colon \mathsf{Op}(\mathcal{P}) \to \mathsf{Op}(\mathcal{O}).$$

Under mild assumptions, this has a left adjoint.

Example 2.3.7. Define a functor $\mathrm{Cut}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathsf{Fin}_*$ that takes $[n]$ to $\langle n\rangle$ and a morphism $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}$ to the map $\mathrm{Cut}(\varphi)\colon \langle m\rangle \to \langle n\rangle$ given by

$$\mathrm{Cut}(\varphi)(i) = \begin{cases} j, \phi(j-1) < i \leq \phi(j), \\ 0, \text{otherwise}. \end{cases}$$

Pulling back along this gives a functor $\mathsf{Op} \to \mathsf{Op}^{\mathrm{ns}}$ that informally "forget symmetric group actions". Its left adjoint is a "symmertization" functor $\mathsf{Op}^{\mathrm{ns}} \to \mathsf{Op}$. In fact, [Barkan et al., 2024, Theorem 5.1.1] proves a comparison result that gives conditions for certain functors as above to induce equivalences.

Let's examine the structure of a Comm-operad $\mathcal{O}$ in more detail. We refer to the fiber $\mathcal{O}_{\langle 1\rangle} := p^{-1}(\langle 1\rangle)$ over the pointed finite set $(\{0,1\},0)$ as the *underlying category of the operad* $\mathcal{O}$. We denote its groupoid core by

$$\mathcal{O}^{\simeq} := \left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)$$

and refer to it as the *anima of colors of* $\mathcal{O}$. For $\langle n\rangle \in \mathsf{Fin}_*$, condition 3 guarantees that every object in $\mathcal{O}^{\otimes}_I$ may be uniquely written as a product $\prod_{i\in I} x_i$ for some colors $x_i \in \mathcal{O}^{\simeq}$. We also denote such a product as an unordered tuple $\{x_i\}_{i\in I}$.

Given another color $y \in \mathcal{O}^{\simeq}$, we define the *anima of multimorphisms(or anima of operations) in* $\mathcal{O}$ from $\{x_i\}_{i\in I}$ to y as the anima of morphisms in $\mathcal{O}^{\otimes}$ that map to the active morphism $\langle n\rangle \to \langle 1\rangle$:

$$\begin{tikzcd} \mathcal{O}(\{x_i\}_{i\in I}; y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, y) \arrow[d, "p"] \\ * \arrow[r] & \mathrm{Hom}_{\mathsf{Fin}_*}(\langle n\rangle, \langle 1\rangle). \end{tikzcd}$$

A multimorphism in an operad $\mathcal{O}$ represents an abstract operation with multiple inputs. When interpreted in a symmetric monoidal category $\mathcal{C}$, this corresponds to a morphism of the form $x_1 \otimes \cdots \otimes x_n \to y$.

At the end of this section, we will give the notion *weakly enrichment*.

Definition 2.3.8. Let $\mathcal{C} \to \mathsf{Assoc}$ be a planar operad, and let $\mathcal{M}$ be a category. we say $\mathcal{M}$ *is weakly enriched over* $\mathcal{C}$ if there exists a LM-operad $\mathcal{O}$ such that $\mathcal{O}^{\otimes}_{\mathfrak{a}} \simeq \mathcal{C}$ and $\mathcal{O}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{M}$.

2.4 $\mathcal{O}$-Monoidal Category And $\mathcal{O}$-Algebra

In this section, we will consider $\mathcal{O}$-monoidal category and $\mathcal{O}$-algebra for some algebraic pattern $\mathcal{O}$.

By the discussion above, it is suitable to consider algebraic objects in the Cartesian setting.

Definition 2.4.1. (Cartesian pattern) A *Cartesian pattern* is an algebraic pattern $\mathcal{O}$ equipped with a morphism of algebraic patterns $|-|\colon \mathcal{O} \to \mathsf{Comm}$ such that for every $O \in \mathcal{O}$, the induced functor

$$\mathcal{O}^{\mathrm{el}}_{O/} \to \mathsf{Comm}^{\mathrm{el}}_{|O|/}$$

is an equivalence.

Remark 2.4.2. All of the examples we considered before are Cartesian patterns.

Now, one can define the $\mathcal{O}$-monoidal categories and $\mathcal{O}$-algebras in them.

Definition 2.4.3. Let $\mathcal{O}$ be a Cartesian pattern. An *$\mathcal{O}$-monoidal category* is a coCartesian fibration $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathcal{O}$ whose associated functor $\mathcal{O} \to \mathsf{Cat}$ is an $\mathcal{O}$-Segal object.

Remark 2.4.4. One can find that every $\mathcal{O}$-monoidal category is an $\mathcal{O}$-operad, if we lift the algebraic pattern structure along $p_{\mathcal{C}}$.

Definition 2.4.5. Let $\mathcal{O}$ be a Cartesian pattern and $\mathcal{V}^{\otimes}$, $\mathcal{W}^{\otimes}$ are $\mathcal{O}$-monoidal categories, then a *lax $\mathcal{O}$-monoidal functor* between them is a commutative triangle

$$\begin{tikzcd} \mathcal{V}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{W}^{\otimes} \arrow[dl] \\ & \mathcal{O}. & \end{tikzcd}$$

such that F preserves inert morphisms. Equivalently, a lax $\mathcal{O}$-monoidal functor is a morphism of algebraic patterns over $\mathcal{O}$.

If the functor is a morphism from $\mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ preserves *all* coCartesian morphisms over $\mathcal{O}$, we call it an *$\mathcal{O}$-monoidal functor.*

Example 2.4.6. A *monoidal category* is a Assoc-monoidal category $\mathcal{C}^{\otimes}$. We will denote the image of [1] by $\mathbb{1}_{\mathcal{C}}$ and refer to it as the unit of the monoidal structure. We also let $\mathcal{C}$ denote the fiber of [1]. In this context, $\mathcal{C}^{\otimes}$ will be referred to as the monoidal structure on $\mathcal{C}$. By Example 2.2.4, the active morphism $[2] \to [1]$ in Assoc corresponds to a functor $-\otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we will refer to *tensor product functor.*

Example 2.4.7. A *symmetric monoidal category* is a Comm-monoidal category $\mathcal{C}^{\otimes}$. We will denote the image of $\langle 1 \rangle$ by $\mathbb{1}_{\mathcal{C}}$ refer to it as the unit of the monoidal structure. We also let $\mathcal{C}$ denote the fiber of $\langle 1 \rangle$. In this context, $\mathcal{C}^{\otimes}$ will be referred to as the symmetric monoidal structure on $\mathcal{C}$. By Example 2.2.3, the active morphism $\langle 2 \rangle \to \langle 1 \rangle$ corresponds to a functor $-\otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we will refer to *tensor product functor.*

Let $X\colon \mathsf{LM} \to \mathsf{Cat}$ be a LM-monoidal categories. Using Grothendieck–Lurie construction, one can get a coCartesian fibration $p\colon \mathcal{O} \to \mathsf{LM}$, let $\mathcal{O}_{\mathfrak{a}}$ and $\mathcal{O}_{\mathfrak{m}}$ be the fiber of $\mathfrak{a}$ and $\mathfrak{m}$, respectively. One can imply the existence of the following structures:

- The fiber $\mathcal{O}_{\mathfrak{a}}$ is a monoidal category.
- The fiber $\mathcal{O}_{\mathfrak{m}}$ is a category that is a left module over $\mathcal{O}_{\mathfrak{a}}$, meaning there is an action functor $\otimes\colon \mathcal{O}_{\mathfrak{a}} \times \mathcal{O}_{\mathfrak{m}} \to \mathcal{O}_{\mathfrak{m}}$ which is well-defined up to homotopy.

Definition 2.4.8. Let $\mathcal{C}$ be a monoidal category, we say a category $\mathcal{M}$ is *$\mathcal{C}$-linear* if there exists an LM-Segal object in Cat given by a coCartesian fibration $p\colon \mathcal{O} \to \mathsf{LM}$, satisfy the following two properties:

- $\mathcal{O}_{\mathfrak{a}} \simeq \mathcal{C}$;
- $\mathcal{O}_{\mathfrak{m}} \simeq \mathcal{M}$.

Remark 2.4.9. One can find that if $\mathcal{M}$ is linear over $\mathcal{C}$, then $\mathcal{M}$ is weakly enriched over $\mathcal{C}$.

Example 2.4.10. For every category $\mathcal{C}$, $\mathcal{C}$ can be regarded as a $\mathsf{Fun}(\mathcal{C},\mathcal{C})$-linear category (where the monoidal structure is composition). The action is given by:

$$(F, c) \mapsto F(c).$$

The required higher coherence data is provided by the natural associativity of functor composition.

Example 2.4.11. For any pair of categories $\mathcal{C}$ and $\mathcal{D}$, the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ can be regarded as being left tensored over the monoidal category $\mathsf{Fun}(\mathcal{C},\mathcal{C})$ (where the monoidal structure is composition). The action is given by precomposition:

$$\otimes\colon \mathsf{Fun}(\mathcal{C},\mathcal{C}) \times \mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathsf{Fun}(\mathcal{C},\mathcal{D}), \quad (T,G) \mapsto G \circ T.$$

The required higher coherence data is provided by the natural associativity of functor composition.

Now, we define the algebra object in a $\mathcal{O}$-monoidal category $\mathcal{C}$.

Definition 2.4.12. Let $\mathcal{P}$ and $\mathcal{P}'$ be Cartesian patterns with a morphism $f\colon \mathcal{P} \to \mathcal{P}'$ over Comm and $\mathcal{O}$ be a $\mathcal{P}$-operad. Then an *$\mathcal{O}$-algebra* in $\mathcal{O}$ is a commutative triangle

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "A"] \arrow[dr, "f"'] & & \mathcal{O} \arrow[dl] \\ & \mathcal{P}' & \end{tikzcd}$$

such that A takes inert morphisms in $\mathcal{P}$ to coCartesian morphisms in $\mathcal{O}$. We write $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{O})$ for the full subcategory of $\mathsf{Fun}_{/\mathcal{P}'}(\mathcal{P},\mathcal{O})$ spanned by the $\mathcal{P}$-algebras. If $f = \mathrm{id}_{\mathcal{P}'}$, we will denote the category $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{O})$ by $\mathsf{Alg}_{/\mathcal{P}'}(\mathcal{C})$. If $\mathcal{P}' = \mathsf{Comm}$, then we will omit $\mathcal{P}'$ in $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{O})$.

Moreover, if $\mathcal{O} = \mathcal{C}^{\otimes}$ be a $\mathcal{P}'$-monoidal category, then we will omit the notation $\otimes$ in $\mathsf{Alg}_{\mathcal{P}/\mathcal{P}'}(\mathcal{C}^{\otimes})$.

Example 2.4.13. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal category. Consider the morphism $|-|\colon \mathsf{Trival} \to \mathsf{Comm}$ send $*$ to the elementary object $\langle 1\rangle$. Then, $\mathsf{Trivial}$-algebra in $\mathcal{C}^{\otimes}$ is just an object in $\mathcal{C} \simeq \mathcal{C}^{\otimes}_{\langle 1\rangle}$.

Now, let $\mathcal{P} = \mathsf{Comm}$, we try to describe what an $\mathcal{O}$-algebra is in a symmetric monoidal category $\mathcal{C}$.

Definition 2.4.14. Let $\mathcal{O}$ be an operad. Then

- Comm-algebra in $\mathcal{O}$ is called *commutative algebra* in $\mathcal{O}$, and denote $\mathsf{Alg}_{\mathsf{Comm}}(\mathcal{O})$ by $\mathsf{CAlg}(\mathcal{O})$.
- Assoc-algebra in $\mathcal{O}$ is called *associative algebra* in $\mathcal{O}$, and denote $\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{O})$ by $\mathsf{Alg}(\mathcal{O})$.
- CM-algebra in $\mathcal{O}$ is called *modules over commutative algebra* in $\mathcal{O}$, and denote $\mathsf{Alg}_{\mathsf{CM}}(\mathcal{O})$ by $\mathsf{Mod}(\mathcal{O})$.
- LM-algebra in $\mathcal{O}$ is called *left modules* in $\mathcal{O}$, and denote $\mathsf{Alg}_{\mathsf{LM}}(\mathcal{O})$ (resp. $\mathsf{Alg}_{\mathsf{RM}}(\mathcal{O})$) by $\mathsf{LMod}(\mathcal{O})$.
- BM-algebra in $\mathcal{O}$ is called *bimodule* in $\mathcal{O}$, and denote $\mathsf{Alg}_{\mathsf{BM}}(\mathcal{O})$ by $\mathsf{BMod}(\mathcal{O})$.

Remark 2.4.15. By Remark 2.1.17, one can also use LM to define right modules in $\mathcal{O}$ (in this case, we will use RM to denote LM) and use $\mathsf{RMod}(\mathcal{O}) := \mathsf{Alg}_{\mathsf{RM}}(\mathcal{O})$ to denote the category of right modules.

Definition 2.4.16. Let $\mathcal{C}$ be a monoidal category and let $q\colon \mathcal{O} \to \mathsf{LM}$ exhibit $\mathcal{M}$ weakly enriched over $\mathcal{C}$. We let $\mathsf{LMod}(\mathcal{M})$ denote the category $\mathsf{Alg}_{/\mathsf{LM}}(\mathcal{O})$. We will refer to $\mathsf{LMod}(\mathcal{M})$ as the category of *left module objects of $\mathcal{M}$*. If A is an associative algebra in $\mathcal{C}$, we let $\mathsf{LMod}_A(\mathcal{M})$ denote the pullback

$$\begin{tikzcd} \mathsf{LMod}_A(\mathcal{M}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{LMod}(\mathcal{M}) \arrow[d] \\ * \arrow[r, "\{A\}"] & \mathsf{Alg}(\mathcal{C}). \end{tikzcd}$$

Remark 2.4.17. One can analogously define Mod_A, RMod_A and ${}_B\mathsf{BMod}_A$ as pullbacks.

2.5 Nonunital $\mathbb{A}_n$-Algebras

From Definition 2.1.11, one can find that for $1 \leq n \leq \infty$, there exists a morphism between algebraic patterns

$$\mathbb{A}_n^{\mathrm{un}} \to \mathbb{A}_n,$$

for arbitary (planar) operad $\mathcal{O}$, which will induce a fully faithful embedding

$$\mathsf{Alg}_{\mathbb{A}_n}(\mathcal{O}) \hookrightarrow \mathsf{Alg}_{\mathbb{A}_n^{\mathrm{nu}}}(\mathcal{C}).$$

Therefore, it suffices to study the properties of nonunital $\mathbb{A}_n$-algebras.

Notice that we have a sequence of algebraic patterns

$$\mathbb{A}_0^{\mathrm{nu}} \to \mathbb{A}_1^{\mathrm{nu}} \to \cdots,$$

Notation 2.5.1. Let $\mathcal{O}$ be a planar operad. For $0 \leq n \leq \infty$, we write $\mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_n}(\mathcal{O})$ to denote the category of $\mathbb{A}_n$-algebras.

Example 2.5.2. When $n = 0$, $\mathbb{A}_n^{\mathrm{nu}} \simeq \varnothing$, therefore, one obtain $\mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_0}(\mathcal{O}) \simeq *$.

Example 2.5.3. When $n = 1$, $\mathbb{A}_n^{\mathrm{nu}} \simeq \mathsf{Triv}$, therefore, one obtain $\mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_1}(\mathcal{O}) \simeq \mathcal{O}_{[1]}$.

Now, we will provide another description of $\mathbb{A}_\infty^{\mathrm{nu}}$-pattern.

Definition 2.5.4. The category $\mathbb{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$ can be endowed with algebraic pattern structure inherited from Assoc, and we will denote it by $\mathsf{Assoc}^{\mathrm{nu}}$ and refer to it as *nonunital associative pattern.*

Let $\mathcal{O}$ be a planar operad, then $\mathsf{Assoc}^{\mathrm{nu}}$-algebra in $\mathcal{O}$ will be referred as the *nonunital associative algebra.* We will write $\mathsf{Alg}^{\mathrm{nu}}(\mathcal{O})$ to denote the category of nonunital associative algebras.

Proposition 2.5.5. *Let $\mathcal{O}$ be a planar operad, then the morphism $q\colon \mathbb{A}_\infty^{\mathrm{nu}} \hookrightarrow \mathsf{Assoc}^{\mathrm{nu}}$ induces an equivalence*

$$\mathsf{Alg}^{\mathrm{nu}}(\mathcal{O}) \xrightarrow{\sim} \mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_\infty}(\mathcal{O}) \simeq \varprojlim{}_n \mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_n}(\mathcal{O})$$

Proof. [Lurie, 2017, Proposition 4.1.4.9]. We only need to demonstrate that the left Kan extension along q induces an equivalence

$$q_!\colon \mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_\infty}(\mathcal{O}) \to \mathsf{Alg}^{\mathrm{nu}}(\mathcal{O}),$$

by the pointwise formula for left Kan extensions and Quillen's Theorem A, this reduces to proving that the category

$$(\mathbb{\Delta}_{\mathrm{inj}})_{[0]/} \times_{\mathbb{\Delta}_{\mathrm{inj}}} \mathbb{A}_\infty$$

is weakly contractible. ∎

In fact, we can show that for $n \geq m+2$, the nonunital $\mathbb{A}_n$-algebra in monoidal $(m,1)$-category is precisely the nonunital $\mathbb{A}_\infty$-algebra.

Definition 2.5.6. We say a category $\mathcal{C}$ is an $(n,1)$-*category* if the mapping anima $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ satisfy for $i > n-1$, $\pi_i \,\mathrm{Hom}_{\mathcal{C}}(X,Y)$ vanishes.

Theorem 2.5.7. *Let $\mathcal{C}$ be a monoidal category. Assume that $\mathcal{C}$ is equivalent to an $(m,1)$-category for $m \geq 1$. Then:*

- *The forgetful functor $\mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_\infty}(\mathcal{C}) \to \mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_n}(\mathcal{C})$ is an equivalence for $n \geq m+2$.*
- *The forgetful functor $\mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_\infty}(\mathcal{C}) \to \mathsf{Alg}^{\mathrm{nu}}_{\mathbb{A}_{m+1}}(\mathcal{C})$ is a fully faithful embedding.*

Proof. [Lurie, 2017, Corollary 4.1.6.17] ∎

Remark 2.5.8. One can obtain the $\mathbb{A}_n$-version of Theorem 2.5.7.

2.6 The Boardman–Vogt Tensor Product

In this section, we introduce the Boardman–Vogt tensor product of operads, which will provide a symmetric monoidal structure of Op. This tensor product was first written down by Boardman and Vogt [Boardman and Vogt, 1973].

Construction 2.6.1. We define the *smash product functor* on Fin_* as follows

$$\mu\colon \mathsf{Fin}_* \times \mathsf{Fin}_* \to \mathsf{Fin}_*, \quad (\langle m\rangle, \langle n\rangle) \mapsto \langle mn\rangle.$$

One can find that it is a morphism of patterns

$$\mu\colon \mathsf{Fin}_*^\natural \times \mathsf{Fin}_*^\natural \to \mathsf{Fin}_*^\natural.$$

Moreover, as a morphism $\mathsf{Comm} \times \mathsf{Comm} \to \mathsf{Comm}$, it gives an isomorphism

$$\mathsf{Comm}^{\mathrm{el}}_{\langle n\rangle/} \times \mathsf{Comm}^{\mathrm{el}}_{\langle m\rangle/} \xrightarrow{\sim} \mathsf{Comm}^{\mathrm{el}}_{\langle mn\rangle/}$$

Therefore, the smash product can induce an adjunction

$$\begin{tikzcd} \mathsf{Op}(\mathsf{Fin}_*^\natural \times \mathsf{Fin}_*^\natural) \arrow[r, bend left, "\mu_!"] \arrow[r, phantom, "\perp"] & \mathsf{Op}^{\mathrm{gen}} \arrow[l, bend left, "\mu^*"] \end{tikzcd}.$$

Let $\mathcal{O}$ and $\mathcal{P}$ be operads. Consider the product $\mathcal{O} \times \mathcal{P}$. It can be observed that this product does not constitute an $(\mathsf{Comm} \times \mathsf{Comm})$-operad, but rather an $\left(\mathsf{Fin}_*^\natural \times \mathsf{Fin}_*^\natural\right)$-operad. This conclusion follows from the fact that the elementary object in $\mathsf{Comm} \times \mathsf{Comm}$ is $(\langle 1\rangle, \langle 1\rangle)$.

So, we can consider $\mu_!(\mathcal{O} \times \mathcal{P})$, it is a generalized operad. Applying the functor $\mathrm{L}_{\mathrm{gen}}$ in Remark 2.3.5, we can obtain an operad $\mathrm{L}_{\mathrm{gen}}\,\mu_!(\mathcal{O} \times \mathcal{P})$ we will refer it to the *Boardman–Vogt tensor product* of $\mathcal{O}$ and $\mathcal{P}$. Next, we present another characterization of the Boardman–Vogt tensor product.

Definition 2.6.2. Let $\mathcal{O}, \mathcal{P}$ and $\mathcal{Q}$ be operads, then a *bifunctor* of operads $(\mathcal{O}, \mathcal{P}) \to \mathcal{Q}$ is a commutative square

$$\begin{tikzcd} \mathcal{O} \times \mathcal{P} \arrow[r, "f"] \arrow[d] & \mathcal{Q} \arrow[d] \\ \mathsf{Comm} \times \mathsf{Comm} \arrow[r, "\mu"] & \mathsf{Comm} \end{tikzcd}$$

such that f takes pairs of inert morphisms to inert morphisms in $\mathcal{Q}$. We denote by

$$\mathsf{BiFun}((\mathcal{O}, \mathcal{P}); \mathcal{Q}) \subseteq \mathsf{Fun}_{/\mathsf{Comm}}(\mathcal{O} \times \mathcal{P}, \mathcal{Q})$$

the full subcategory spanned by the bifunctors of operads.

Definition 2.6.3. (Boardman–Vogt Tensor Product) We say f exhibits $\mathcal{Q}$ as a *Boardman–Vogt tensor product* of $\mathcal{O}$ and $\mathcal{P}$ if for every operad $\mathcal{R}$, precomposition with f induces an equivalence

$$f^*\colon \mathsf{Fun}_{\mathsf{Op}}(\mathcal{Q}, \mathcal{R}) \simeq \mathsf{BiFun}((\mathcal{O}, \mathcal{P}); \mathcal{R}).$$

In this case, we will denote the Boardman–Vogt tensor product of $\mathcal{O}$ and $\mathcal{P}$ by $\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{P}$.

Informally speaking, $\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{P}$ is the operad whose colors are pairs (x, y) of colors of $\mathcal{O}$ and $\mathcal{P}$, and whose multimorphisms are 'freely generated' by multimorphisms of the form $\varphi \otimes y\colon \{(x_i, y)\}_{i\in I} \to (z, y)$ for $\varphi \in \mathcal{O}(\{x_i\}_{i\in I}; z)$ and $x \otimes \psi\colon \{(x, y_j)\} \to (x, w)$ for $\psi \in \mathcal{P}(\{y_j\}_{j\in J}; w)$. Therefore, one can find that there exists an equivalence

$$\mathrm{L}_{\mathrm{gen}}(\mu_!(\mathcal{O} \times \mathcal{P})) \simeq \mathcal{O} \otimes_{\mathrm{BV}} \mathcal{P}.$$

Theorem 2.6.4. *The category* Op *has a symmetric monoidal structure given by the Boardman–Vogt tensor product, which preserves colimits in each variable. The unit of* $\mathsf{Op}^{\otimes_{\mathrm{BV}}}$ *is* Fin_*^{inv}.

Proof. [Lurie, 2017, Proposition 2.2.5.13] □

In fact that $\mathsf{Op}^{\otimes_{\mathrm{BV}}}$ is a closed symmetric monoidal category. That is, for operads $\mathcal{O}$ and $\mathcal{P}$, there exists an internal hom $\mathcal{F}un_{\mathsf{Op}}(\mathcal{O}, \mathcal{P})$ which comes with an equivalence

$$\mathsf{Fun}_{\mathsf{Op}}(\mathcal{Q} \otimes_{\mathrm{BV}} \mathcal{P}, \mathcal{P}) \simeq \mathsf{Fun}_{\mathsf{Op}}(\mathcal{Q}, \mathcal{F}un_{\mathsf{Op}}(\mathcal{O}, \mathcal{P}))$$

for every operad $\mathcal{Q}$. Furthermore, if $\mathcal{P} = \mathcal{C}^{\otimes}$ be a symmetric monoidal category, we set

$$\mathcal{A}lg_{\mathcal{O}}(\mathcal{C}) \coloneqq \mathcal{F}un_{\mathsf{Op}}(\mathcal{O}, \mathcal{C}^{\otimes}).$$

so that we get an equivalence

$$\mathsf{Alg}_{\mathcal{P}}(\mathcal{A}lg_{\mathcal{O}}(\mathcal{C})) \simeq \mathsf{Alg}_{\mathcal{P} \otimes_{\mathrm{BV}} \mathcal{O}}(\mathcal{C}).$$

2.7 Lurie–Dunn Additive Theorem

Definition 2.7.1. Fix an integer $k \geq 0$. We let $\square^k = (-1, 1)^k$ to denote an open cube of dimension k. We will say that a map $f\colon \square^k \to \square^k$ is a *rectilinear embedding* if it is given by the formula

$$f(x_1, \cdots, x_k) = (a_1 x_1 + b_1, \cdots, a_k x_k + b_k)$$

for some real constants a_i and b_i, with $a_i \geq 0$. More generally, if S is a finite set, then we will say that a map $\square^k \times S \to \square^k$ is a *rectilinear embedding* if it is an open embedding whose restriction to each connected component of $\square^k \times S$ is rectilinear. Let $\mathrm{Rect}(\square^k \times S, \square^k)$ denote the collection of all rectilinear embedding from $\square^k \times S$ into $\square^k$. We will regard $\mathrm{Rect}(\square^k \times S, \square^k)$ as a open subset of $(\mathbb{R}^{2k})^{|S|}$.

Definition 2.7.2. We define a topological category ${}^t\mathbb{E}_k^{\otimes}$ as follows:

1. The objects of ${}^t\mathbb{E}_k^{\otimes}$ are the objects $\langle n \rangle \in \mathsf{Fin}_*$.
2. Given a pair of objects $\langle m \rangle, \langle n \rangle \in^t \mathbb{E}_k^{\otimes}$, a morphism from $\langle m \rangle$ to $\langle n \rangle$ in ${}^t\mathbb{E}_k^{\otimes}$ consists of the following data:
   - A morphism $\alpha\colon \langle m \rangle \to \langle n \rangle$ in Fin_*.
   - For each $j \in \langle n \rangle^{\circ}$ a rectilinear embedding $\square^k \times \alpha^{-1}(\{j\}) \to \square^k$.
3. For every pair of objects $\langle m \rangle, \langle n \rangle \in^t \mathbb{E}_k^{\otimes}$, we regard $\mathrm{Hom}_{{}^t\mathbb{E}_k^{\otimes}}(\langle n \rangle, \langle m \rangle)$ as endowed with the topology induced by the presentation

$$\mathrm{Hom}_{{}^t\mathbb{E}_k^{\otimes}}(\langle m \rangle, \langle n \rangle) = \coprod_{f\colon \langle m \rangle \to \langle n \rangle} \prod_{1 \leq j \leq n} \mathrm{Rect}\left(\square^k \times f^{-1}(\{j\}), \square^k\right)$$

4. Composition of morphisms in ${}^t\mathbb{E}_k^{\otimes}$ is defined in the obvious way.

We let $\mathbb{E}_k$ to denote the nerve of ${}^t\mathbb{E}_k^{\otimes}$ and refer to it as $\mathbb{E}_k$-*operad* or *operad of little* k-*cubes.*

Proposition 2.7.3. $\mathbb{E}_k \to \mathsf{Comm}$ *exhibits* $\mathbb{E}_k$ *as an operad.*

Proof. [Lurie, 2017, Proposition 5.1.0.3] □

Remark 2.7.4. We have a sequence of operads

$$\mathbb{E}_0 \to \mathbb{E}_1 \to \mathbb{E}_2 \to \cdots .$$

One can then find that Comm is precisely the colimit of this diagram. Therefore, we also use $\mathbb{E}_\infty$ to denote Comm.

In fact, one can find that $\mathbb{E}_1$ is equivalent to Assoc in the higher algebra sense.

Remark 2.7.5. Consider the functor Cut defined in Example 2.3.7. We can regard Assoc as an operad. In fact, we have an equivalence $\mathrm{Cut}(\mathsf{Assoc}) \simeq \mathbb{E}_1$ in the category Op.

Theorem 2.7.6. (Lurie–Dunn Additivity Theorem) *The Boardman–Vogt tensor product $\mathbb{E}_n \otimes \mathbb{E}_m$ is the operad $\mathbb{E}_{n+m}$.*

Proof. [Lurie, 2017, Theorem 5.1.2.2] □

Now, let's discuss the relation between algebraic pattern E_k (Definition 2.1.12) and the operad $\mathbb{E}_k$.

Remark 2.7.7. Since $\mathsf{E}_k \simeq \mathsf{Assoc}^{\times k}$ in $\mathsf{AlgPatt}$. The generalized operad $\mathrm{Cut}(\mathsf{Assoc})^{\times k}$ encodes same data with E_k. Using the k-fold smash product $\mu^k \colon \mathsf{Fin}_*^{\times k} \to \mathsf{Fin}_*$ of Fin_* and Remark 2.3.5, one can obtain the operad $\mathrm{L}_{\mathrm{gen}}\left(\mu_!^k \,\mathrm{Cut}(\mathsf{Assoc})^k\right)$. By Theorem 2.7.6, we obtain that there exists an equivalence

$$\mathrm{L}_{\mathrm{gen}}\left(\mu_!^k \,\mathrm{Cut}(\mathsf{Assoc})^k\right) \simeq \mathbb{E}_k$$

in Op. Therefore, E_k encodes same data with $\mathbb{E}_k$.

The Eckmann–Hilton argument shows that a set (or more generally an object in 1-category) with two compatible associative multiplications is just a Comm-Segal object. In the general category $\mathcal{C}$, the argument above says that the algebras of $\mathbb{E}_n$ are objects with n compatible associative multiplications, so here we get an infinite hierarchy of algebraic structures that lie between associative and commutative algebras. We can interpolate between these situations by considering $(m,1)$-categories.

Corollary 2.7.8. *If $\mathcal{C}$ is a symmetric monoidal $(m,1)$-category, then*

$$\mathsf{Alg}_{\mathbb{E}_k}(\mathcal{C}) \simeq \mathsf{CAlg}(\mathcal{C})$$

whenever $k \geq m+1$.

The key idea of the proof is that since the animae in the operad $\mathbb{E}_k$ become increasingly connected as k increases, we eventually can' t tell them apart from points when we map them into the truncated mapping animae of $\mathcal{C}$.

By Remark 2.7.7, we note another surprisingly useful consequence of additivity

Corollary 2.7.9. *To define a symmetric monoidal n-category, it suffices to define a compatible sequence of Segal E_k-objects in n-categories for all k.*

This is useful to construct symmetric monoidal structures on bordism n-categories, for example. Without Remark 2.7.7, it is not clear that this data should "converge" to a symmetric monoidal structure.

2.8 Some Symmetric Monoidal Structure

In this section, we define and study several symmetric monoidal categories.

2.8.1 Cartesian Symmetric Monoidal Structure

Definition 2.8.1. We say a symmetric monoidal structure $\mathcal{C}^{\otimes} \to \mathsf{Comm}$ on $\mathcal{C}$ is *Cartesian* if it satisfies the following properties:

1. The unit object $\mathbb{1}_{\mathcal{C}}$ is the final.
2. For objects $X, Y \in \mathcal{C}$, the morphisms
$$X \xleftarrow{\sim} X \otimes \mathbb{1}_{\mathcal{C}} \leftarrow X \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes Y \xrightarrow{\sim} Y$$
exhibits $X \otimes Y$ as a product of X and Y in $\mathcal{C}$.

Theorem 2.8.2. *The functor $\mathcal{C}^{\otimes} \mapsto \mathcal{C}$ defines an equivalence between*

- *The category of (small) Cartesian symmetric monoidal categories and symmetric monoidal functors between them.*
- *The category of categories that admits finite products, and finite-product-preserving functors between them.*

Proof. [Lurie, 2017, Corollary 2.4.1.9] □

In particular, for a category $\mathcal{C}$ that admits finite products, there exists an essentially unique Cartesian symmetric monoidal structure on it, which we denote by $\mathcal{C}^{\times} \to \mathsf{Comm}$. Note that when using this notation, we need to equip it with an identification $\mathcal{C}^{\times}_{\langle 1 \rangle} \simeq \mathcal{C}$.

Now, consider the category of categories Cat, by the axiomation of category theory ([Cnossen, 2025, § 1.3.1, Axiom B.3]), we know that Cat admits finite products. So, we can equip Cat with Cartesian symmetric monoidal structure $\mathsf{Cat}^{\times}$.

One can find that for a Cartesian pattern $\mathcal{O}$, $\mathcal{O}$-monoidal category is just an $\mathcal{O}$-object in $\mathsf{Cat}^{\times}$.

2.8.2 Lurie Tensor Product

In this section, we are going to introduce a symmetric monoidal structure of Pr^{L}(or more generally, cocompleted categories). This symmetric monoidal structure is due to [Lurie, 2017, § 4.8.1], and generally all results in this section are due to him. Therefore, this symmetric monoidal structure is also known as the *Lurie tensor product.*

Let $\widehat{\mathsf{Cat}}$ denote the (very big) category of (big) categories. One can equip $\widehat{\mathsf{Cat}}$ with a Cartesian symmetric monoidal structure.

Definition-Proposition 2.8.3. Consider the full subcategory
$$\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes} \subseteq \left(\widehat{\mathsf{Cat}}\right)^{\times}$$
defined as follows:

- An object $(\mathcal{C}_1, \cdots, \mathcal{C}_n)$ is contained in $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$ if and only if each $\mathcal{C}_i$ is presentable.
- A morphism $(\mathcal{C}_1, \cdots, \mathcal{C}_n) \to (\mathcal{D}_1, \cdots, \mathcal{D}_m)$ defined over $\alpha\colon \langle n \rangle \to \langle m \rangle$ is contained in $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$ if and only if for each $j \in \langle n \rangle^{\circ}$, the corresponding functor
$$\prod_{i \in \alpha^{-1}(j)} \mathcal{C}_i \to \mathcal{D}_j$$
preserves colimits in each variable.

Then $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes} \to \mathsf{Comm}$ defines a symmetric monoidal structure on Pr^{L}. We call the multiplication functor

$$- \otimes -\colon \mathsf{Pr}^{\mathrm{L}} \times \mathsf{Pr}^{\mathrm{L}} \to \mathsf{Pr}^{\mathrm{L}}$$

the *Lurie tensor product*.

We can observe the following analogy relationship between the Lurie tensor product and the tensor product of modules.

$\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$	**Linear Algebra**
Presentable category $\mathcal{C}$	Module M
Colimit-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$	Linear map $M \to N$
$F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserving colimits in each variable separately	Bilinear map $M \times N \to P$
Lurie tensor product	Tensor product of modules

Next, we will provide a detailed explanation of this analogy.

Notation 2.8.4. For presentable categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, write $\mathsf{Fun}^{\mathrm{Bil}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \subseteq \mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ be the full subcategory of functor category consisting of all the functors that preserve colimits in each variable.

Now, we can regard the Lurie tensor product of $\mathcal{C}$ and $\mathcal{D}$ by a presentable category $\mathcal{C} \otimes \mathcal{D}$ together with a functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

preserving colimit in each variable separately, such that the precomposition induces an equivalence

$$\mathsf{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathsf{Fun}^{\mathrm{Bil}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

for each presentable category $\mathcal{E}$.

We also see directly that

$$\mathsf{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathsf{Fun}^{\mathrm{Bil}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathsf{Fun}^{\mathrm{L}}(\mathcal{C}, \mathsf{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}))$$

so that the Lurie tensor product exhibits $\mathsf{Fun}^{\mathrm{L}}$ as corrsponding internal Hom.

Since $\mathsf{Fun}^{\mathrm{L}}(\mathsf{Ani}, \mathcal{C}) \simeq \mathcal{C}$, we see that Ani is the unit object of $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$.

Next, we will discuss some properties of the Lurie tensor products.

Proposition 2.8.5. *For small categories $\mathcal{C}_0$ and $\mathcal{D}_0$, we have*

$$\mathsf{PSh}(\mathcal{C}_0 \times \mathcal{D}_0) \simeq \mathsf{PSh}(\mathcal{C}_0) \otimes \mathsf{PSh}(\mathcal{D}_0).$$

Proof. We use the universal property of the Lurie tensor product to show this equivalence. For any category $\mathcal{E}$, we have

$$\mathsf{Fun}^{\mathrm{Bil}}(\mathsf{PSh}(\mathcal{C}_0) \times \mathsf{PSh}(\mathcal{D}_0), \mathcal{E}) \simeq \mathsf{Fun}^{\mathrm{L}}(\mathsf{PSh}(\mathcal{C}_0), \mathsf{Fun}^{\mathrm{L}}(\mathsf{PSh}(\mathcal{D}_0), \mathcal{E}).$$

Since for a category $\mathcal{C}$, $\mathsf{PSh}(\mathcal{C})$ can be regarded as a free cocompleted category on $\mathcal{C}$, we have

$$\mathsf{Fun}^{\mathrm{L}}(\mathsf{PSh}(\mathcal{C}_0), \mathsf{Fun}^{\mathrm{L}}(\mathsf{PSh}(\mathcal{D}_0), \mathcal{E}) \simeq \mathsf{Fun}(\mathcal{C}_0, \mathsf{Fun}(\mathcal{D}_0, \mathcal{E})) \simeq \mathsf{Fun}(\mathcal{C}_0 \times \mathcal{D}_0, \mathcal{E}) \simeq \mathsf{Fun}^{\mathrm{L}}(\mathsf{PSh}(\mathcal{C}_0 \times \mathcal{D}_0), \mathcal{E}).$$

□

To get more properties, we consider the following notion:

Definition 2.8.6. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$ be an adjunction.

- If R is fully faithful, then L is called *left Bousfield localization.*
- If L is fully faithful, then R is called *right Bousfield localization.*

The Bousfield localization is a categorical localization by the following proposition.

Proposition 2.8.7. *Let $L\colon \mathcal{C} \to \mathcal{D}$ be a left Bousfield localization and $W \subseteq \mathsf{Ar}(\mathcal{C})$ be a collection of morphisms that are sent to equivalence by L. Then, for any category $\mathcal{E}$, the precomposition induces an categorical equivalence*

$$\mathsf{Fun}(\mathcal{D}, \mathcal{E}) \to \mathsf{Fun}^{W\text{-}loc}(\mathcal{C}, \mathcal{E}),$$

where $\mathsf{Fun}^{W\text{-}loc}$ denotes the full subcategory spanned by those functors sending W to equivalence. The equivalence also restricts to an equivalence.

$$\mathsf{Fun}^{\mathrm{colim}}(\mathcal{D}, \mathcal{E}) \to \mathsf{Fun}^{W\text{-}loc,\mathrm{colim}}(\mathcal{C}, \mathcal{E}).$$

Proof. [Lurie, 2009, Proposition 5.2.7.12] □

Now, we will show that the Lurie tensor product commutes with Bousfield localization.

Proposition 2.8.8. *Given a left Bousfield localization $\mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}' \subseteq \mathcal{C}$ consisting of the full subcategory of $W_{\mathcal{C}}$-local objects for a collection of morphisms $W_{\mathcal{C}}$, and analogously for $\mathcal{D} \to \mathcal{D}'$, then*

$$\mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D}'$$

is also a left Bousfield localization, where $\mathcal{C}' \otimes \mathcal{D}'$ can explicitly be described as the full subcategory local with respect to $W_{\mathcal{C}} \otimes \mathcal{D}^{\kappa} \cup \mathcal{C}^{\kappa} \otimes W_{\mathcal{D}}$ with κ enough that $\mathcal{C}$ and $\mathcal{D}$ are κ-compactly generated.

Proof. By Proposition 2.8.7, we can obtain

$$\mathsf{Fun}^{\mathrm{L}}(\mathcal{C}' \otimes \mathcal{D}', \mathcal{E}) \simeq \mathsf{Fun}^{\mathrm{L}}(\mathcal{C}', \mathsf{Fun}^{\mathrm{L}}(\mathcal{D}', \mathcal{E})) \simeq \mathsf{Fun}^{W_{\mathcal{C}}\text{-loc,L}}(\mathcal{C}, \mathsf{Fun}^{W_{\mathcal{D}}\text{-loc,L}}(\mathcal{D}, \mathcal{E})).$$

Using uncurrying, one can find that the functor F in $\mathsf{Fun}^{W_{\mathcal{C}}\text{-loc}}(\mathcal{C}, \mathsf{Fun}^{W_{\mathcal{D}}\text{-loc}}(\mathcal{D}, \mathcal{E}))$ can be uncurried as a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that sending both $W_{\mathcal{C}} \otimes \mathcal{D}^{\kappa}$ and $W_{\mathcal{D}} \otimes \mathcal{C}^{\kappa}$ into equivalence. Let $W_{\mathcal{C}\otimes\mathcal{D}}$ denote $W_{\mathcal{C}} \otimes \mathcal{D}^{\kappa} \cup W_{\mathcal{D}} \otimes \mathcal{C}^{\kappa}$, we have

$$\mathsf{Fun}^{W_{\mathcal{C}}\text{-loc}}(\mathcal{C}, \mathsf{Fun}^{W_{\mathcal{D}}\text{-loc,L}}(\mathcal{D}, \mathcal{E})) \simeq \mathsf{Fun}^{W_{\mathcal{C}\otimes\mathcal{D}}\text{-loc,Bil}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

Hence, we obtain the desired result by the universal property. □

Since every presentable category can be written as a Bousfield localization of $\mathsf{PSh}(\mathcal{C}_0)$, the two examples combine to give the existence of tensor products generally. In fact, we can provide a more useful formula.

Proposition 2.8.9. *For presentable categories $\mathcal{C}$ and $\mathcal{D}$, we have*

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathsf{Fun}^{lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

Proof. Write $\mathcal{C}$ and $\mathcal{D}$ as left Bousfield localizations of $\mathsf{PSh}(\mathcal{C}_0)$ and $\mathsf{PSh}(\mathcal{D}_0)$ with respect to some generating equivalences, we have

$$\begin{aligned} \mathsf{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) &\subseteq \mathsf{Fun}^{\lim}(\mathsf{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{Ani})^{\mathrm{op}}, \mathcal{D}) \\ &= \mathsf{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{D}) \\ &\subseteq \mathsf{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{Fun}(\mathcal{D}_0^{\mathrm{op}}, \mathsf{Ani})) \\ &= \mathsf{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathsf{Ani}) = \mathsf{PSh}(\mathcal{C}_0 \times \mathcal{D}_0). \end{aligned}$$

where both inclusions are characterized by locality conditions.

Tracing these through to the rightmost term, one sees that $\mathsf{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ agrees with the full subcategory of $\mathsf{Fun}((\mathcal{C}_0 \times \mathcal{D}_0)^{\mathrm{op}}, \mathsf{Ani})$ on local objects, i.e. agrees with the left Bousfield localization. □

Using the formula, one can obtain the following result.

Proposition 2.8.10. *Let Ani_* be the category of pointed anima. For every presentable category $\mathcal{C}$, we have*

$$\mathsf{Ani}_* \otimes \mathcal{C} \simeq \mathcal{C}_*.$$

Proof. One can regard Ani_* as the Bousfield localization of $\mathsf{Ar}(\mathsf{Ani})$, consisting of the full subcategory of $\mathsf{Ar}(\mathsf{Ani})$ on all arrows whose first entry is $*$(The left adjoint takes $A \to X$ to X/A). This exhibits $\mathsf{Fun}^{\lim}((\mathsf{Ani}_*)^{\mathrm{op}}, \mathcal{C})$ as full subcategory of $\mathsf{Fun}([1], \mathcal{C})$ with objects characterized by some locality condition. Unwrapping things, we exactly find the full subcategory of arrows where the first entry is $*$. □

Using Proposition 2.8.10, one show that Sp is an commutative algebra in $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$.

Proposition 2.8.11.

- *The canonical functor $\Sigma^{\infty}[-]\colon \mathsf{Ani} \to \mathsf{Sp}$ induces an equivalence $\mathsf{Sp} \otimes \mathsf{Sp}$. The inverse makes Sp be a commutative algebra over $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$.*
- *A presentable category $\mathcal{C}$ is stable if and only if the canonical functor*

$$\mathcal{C} \to \mathsf{Sp} \otimes \mathcal{C}$$

is an equivalence. This makes $\mathcal{C}$ into a module over commutative algebra Sp in $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$.

Proof. Since Ani is presentable, Sp can be viewed as the colimit

$$\mathsf{Sp} \simeq \varinjlim \left(\mathsf{Ani}_* \xrightarrow{\Sigma} \mathsf{Ani}_* \xrightarrow{\Sigma} \cdots\right)$$

in Pr^{L}. Now, tensoring Sp with $\mathcal{C}$, we obtain

$$\mathsf{Sp} \otimes \mathcal{C} \simeq \varinjlim \left(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \cdots\right)$$

is the stabilization $\mathsf{Sp}(\mathcal{C}_*)$ of $\mathcal{C}_*$. Therefore $\mathcal{C} \otimes \mathsf{Sp}$ equivalent to $\mathcal{C}$ is equal to say $\mathsf{Sp}(\mathcal{C}) \simeq \mathcal{C}$, and hence $\mathcal{C}$ is stable.

Now use $p\colon \left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes} \to \mathsf{Comm}$ denote the coCartesian fibration which exhibits $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$ be a symmetric monoidal category. Consider the functor

$$\mathsf{Sp}^{\otimes -}\colon \mathsf{Comm} \to \left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$$

that sends $\langle 1 \rangle$ to Sp, $\langle n \rangle$ to $\mathsf{Sp}^{\otimes n}$ and sends the inert morphism $\langle n \rangle \to \langle m \rangle$ to the equivalence $\mathsf{Sp}^{\otimes n} \simeq \mathsf{Sp}^{\otimes m}$. Since equivalence is canonically a p-coCartesian morphism by the universal property of p-coCartesian morphism, we have $\mathsf{Sp}^{\otimes -}$ is a commutative algebra in $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$ by Definition 2.4.12. □

2.8.3 The Tensor Product Of Spectra

Now, we construct the symmetric monoidal structure on Sp.

Definition 2.8.12. Let $\mathcal{C}^{\otimes} \to \mathsf{Comm}$ be a symmetric monoidal category. We say a commutative algebra $A \in \mathsf{CAlg}(\mathcal{C})$ is *idempotent* if $A \otimes A \to A$ is an equivalence. We use $\mathsf{CAlg}^{\mathrm{idem}}(\mathcal{C}) \subseteq \mathsf{CAlg}(\mathcal{C})$ to denote the full subcategory spanned by idempotent commutative algebras in $\mathcal{C}$.

By Proposition 2.8.11, Sp is an idempotent commutative algebra in Pr^{L}.

Notation 2.8.13. Let $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ for the full subcategory of Pr^{L} spanned by stable presentable categories.

Proposition 2.8.14. *Let $\mathcal{C}^{\otimes} \to \mathsf{Comm}$ be a symmetric monoidal category and let $A \in \mathsf{CAlg}^{\mathrm{idem}}(\mathcal{C})$ be an idempotent commutative algebra. Let $L = - \otimes A\colon \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$ be the associated localization functor, so that $L\mathcal{C}^{\otimes}$ inherits the symmetric monoidal structure. Then the forgetful functor $G\colon \mathsf{Mod}_A(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ determines an equivalence*

$$\mathsf{Mod}_A(\mathcal{C})^{\otimes} \simeq (L\mathcal{C})^{\otimes}.$$

Proof. [Lurie, 2017, Proposition 4.8.2.10] □

We now specialize to the case where $\mathcal{C} = \left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$. By Proposition 2.8.14, we obtain a symmetric monoidal structure in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ with unit Sp.

So we can obtain a symmetric monoidal structure on Sp, it is uniquely characterized by the following properties:

1. The bifunctor $\otimes\colon \mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$ preserves small colimits separately in each variable.

2. The unit object of $\mathsf{Sp}^{\otimes}$ is the sphere spectrum $\mathbb{S}$.

We will refer to this symmetric monoidal structure on Sp as the *tensor product of spectra* or *smash product of spectra*.

Remark 2.8.15. One can also use Day convolution to provide the tensor product of Sp.

Chapter 3

Monad

This chapter introduces the concept of a monad in the context of higher categories. Our primary goal is to present and prove the Lurie–Barr–Beck theorem, which provides sufficient conditions for an adjunction to exhibit a category $\mathcal{D}$ as equivalent to a category of modules over a monad.

3.1 Monads

Definition 3.1.1. Let $\mathcal{C}$ be a category, then the category of endofunctors of $\mathcal{C}$ can be equipped with a monoidal structure given by composition. We will denote it by $\mathsf{Fun}(\mathcal{C}, \mathcal{C})^{\circ}$.

1. A *monad* on $\mathcal{C}$ is an algebra object in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})^{\circ}$.
2. If T is a monad on $\mathcal{C}$, by Example 2.4.10 and Definition 2.4.16, one can use $\mathsf{LMod}_T(\mathcal{C})$ denote the category of left *T-modules* in $\mathcal{C}$.

Remark 3.1.2. The category $\mathsf{LMod}_T(\mathcal{C})$ described in Definition 3.1.1 is the higher categorical generalization of the classic category of T-modules (as seen in § 1.1). For a detailed construction, see Lurie's *Higher Algebra* [Lurie, 2017, § 4.2.2].

We also have the dual version of Definition 3.1.1.

Definition 3.1.3. Let $\mathcal{D}$ be a category, then the category of endofunctors of $\mathcal{D}$ can be equipped with a monoidal structure given by composition. We will denote it by $\mathsf{Fun}(\mathcal{D}, \mathcal{D})^{\circ}$.

1. A *comonad* on $\mathcal{D}$ is an algebra object in $(\mathsf{Fun}(\mathcal{D}, \mathcal{D})^{\circ})^{\mathrm{op}}$
2. If U is a comonad on $\mathcal{D}$, by Example 2.4.10 and Definition 2.4.16, one can use $\mathsf{LMod}_U(\mathcal{D}^{\mathrm{op}})$ denote the category of left *U-comodules* in $\mathcal{D}^{\mathrm{op}}$.

The main theorem of this chapter is the following powerful result.

Theorem 3.1.4. (Lurie–Barr–Beck Theorem) *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be a pair of adjoint functors between categories, with $F \dashv G$. Suppose that:*

- *The functor G is conservative (i.e., a morphism f in $\mathcal{D}$ is an equivalence if and only if $G(f)$ is an equivalence in $\mathcal{C}$).*
- *The category $\mathcal{D}$ admits geometric realizations of simplicial objects.*

- *The functor G preserves these geometric realizations.*

Then the endofunctor $T \coloneqq G \circ F$ on $\mathcal{C}$ is a monad, and there is an equivalence of categories $G' \colon \mathcal{D} \xrightarrow{\sim} \mathsf{LMod}_T(\mathcal{C})$ such that the functor G is equivalent to the composition:

$$\mathcal{D} \xrightarrow{G'} \mathsf{LMod}_T(\mathcal{C}) \xrightarrow{U_T} \mathcal{C},$$

where U_T is the forgetful functor.

The proof of Theorem 3.1.4 naturally breaks into two parts:

1. Construct the monad structure on $T = G \circ F$ and the comparison functor $G' \colon \mathcal{D} \to \mathsf{LMod}_T(\mathcal{C})$.
2. Prove that this comparison functor G' is an equivalence of categories.

Unlike the 1-categorical case discussed in § 1.1, part 1 is considerably more difficult in the general-categorical setting. To define an algebra structure on the endofunctor T, it is not enough to provide a single multiplication map $T \circ T \to T$; one must also supply an infinite hierarchy of coherence data, which is difficult to describe explicitly.

Therefore, we will adopt a different strategy. Rather than constructing the monad structure on T directly, we will characterize it by a universal property. Once the monad T and the functor G' have been constructed, we will address part 2 by proving a slightly stronger result that establishes the equivalence.

3.2 Split Simplicial Objects

Notation 3.2.1. We define a category $\Delta_{-\infty}$ as follows:

- The objects of $\Delta_{-\infty}$ are integers $n \geq -1$
- A morphism from $[m]$ to $[n]$ in $\Delta_{-\infty}$ is a nondecreasing function $\alpha \colon [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ satisfying $\alpha(-\infty) = -\infty$.

Remark 3.2.2. One can easily find that we have inclusion

$$\Delta \subseteq \Delta_+ \subseteq \Delta_{-\infty}$$

Definition 3.2.3. Let $\mathcal{C}$ be a category and $X_\bullet \colon \Delta_+^{\mathrm{op}} \to \mathcal{C}$ be an augmented simplicial object of $\mathcal{C}$. We say $X_\bullet$ is *split* if X extends to $\Delta_{-\infty}^{\mathrm{op}} \to \mathcal{C}$.

We will say that a simplicial object $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is *split* if it extends to a split augmented simplicial object.

Given a functor $F \colon \mathcal{C} \to \mathcal{D}$, we will say the (augmented) simplicial object X is *F-spilt* if $F \circ X$ is split.

Remark 3.2.4. According to [Lurie, 2009, Lemma 6.1.3.16], every split augmented simplicial object is a colimit diagram. Moreover, let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor, and let V be a split simplicial object of $\mathcal{C}$. Then $F \circ V$ is a split simplicial object of $\mathcal{D}$, it says that F preserves that colimit.

Construction 3.2.5. (Bar construction) Let A be an associative algebra in monoidal category $\mathcal{C}$, N be a right A module, and M be a left A-module. Then the *two-sided bar construction* for N and M over A, denoted $\mathrm{Bar}_A(N, M)_\bullet$, is a simplicial object in $\mathcal{C}$ described as follows:

- In degree $[n] \in \Delta^{\mathrm{op}}$, it is given by $\mathrm{Bar}_A(N, M)_n \coloneqq N \otimes A^{\otimes n} \otimes M$.
- The face morphism $\mathrm{d}_i \colon \mathrm{Bar}_A(N, M)_n \to \mathrm{Bar}_A(N, M)_{n-1}$ are given by:

  - For $i = 0$, d_i is induced by right action morphism $\mathrm{act}_N\colon A \otimes N \to N$.
  - For $0 < i < n$, d_i is induced by multiplication morphism $m\colon A \otimes A \to A$.
  - For $i = n$, d_n is induced by left action morphism $\mathrm{act}_M\colon A \otimes M \to M$.

- The degeneracy morphism $\mathrm{s}_i\colon \mathrm{Bar}_A(N, M)_n \to \mathrm{Bar}_A(N, M)_{n+1}$ are induced by inserting the unit $e\colon \mathbb{1} \to B$ at the $(i+1)$-th position.

Example 3.2.6. Consider the functor $\Phi\colon \Delta^{\mathrm{op}}_{-\infty} \to \mathsf{LM}$ such that $\Phi(\{-\infty\} \cup [n]) = ([n], 0)$. Note that $\{-\infty\}$ is an initial object of $\Delta_{-\infty}$, there is a canonical tranformation $\Phi \Rightarrow \Phi_0$, where Φ_0 is the constant functor taking the value $\mathfrak{m}$.

Now, let $q\colon \mathcal{O}^{\otimes} \to \mathsf{LM}^{\otimes}$ be the coCartesian fibration corresponding to a category $\mathcal{M}$ linear over a monoidal category $\mathcal{C}$. Let $X\colon \mathsf{LM} \to \mathcal{O}^{\otimes}$ be an object of $\mathsf{LMod}(\mathcal{M})$. We can identify X with a pair (A, M) where A is an associative algebra and M is a left module over A. Then $X \circ \Phi$ defines a functor $\Delta^{\mathrm{op}}_{-\infty} \to \mathcal{O}^{\otimes}$. Since q is a coCartesian fibration, we can lift α to a q-coCartesian natural transformation $\overline{\alpha}\colon (X \circ \Phi) \to X'$ for some $X'\colon \Delta^{\mathrm{op}}_{-\infty} \to \mathcal{M}$. Unwinding the definition, we see that $X' \,|_{\Delta^{\mathrm{op}}}$ is the bar construction $\mathrm{Bar}_A(A, M)_\bullet$. It follows that $\mathrm{Bar}_A(A, M)_\bullet$ is a split simplicial object of $\mathcal{M}$. Moreover, one can find that $X' \,|_{\Delta^{\mathrm{op}}_+}$ is a colimit diagram, which says that $|\,\mathrm{Bar}_A(A, M)_\bullet| \simeq X'(\{-\infty\}) = M$.

Definition 3.2.7. (Relative Tensor Product) Let $\mathcal{C}$ be a monoidal category. Assume that $\mathcal{C}$ admits geometric realizations and the tensor product $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ preserves geometric realizations in both variables. For an associative algebra A, a right A-module N and a left A-module M, their *relative tensor product* is the object of $\mathcal{C}$ given by the geometric realization of their bar construction

$$N \otimes_A M \coloneqq |\,\mathrm{Bar}_A(N, M)_\bullet|$$

Therefore, Example 3.2.6 figure outs that $A \otimes_A M \simeq M$.

The construction $[n] \mapsto [0] \star [n] \simeq [n+1]$ determines a functor $\rho\colon \Delta_+ \to \Delta_+$. For any category $\mathcal{C}$, composition with ρ determines a functor from the category of augmented simplicial objects of $\mathcal{C}$ to itself. We will denote it by T. The natural transformation $\mathrm{id} \Rightarrow T$ determines a natural morphism $TX_\bullet \to X_\bullet$.

Proposition 3.2.8. *Let $\mathcal{C}$ be a category and $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$. The following conditions are equivalent:*

1. *$X_\bullet$ is split;*
2. *The canonical morphism $TX_\bullet \to X_\bullet$ is a section, that is, it admits a right inverse s.*

Proof. [Lurie, 2017, Corollary 4.7.2.9] □

3.3 Endomorphism Categories

Remark *This section provides an informal introduction to the concepts. For a rigorous and detailed treatment, readers should consult Lurie's Higher Algebra [Lurie, 2017, § 4.7.1].*

Let's begin with a familiar example. For an abelian group M, the set $\mathsf{End}(M)$ of group homomorphisms from M to itself forms an associative ring, where addition is pointwise and multiplication is function composition. This ring has a universal property: for any associative ring R, a ring homomorphism $R \to \mathsf{End}(M)$ is equivalent to giving M the structure of an R-module.

We now generalize this by replacing the monoidal category of abelian groups with a general monoidal category $\mathcal{C}$.

Definition 3.3.1. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category and let $M \in \mathcal{C}$ be an object. An *endomorphism object* of M is an object $E \in \mathcal{C}$, equipped with a morphism $a\colon E \otimes M \to M$, that is universal in the following sense: for any other object $C \in \mathcal{C}$, precomposition with a induces an equivalence of mapping spaces:

$$\mathrm{Hom}_{\mathcal{C}}(C, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(C \otimes M, M).$$

To apply this idea to the proof of the Barr–Beck theorem (Part 1), we must generalize further from a single monoidal category to a category that is acted upon by a monoidal category. This notion is captured by the concept of being linear over this monoidal category.

Now, let $\mathcal{M}$ be a category linear over a monoidal category $\mathcal{C}$. For any object $M \in \mathcal{M}$, we can define its endomorphism object.

Definition 3.3.2. Let $\mathcal{M}$ be linear over a monoidal category $\mathcal{C}$, and let $M \in \mathcal{M}$ be an object. The *endomorphism object* of M, denoted $\mathsf{End}(M)$, is an object in $\mathcal{C}$ that represents the functor of actions on M. That is, for every object $C \in \mathcal{C}$, there is a natural equivalence:

$$\mathrm{Hom}_{\mathcal{C}}(C, \mathsf{End}(M)) \simeq \mathrm{Hom}_{\mathcal{M}}(C \otimes M, M).$$

If $\mathsf{End}(M)$ exists, it inherits the structure of an associative algebra in $\mathcal{C}$. Moreover, it is the universal associative algebra acting on M. This is made precise by the following equivalence, where A is any associative algebra object in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathsf{Alg}(\mathcal{C})}(A, \mathsf{End}(M)) \simeq \{A\} \times_{\mathsf{Alg}(\mathcal{C})} \mathsf{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

A key step in constructing $\mathsf{End}(M)$ is to analyze the forgetful functor from the category of algebras acting on M to the category of all algebras.

Proposition 3.3.3. *Let $\mathcal{M}$ be linear over a monoidal category $\mathcal{C}$, and let $M \in \mathcal{M}$. The forgetful functor*

$$U\colon \mathsf{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\} \to \mathsf{Alg}(\mathcal{C})$$

is a right fibration.

Proof. See [Lurie, 2017, Corollary 4.7.1.42]. □

Since U is a right fibration, it has a chance of being representable. If we can find an object $\mathsf{End}(M) \in \mathsf{Alg}(\mathcal{C})$ that represents this fibration, then we have an equivalence

$$\mathsf{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\} \simeq \mathsf{Alg}(\mathcal{C})_{/\mathsf{End}(M)},$$

and we have found our universal endomorphism algebra. The construction of this representing object relies on the technical notion of the "endomorphism category" of M, denoted $\mathcal{C}[M]$. Informally, its objects are pairs $(C, \eta\colon C \otimes M \to M)$ consisting of an object $C \in \mathcal{C}$ and an action map η. The monoidal structure is given by composition of actions:

$$(C, \eta) \otimes (C', \eta') = (C \otimes C', \eta''),$$

where η'' is the composite $C \otimes C' \otimes M \xrightarrow{\mathrm{id} \otimes \eta'} C \otimes M \xrightarrow{\eta} M$.

Proposition 3.3.4. *There is a categorical equivalence*

$$\mathsf{Alg}(\mathcal{C}[M]) \simeq \mathsf{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}.$$

Proof. See [Lurie, 2017, Theorem 4.7.1.34]. □

This equivalence reveals that the endomorphism algebra $\mathsf{End}(M)$ can be constructed in a universal way: it is the image of the *final object* of the monoidal category $\mathcal{C}[M]$ under the forgetful functor $\mathcal{C}[M] \to \mathcal{C}$. The fact that a final object in a monoidal category is automatically an algebra object (by [Lurie, 2017, Corollary 3.2.2.4]) guarantees that $\mathsf{End}(M)$ inherits the desired associative algebra structure.

3.4 Lurie–Barr–Beck Theorem

In this section, we will provide a proof of Theorem 3.1.4. Using Example 2.4.11 to regard $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ as the category linear over $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$.

Lemma 3.4.1. *Let $F \colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G$ with $F \dashv G$ be an adjunction. Let $c \colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ be its counit. Then v induces a morphism*

$$(G \circ F) \circ G \simeq G \circ (F \circ G) \xrightarrow{c} G$$

which exhibits $G \circ F \in \mathsf{Fun}(\mathcal{C}, \mathcal{C})$ as a endomorphism object of G.

Proof. It suffices to show that for every $U \colon \mathcal{C} \to \mathcal{C}$, composition v induces an equivalence $\alpha \colon \mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{C})}(U, G \circ F) \to \mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(U \circ G, G)$. Let u denote the unit of $F \dashv G$. Then u induces the composition

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(U \circ G, G) \to \mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{C})}(U \circ G \circ F, G \circ F) \xrightarrow{- \circ u} \mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{C})}(U, G \circ F).$$

Since u and c are the unit and counit for the same adjunction, β is the inverse of α. $\square$

Definition 3.4.2. Let $G \colon \mathcal{D} \to \mathcal{C}$ be a functor. A *endomorphism monad* consists of a monad $T \in \mathsf{Alg}(\mathsf{Fun}(\mathcal{C}, \mathcal{C})$ together with a left T-module $\overline{G} \in \mathsf{LMod}_T(\mathsf{Fun}(\mathcal{D}, \mathcal{C}))$ whose image in $\mathsf{Fun}(\mathcal{D}, \mathcal{C})$ coincides with G, such that the composition morphism $T \circ G \to G$ exhibits T as a endomorphism object of G.

Combining Lemma 3.4.1 and Proposition 3.3.4, we can give the equivalent description of the endomorphism monad.

Proposition 3.4.3. *Let $G \colon \mathcal{D} \to \mathcal{C}$ be a functor. Assume that there exists a functor $F \colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $u \colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is the unit for an adjunction between G and F. Then:*

1. *There exists an endomorphism monad of G.*
2. *Let T be a monad on $\mathcal{C}$ and $\overline{G} \in \mathsf{LMod}_T(\mathsf{Fun}(\mathcal{D}, \mathcal{C}))$ be a lifting of G. Then $\overline{G}$ exhibits T as an endomorphism monad of G if and only if the composition*

$$T \xrightarrow{\mathrm{id}_T \times u} T \circ G \circ F \xrightarrow{a \times \mathrm{id}_F} G \circ F$$

is an equivalence in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$, where $a \colon T \circ G \to G$ denotes the action of T in G.

Now, suppose that $G \colon \mathcal{D} \to \mathcal{C}$ is a functor between categories which is a left module over a monad $T \in \mathsf{Alg}(\mathsf{Fun}(\mathcal{C}, \mathcal{C})$. The action of T on G determines a functor $G' \colon \mathcal{D} \to \mathsf{LMod}_T(\mathcal{C})$ such that G is obtained by composing G' with the forgetful functor $\mathsf{LMod}_T(\mathcal{C}) \to \mathcal{C}$. Now we finish the Part 1.

Definition 3.4.4. Let $G \colon \mathcal{D} \to \mathcal{C}$ be a functor. Assume that G has a left adjoint F, so by Proposition 3.4.3 G admits an endomorphism monad T. We will say that $\mathcal{D}$ is *monadic over* $\mathcal{C}$ if the induced functor $G' \colon \mathcal{D}' \to \mathsf{LMod}_T(\mathcal{C})$ is an equivalence.

Example 3.4.5. Monoid is monoadic over Set.

We also have the dual version of Definition 3.4.4.

Definition 3.4.6. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor and U be a comonad on $\mathcal{D}$. Then by the dual version of Proposition 3.4.3, one can factor F as a composition $\mathcal{C} \xrightarrow{F'} \mathsf{LMod}_U(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{F''} \mathcal{D}$. We will say that $\mathcal{C}$ is *comonadic over* $\mathcal{D}$ if F' is an equivalence.

Now, we will give a precise version of Theorem 3.1.4.

Theorem 3.4.7. (Lurie–Barr–Beck,precise version) *Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor which admits a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$. The following are equivalent:*

1. *The functor G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.*
2. *There exists a monoidal category $\mathcal{E}$, a left action of $\mathcal{E}$ on $\mathcal{C}$, an associative algebra $T \in \mathsf{Alg}(\mathcal{E})$ and an equivalence $G'\colon \mathcal{D} \to \mathsf{LMod}_T(\mathcal{C})$ such that G is equivalent to the composition*

$$\mathcal{D} \xrightarrow{G'} \mathsf{LMod}_T(\mathcal{C}) \xrightarrow{U_A} \mathcal{C},$$

   where U_A is the forgetful functor.
3. *The functor G satisfies the following conditions:*
   a) *The functor $G\colon \mathcal{D} \to \mathcal{C}$ is conservative; that is, a morphism $f\colon D \to D'$ in $\mathcal{D}$ is an equivalence if and only if $G(f)$ is an equivalence in $\mathcal{C}$.*
   b) *Let V be a simplicial object of $\mathcal{D}$ which is G-split. Then V admits a colimit in $\mathcal{D}$, and that colimit is preserved by G.*

Remark 3.4.8. The Hypotheses 3a) and 3b) can be rephrased as the following single condition:

- Let $V\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{D}$ be a simplicial object of $\mathcal{D}$ which is G-split. Then V has a colimit in $\mathcal{D}$. Moreover, an arbitrary extension $\overline{V}\colon \mathbb{\Delta}^{\mathrm{op},\triangleright} \to \mathcal{D}$ is a colimit diagram if and only if $G \circ \overline{V}$ is a colimit diagram.

To prove Theorem 3.4.7, we need following lemma.

Lemma 3.4.9. *Let $\mathcal{C}$ be a monoidal category, $\mathcal{M}$ be a $\mathcal{C}$-linear category, A be an associative algebra object of $\mathcal{C}$ and $\theta\colon \mathsf{LMod}_A(\mathcal{M}) \to \mathcal{M}$ be the forgetful functor. Then:*

1. *Every θ-split simplicial object of $\mathsf{LMod}_A(\mathcal{M})$ admits a colimit in $\mathsf{LMod}_A(\mathcal{M})$.*
2. *θ preserves colimits of θ-split simplicial objects.*

Proof. 1. By Definition 2.4.16 and Example 2.2.7, one can find that the simplicial object in $\mathsf{LMod}_A(\mathcal{M})$ can be regarded as a bisimplicial object in $\mathcal{M}^{\otimes}$. In order to avoid confusion, we let K to denote the indexing category $\mathbb{\Delta}^{\mathrm{op}}$ of θ-split simplicial objects.

Consider the pullback

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}^{\otimes} \arrow[d] \\ \mathbb{\Delta}^{\mathrm{op}} \arrow[r, "A"] & \mathcal{C}^{\otimes}. \end{tikzcd}$$

p is a locally coCartesian fibration by Lemma B.1.1. Now, let $v\colon K \to \mathsf{LMod}_A(\mathcal{M})$ be a θ-split simplicial object, corresponding to a bisimplicial object $V\colon K \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{N}$.

One can observe that every fiber of p is equivalent to $\mathcal{M}$, and each of the induced morphism $V_{[n]}\colon K \times \{[n]\} \to \mathcal{N}_{[n]}$ can be identified with the composition

$$K \xrightarrow{V} \mathsf{LMod}_A(\mathcal{M}) \xrightarrow{\theta} \mathcal{N}.$$

It follows that each of the simplicial objects $V_{[n]}$ is split. Using [Lurie, 2009, Lemma 6.1.3.16], we deduce that each $V_{[n]}$ admits a colimit $\overline{V}_{[n]}\colon K^{\triangleright} \to \mathcal{N}_{[n]}$ and that these colimits are preserved by each of the associated functors $\mathcal{N}_{[n]} \to \mathcal{M}_{[n]}$. Applying [Lurie, 2009, Proposition 4.3.1.10], we conclude that each $\overline{V}_{[n]}$ is a p-colimit diagram.

Therefore, we have the commutative diagram

$$\begin{tikzcd} K \times \mathbb{\Delta}^{\mathrm{op}} \arrow[r, "V"] \arrow[d, hook] & \mathcal{N} \arrow[d, "p"] \\ K^{\triangleright} \times \mathbb{\Delta}^{\mathrm{op}} \arrow[r] \arrow[ur, dashed, "\overline{V}"] & \mathbb{\Delta}^{\mathrm{op}}. \end{tikzcd}$$

By [Lurie, 2017, Lemma 3.2.2.9], there exists a functor $\overline{V}\colon K^{\triangleright} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{N}$ rendering the above diagram commutative, with the property that for each $[n] \in \mathbb{\Delta}$, $\overline{V}_{[n]}$ is p-colimit diagram. Moreover, the adjoint morphism $\overline{v}\colon K^{\triangleright} \to \mathsf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{N})$ is a $p^{\mathbb{\Delta}^{\mathrm{op}}}$-colimit diagram.

Since $v = \overline{v}\mid_K$ factor through $\mathsf{LMod}_A(\mathcal{M}) \subseteq \mathsf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{N})$, it is easy to see that $\overline{v}$ also factor through $\mathsf{LMod}_A(\mathcal{M})$. It follows that $\overline{v}$ is a colimit of v.

2. It follows from $\theta \circ \overline{v} = \overline{V}\mid_{K \times \{[0]\}}$.

□

We also need the following lemma:

Lemma 3.4.10. *Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor which admits a left adjoint F and let T be an endomorphism monad for G. Assume that*

- *Every G-split simplicial object of $\mathcal{D}$ admits a colimit in $\mathcal{D}$, and this colimit is preserved by G.*

Then $G'\colon \mathcal{D} \to \mathsf{LMod}_T(\mathcal{C})$ admits a fully faithful left adjoint.

Proof. Let $\mathcal{X} \subseteq \mathsf{LMod}_T(\mathcal{C})$ be the full subcategory spanned by those left T-modules $M \in \mathsf{LMod}_T(\mathcal{C})$ such that the presheaf $\mathrm{Hom}_{\mathsf{LMod}_T(\mathcal{C})}(M, G'(-)) \in \mathrm{PSh}(\mathcal{D})$ is corepresentable by $F'(M) \in \mathcal{D}$. In this case, $M \mapsto F'(M)$ determines a functor $F'\colon \mathcal{X} \to \mathcal{D}$, which we can regard as a partially-defined left adjoint to G'.

Now, we try to show that there exists a category equivalence $\mathcal{X} \simeq \mathsf{LMod}_T(\mathcal{C})$. Let $U_T\colon \mathsf{LMod}_T(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. One can find that it admits a left adjoint $F''\colon \mathcal{C} \to \mathsf{LMod}_T(\mathcal{C})$, given informally by $C \mapsto T(C)$. So, we have $U_T \circ G' \simeq G$ by definition of G'.

Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those M such that the unit morphism $u_M\colon U_T(M) \to (U_T G' F')(M) \simeq GF'(M)$ is equivalence. Therefore, for every object $C \in \mathcal{C}$, the object $F''(C)$ belongs to $\mathcal{X}$ and we have canonical equivalence $F(C) \simeq F'F''(C)$.

Therefore, $u_{F''(C)}$ can be identified the morphism

$$T(C) \simeq U_T F''(C) \to U_T G' F' F''(C) \simeq GF(C)$$

which is an equivalence by Proposition 3.4.3. It follows that $\mathcal{X}_0$ contains the essential image of F''.

We want to show that $\mathcal{X}_0 \simeq \mathsf{LMod}_T(\mathcal{C})$(then F is fully faithful left adjoint.). Only need to show the essential surjectivity, let M be an arbitrary object of $\mathsf{LMod}_T(\mathcal{C})$, and $M_\bullet = \mathrm{Bar}_T(T, M)$ denote the simplicial object of $\mathsf{LMod}_T(\mathcal{C})$ given by bar construction(Construction 3.2.5). Then $M \simeq |M_\bullet| \simeq T \otimes_T M$, and $M_\bullet$ is G''-split by Example 3.2.6. Each M_n belongs to the essential image of F'' and hence to $\mathcal{X}_0$, so that $G(F'(M_\bullet)) \simeq G''(M_\bullet)$ is a split simplicial object of $\mathcal{C}$ and therefore $F'(M_\bullet)$ is a G-simplicial object of $\mathcal{D}$. It follows from the assumption that $F'(M_\bullet)$ admits a colimit $D \in \mathcal{D}$, so that $M \in \mathcal{X}$ and $F'(M) \simeq D$. The morphism u_M is given by the composition

$$G''(M) \simeq |G''(M_\bullet)| \simeq |G(F'(M_\bullet))| \to G|F'(M_\bullet)| \simeq GF'(M)$$

is an equivalence, therefore, $\mathcal{X}_0 \simeq \mathsf{LMod}_T(\mathcal{C})$ and hence F' is an fully faithful left adjoint. □

Proof of Theorem 3.4.7. The implication 1. to 2. is obvious (take $\mathcal{E} = \mathsf{End}(\mathcal{C})$).

The implication 2. to 3. follows from Lemma 3.4.9 and Lemma B.1.2.

Now, we try to show 3. implies 1. Assume that $G\colon \mathcal{D} \to \mathcal{C}$ is a functor that admits a left adjoint and therefore an endomorphism monad T, and let $G'\colon \mathcal{D} \to \mathsf{LMod}_T(\mathcal{C})$ be the induced functor. Note that G' is an equivalence if and only if it is conservative and admits a fully faithful left adjoint. The conservativity of G follows from $G = U_A \circ G'$, and the existence of fully faithful left adjoint follows from Lemma 3.4.10. □

In practical applications, we often use the dual version of Theorem 3.1.4.

Corollary 3.4.11. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$. The following are equivalent:*

- *The functor F exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$.*
- *The functor F satisfies the following conditions:*
  - *a') The functor F is conservative.*
  - *b') Let V be a cosimplicial obejct of $\mathcal{C}$ which is F-split, then V admits a limit in $\mathcal{C}$, and that limit is preserved by F.*

3.5 Examples

In this section, we will provide some examples of monads.

First, we will give an observation of Theorem 3.4.7.

Let $\mathcal{C}$ be a category and regarded $\mathcal{C}$ linear over $\mathsf{Fun}(\mathcal{C},\mathcal{C})$. We have commutative diagram

$$\begin{tikzcd}
\mathsf{LMod}(\mathcal{C}) \arrow[rr, "\theta"] \arrow[dr, "p"'] & & \mathcal{C} \times \mathsf{Alg}(\mathsf{End}(\mathcal{C})) \arrow[dl, "p'"] \\
& \mathsf{Alg}(\mathsf{End}(\mathcal{C})), &
\end{tikzcd}$$

where θ is determined by p and the forgetful functor $\mathsf{LMod}(\mathcal{C}) \to \mathcal{C}$.

The functor p' is obviously a Cartesian fibration, and Lemma B.1.2 implies that p is a Cartesian fibration as well. [Lurie, 2017, Proposition 4.2.3.1] shows that θ carries p-Cartesian morphisms to p'-Cartesian morphisms, that is $\theta \in \mathrm{Cart}(\mathsf{Alg}(\mathsf{End}(\mathcal{C}))$. Consequently, θ classfies a natural tranformation of functors $F \Rightarrow F'$, where $F = \mathrm{St}(p)$ (i.e. $F(T) \simeq \mathsf{LMod}_T(\mathcal{C})$ for every monad $T \in \mathsf{Alg}(\mathsf{End}(\mathcal{C}))$) and $F' = \mathrm{St}(p')$ is the constant functor taking the value $\mathcal{C}$.

Therefore, we can identify this transformation with a functor

$$\alpha\colon \mathsf{Alg}(\mathsf{End}(\mathcal{C}))^{\mathrm{op}} \to \mathsf{Cat}_{/\mathcal{C}}.$$

Then, we can interpret Theorem 3.4.7 as describing the essential image of α: namely, a functor $G\colon \mathcal{D} \to \mathcal{C}$ belongs to the essential image of α if and only if G admits a left adjoint satisfying conditions 3a) and 3b) of Theorem 3.4.7. With more effort, one can show that the functor α is fully faithful. In other words, we may identify monads on $\mathcal{C}$ with (certain) categories lying over $\mathcal{C}$.

Example 3.5.1. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor which exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$. Assume that $\mathcal{D}$ and $\mathcal{C}$ admit finite limits. The functor G is left exact, and hence induces a functor

$$g\colon \mathsf{Sp}(\mathcal{D}) \to \mathsf{Sp}(\mathcal{C}).$$

Suppose that g admits a left adjoint f. Then g exhibits $\mathsf{Sp}(\mathcal{D})$ as monadic over $\mathsf{Sp}(\mathcal{C})$. To prove this, it suffices to show that g satisfies conditions 3a) and 3b) of Theorem 3.4.7.

- The conservativity of g follows from G ehibits $\mathcal{D}$ as monadic over $\mathcal{C}$, so G is conservative, and for every morphism $\varphi\colon X \to Y$ in $\mathsf{Sp}(\mathcal{D})$, $g(\varphi)$ can be regarded as the limit of $G(\Omega^{\infty-n}(\varphi))$.

- Let $U_\bullet$ be a g-split simplicial object of $\mathsf{Sp}(\mathcal{D})$, we need to show that $U_\bullet$ admits a colimit in $\mathsf{Sp}(\mathcal{D})$ and this colimit is preserved by g.

  Recall that we can describe the stablization via reduced and excisive functors, that is

  $$\mathsf{Sp}(\mathcal{D}) \simeq \mathrm{Exc}_*(\mathsf{Ani}_*^{\mathrm{fin}}, \mathcal{D}).$$

  So, we can regard $U_\bullet$ as a simplicial object of the category of both reduced and excisive functors. For any finite pointed anima $K \in \mathsf{Ani}_*^{\mathrm{fin}}$, we have $U_\bullet(K)$ is a G-split simplicial objects of $\mathcal{D}$. It follows from Theorem 3.4.7 that $U_\bullet(K)$ admits a colimit $V(K) \in \mathcal{D}$ and this colimit is preserved by G. Therefore, the construction $K \mapsto V(K)$ determines a functor $V\colon \mathsf{Ani}_*^{\mathrm{fin}} \to \mathcal{D}$, we need to show that it is both reduced and excisive. Note that we have the equivalence of functors $G \circ V \simeq |g(U_\bullet)|$, which is reduced and excisive. Hence, we know that V is left exact(so reduced) and excisive by G, which is left exact and conservative. Since V is a colimit of the diagram $U_\bullet$ in $\mathsf{Fun}(\mathsf{Ani}_*^{\mathrm{fin}}, \mathcal{D})$, it is the colimit of $U_\bullet$ in $\mathrm{Exc}_*(\mathsf{Ani}_*^{\mathrm{fin}}, \mathcal{D}) \simeq \mathsf{Sp}(\mathcal{D})$, and this colimit is preserved by g by construction.

Recall Definition 2.4.12

Example 3.5.2. Let $q\colon \mathcal{C} \to \mathcal{O}$ be a coCartesian fibration of operads and let $\mathcal{O}' \to \mathcal{O}$ be a morphism between operads which is essentially surjective. Assume that:

1. For each $X \in \mathcal{O}$, $\mathcal{C}_X$ admits geometric realizations of simplicial objects.
2. For each multimorphism $\alpha \in \mathcal{O}(\{X_i\}_{i\in I}; Y)$, the induced functor $\prod_{i\in I} \mathcal{C}_{X_i} \to \mathcal{C}_Y$ preserves geometric realizations of simplicial objects.
3. The forgetful functor $p\colon \mathsf{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathsf{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ admits a left adjoint.

Then p exhibits $\mathsf{Alg}_{/\mathcal{O}}(\mathcal{C})$ as monadic over $\mathsf{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$.

3.6 Miscellanea

3.6.1 Some Consequences Of The Proof Of Lemma 3.4.9

In this section, we will provide some useful consequences of the proof of Lemma 3.4.9.

Proposition 3.6.1. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ *be a pair of adjoint functors which satisfies conditions 3a) and 3b) of Theorem 3.4.7. For every $D \in \mathcal{D}$, there exists a G-split simplicial object $D_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{D}$ having colimit D, such that each D_n lies in the essential image of F.*

Remark 3.6.2. In the situation of Proposition 3.6.1, the simplicial object $D_\bullet$ can be chosen to depend functionally on D, and for each n, we can choose $C \in \mathcal{C}$ such that $D_n \simeq F(C)$ which depends functorially on D. This follows either the proof of Lemma 3.4.9 or from applying Proposition 3.6.1 to the adjunction

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{D}, \mathcal{C}) \arrow[r, bend left, "F'"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{D}, \mathcal{D}) \arrow[l, bend left, "G'"] \end{tikzcd}$$

We now describe a few applications of Proposition 3.6.1.

Corollary 3.6.3. *Suppose we are given a commutative diagram of categories*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "U"] \arrow[dr, "G"'] & & \mathcal{C}' \arrow[dl, "G'"] \\ & \mathcal{D} & \end{tikzcd}$$

Assume that:

1. *The functors G and G' admit left adjoints F and F'.*

2. *Every G-split simplicial object of $\mathcal{C}$ admits a colimit in $\mathcal{C}$, which is preserved by G.*

3. *Every G'-split simplicial object of $\mathcal{C}'$ admits a colimit in $\mathcal{C}'$, which is preserved by G'.*

4. *G' is conservative.*

5. *For each object $D \in \mathcal{D}$, the unit morphism $u_D \colon D \to GF(D) \simeq G'(UF(D))$ induces an equigalence $\alpha_D \colon F'(D) \to UF(D)$ in $\mathcal{C}'$.*

Then U admits a fully faithful left adjoint. Moreover, U is an equivalence of categories if and only if G is conservative.

Proof. [Lurie, 2017, Corollary 4.7.3.16] □

Chapter 4

Descent

Remark 4.0.1. Recall that in § 1.2, we use "static" to indicate the mathematical object in ordinary mathematics.

Let $f\colon A \to B$ be a homomorphism of static commutative rings, and let M^0 be a static B-module. The classical theory of *descent* is an attempt to answer the following question.

- Under what circumstances can one find a static A-module M and an isomorphism $\eta\colon M \otimes_A B \simeq M^0$?

A choice of such an isomorphism η determines an isomorphism of static $B \otimes_A B$-modules

$$\tau\colon B \otimes_A M^0 \simeq M^0 \otimes_A B$$

(since both sides are canonically isomorphic to $M \otimes_A (B \otimes_A B)$). Moreover, the morphism τ satisfies a "cocycle condition": namely, it renders the diagram

$$\begin{tikzcd} & B \otimes_A M^0 \otimes_A B \arrow[dr, "\tau \otimes \mathrm{id}"] & \\ B \otimes_A B \otimes_A M^0 \arrow[ur, "\mathrm{id} \otimes \tau"] \arrow[rr, "\tau'"] & & M^0 \otimes_A B \otimes_A B \end{tikzcd}$$

commutative, where τ' is given by τ on the outer factors and the identity id_B on the middle factor.

A *descent datum* is a pair (M^0, τ), where M^0 is a static B-module and $\tau\colon B \otimes_A M^0 \to M^0 \otimes_A B$ is an isomorphism of static $B \otimes_A B$-modules making the above diagram commute. The collection of descent data can be organized into a 1-category $\mathsf{Desc}(f)$, and the construction $M \mapsto M \otimes_A B$ determines a functor F from the category of static A-modules $\mathsf{Mod}^{\mathrm{static}}(A)$ into $\mathsf{Desc}(f)$.

In this chapter, we will set up a higher categorical version of the machinery of descent theory. This discussion will require some general observations concerning the functoriality of the formation of adjoint functors.

4.1 Some Tools

In this section, we will introduce some tools that can help us to establish descent theory in higher category theory.

4.1.1 Adjointable Squares

Definition 4.1.1. (Adjointable Squares) Consider a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G^*"] \arrow[d, "F^*"] & \mathcal{D} \arrow[d, "K^*"] \\ \mathcal{C}' \arrow[r, "H^*"] & \mathcal{D}' \end{tikzcd}$$

in Cat.

1. The square is called *(horizonally) left adjointable* if G^* and H^* admit left adjoint $G_\sharp$ amd $H_\sharp$, and the *Beck–Chevalley transformation*

$$\mathrm{BC}_\sharp\colon H_\sharp \circ K^* \Rightarrow H_\sharp \circ K^* \circ G^* \circ G_\sharp \simeq H_\sharp \circ H^* \circ F^* \circ G_\sharp \Rightarrow F^* \circ G_\sharp$$

   is an equivalence.

2. The square is called *(horizonally) right adjointable* if G^* and H^* admit right adjoint G_* and H_*, and the *Beck–Chevalley transformation*

$$\mathrm{BC}_*\colon F^* \circ G_* \Rightarrow H_* \circ H^* \circ F^* \circ G_* \simeq H_* \circ K^* \circ G^* \circ G_* \Rightarrow H_* \circ K^*$$

   is an equivalence.

Remark 4.1.2. The definition above can naturally be extended to an arbitrary 2-category $\mathbb{D}$. And one can find that if we are given a commutative diagram in Cat:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G^*"] \arrow[d, "F^*"] & \mathcal{D} \arrow[d, "K^*"] \\ \mathcal{C}' \arrow[r, "H^*"] & \mathcal{D}' \end{tikzcd}$$

where G^* and H^* admit left adjoints, F^* and K^* admit right adjoints. Then the diagram is left adjointable if and only if

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F^*"] \arrow[d, "G^*"] & \mathcal{C}' \arrow[d, "H^*"] \\ \mathcal{D} \arrow[r, "K^*"] & \mathcal{D}' \end{tikzcd}$$

is right adjointable.

Definition 4.1.3. Let $\mathcal{C}$ be a category. We define subcategories

$$\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}, \mathsf{Cat}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$$

as follows:

1. Let $F\colon \mathcal{C} \to \mathsf{Cat}$ be a functor. Then F belongs to $\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}, \mathsf{Cat})$ if and only if, for every morphism $X \to Y$ in $\mathcal{C}$, the induced functor $F(X) \to F(Y)$ admits a left adjoint.

2. Let $\alpha\colon F \Rightarrow F'$ be a morphism in $\mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$, where F and F' belongs to $\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}, \mathsf{Cat})$. Then α is a morphism in $\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}, \mathsf{Cat})$ if and only if, for every morphism $X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ F'(X) \arrow[r] & F'(Y) \end{tikzcd}$$

   is left adjointable.

If we replace all the "left" in the above statement with "right", we obtain the category $\mathsf{Fun}^{\mathrm{RAd}}(\mathcal{C}, \mathsf{Cat})$.

Proposition 4.1.4. *Let $\mathcal{C}$ be a category. Then:*

1. $\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}, \mathsf{Cat})$ *and* $\mathsf{Fun}^{\mathrm{RAd}}(\mathcal{C}, \mathsf{Cat})$ *are presentable.*

2. *The inclusion* $\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}, \mathsf{Cat}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$ *and* $\mathsf{Fun}^{\mathrm{RAd}}(\mathcal{C}, \mathsf{Cat}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathsf{Cat})$ *admit left adjoints; in particular, they preserves small limits.*

3. *There is a canonical categorical equivalence*

$$\mathsf{Fun}^{\mathrm{LAd}}(\mathcal{C}^{\mathrm{op}}, \mathsf{Cat}) \simeq \mathsf{Fun}^{\mathrm{RAd}}(\mathcal{C}, \mathsf{Cat})$$

Proof. [Lurie, 2017, Corollary 4.7.4.18] □

4.1.2 Čech Nerve

Definition 4.1.5. (Čech Nerve) Let $\mathcal{C}$ be a category that admits finite product, the functor $\mathrm{ev}_{[0]} \colon \mathsf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$, $X_\bullet \mapsto X_0$ admits a right adjoint

$$\check{C}_\bullet \colon \mathcal{C} \to \mathsf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$$

given by right Kan extension along the functor $i \colon [0] \hookrightarrow \mathbb{\Delta}^{\mathrm{op}}$. We refer to the simplicial anima $\check{C}_\bullet(X)$ as the *Čech nerve* of X.

Remark 4.1.6. Using the pointwise formula of right Kan extensions, we may concretely describe $\check{C}(X)$ by $\check{C}(X)_n$ satisfying the following property:

- For $0 \leq i \leq n$, let $\rho_i \colon \check{C}_n(X) \to \check{C}_0(X)$ be the morphism in $\mathcal{C}$ induced by $[0] \simeq \{i\} \subseteq [n]$. Then the morphisms $\{\rho_i\}_{0 \leq i \leq n}$ exhibit $\check{C}_n(X)$ as a product of $(n+1)$-copies of $\check{C}_0(X) = X$.

Now, we will consider the relative version of Definition 4.1.5.

Definition 4.1.7. (Čech Nerve,Relative version) Let $\mathcal{C}$ be a category that admits pullbacks and $f \colon X \to Y$ be a morphism in $\mathcal{C}$. Then, f can be regarded as an object in $\mathcal{C}_{/Y}$. The *Čech nerve* $\check{C}_\bullet(X/Y)$ of f is the Čech nerve of f in $\mathcal{C}_{/Y}$.

Remark 4.1.8. Using Remark 4.1.6, one can concretely describe $\check{C}_\bullet(X/Y)$ by $\check{C}_n(X/Y)$ satisfying the following property:

- For $0 \leq i \leq n$, let $\rho_i \colon \check{C}_n(X/Y) \to \check{C}_0(X/Y) = X$ be the morphism in $\mathcal{C}$ induced by $[0] \simeq \{i\} \subseteq [n]$. Then the morphisms $\{\rho_i\}_{0 \leq i \leq n}$ exhibit $\check{C}_n(X/Y)$ as an iterated fiber product

$$X \times_Y \times X \times_Y \cdots \times_Y X$$

(where the factor X appears $n+1$ times).

Remark 4.1.9. One can also regard $\check{C}_\bullet(X/Y)$ as an augmented simplicial object of $\mathcal{C}$ with $\check{C}_{-1}(X/Y) = Y$.

4.2 Descent And Lurie–Beck–Chevalley Condition

Now, let's take a fresh look at the issue we discussed at the very beginning of this chapter. Let $f \colon A \to B$ be a ring homomorphism. Taking the Čech nerve of f in $(\mathsf{CRing}^{\mathsf{static}})^{\mathrm{op}}$(Since tensor product is pushout), we obtain a cosimplicial static commutative ring $B^\bullet$, where B^n denotes the n-th tensor power of B over A.

The category $\mathsf{Desc}(f)$ now can be identified with the category of cosimplicial modules $M^\bullet$ over $B^\bullet$ satisfying the requirement that for every $[m] \to [n]$ in $\mathbb{\Delta}$, the induced morphism $M^m \otimes_{B^m} B^n \to M^n$ is an isomorphism. Put another way, $\mathsf{Desc}(f)^{\mathrm{op}}$ can be identified with the limit of cosimplicial category $\mathcal{C}^\bullet$, where each $\mathcal{C}^n$ is given by the opposite of the category of modules over the static commutative ring B^n. In good cases(for example, if f is faithfully flat), one can show that this limit is equivalent to the opposite of the category of A-modules. Next, we will develop some general tools for proving results of this kind.

Very broadly, we can state our main problem as follows: given a cosimplicial object $\mathcal{C}^\bullet$ in Cat, we would like to understand the totalization $\varprojlim \mathcal{C}^\bullet$. There is an evident forgetful functor $\varprojlim \mathcal{C}^\bullet \to \mathcal{C}^0$. By analogy with the situation considered above, it is natural to expect that an object of $\varprojlim \mathcal{C}^\bullet$ can be identified with an object of $\mathcal{C}^0$ together with some sort of "descent data".

One way to articulate this idea precisely is to show that the forgetful functor is $\varprojlim \mathcal{C}^\bullet \to \mathcal{C}^0$ is *monadic*(Definition 3.4.4): that is, it exhibits $\varprojlim \mathcal{C}^\bullet$ as the category $\mathsf{LMod}_T(\mathcal{C}^0)$ for some monad T on $\mathcal{C}^0$.

We begin with a very general result about the limit of a diagram of categories.

Proposition 4.2.1. *Let $\mathcal{I}$ be a small category and let $q\colon \mathcal{I} \to \mathsf{Cat}$ be a diagram with limit $\mathcal{C} \in \mathsf{Cat}$. Let $0 \in \mathcal{I}$ be an object such that $\mathcal{C}^0 = q(0)$. Suppose that the following conditions are satisfied:*

1. *The canonical functor $G\colon \mathcal{C} \to \mathcal{C}^0$ admits a left adjoint F.*
2. *For every $I \in \mathcal{I}$, there exists a morphism $0 \to I$ in $\mathcal{I}$.*

Then, the adjoint functor $\begin{tikzcd} \mathcal{C}^0 \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$ *satisfy the conditions 3a) and 3b) of Theorem 3.4.7, so that $\mathcal{C}$ is equivalent to the category of left modules $\mathsf{LMod}_T(\mathcal{C}^0)$ for the induced monad $T \simeq G \circ F$ on $\mathcal{C}^0$.*

Proof. Let's verify G satisfy conditions 3a) and 3b) of Theorem 3.4.7.

1. We first show that G is consevative. Let $\alpha\colon X \to Y$ be a morphism in $\mathcal{C}$ such that $G(\alpha)$ is an equivalence. Since for every $I \in \mathcal{I}$, there exists a functor $H_I\colon q(0) \to q(I)$. Therefore, $H(\alpha)$ is an equivalence for all H_J, and hence α is an equivalence.

2. Now, suppose that $X_\bullet$ is a G-split simplicial object of $\mathcal{C}$. We want to show that $X_\bullet$ admits a colimit in $\mathcal{C}$ and the colimit is preserved by G. Using the straightening theorem, we get a coCartesian fibration $p\colon \mathcal{U} \to \mathcal{I}$. Using $\mathsf{Fun}_{\mathcal{I}}(\mathcal{I},\mathcal{U})$ to denote the pullback

$$\begin{tikzcd} \mathsf{Fun}_{\mathcal{I}}(\mathcal{I},\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}(\mathcal{I},\mathcal{U}) \arrow[d, "p_*"] \\ * \arrow[r, "\{\mathrm{id}_{\mathcal{I}}\}"] & \mathsf{Fun}(\mathcal{I},\mathcal{I}), \end{tikzcd}$$

and let

$$\mathsf{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I},\mathcal{U}) \subseteq \mathsf{Fun}_{\mathcal{I}}(\mathcal{I},\mathcal{U})$$

be the full subcategory spanned by those $\mathcal{I} \to \mathcal{U}$ such that all morphisms in $\mathcal{I}$ are sent to p-coCartesian morphisms. By [Cnossen, 2025, Proposition 21.5.2], we get category equivalence

$$\mathcal{C} \simeq \mathsf{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I},\mathcal{U}).$$

Therefore, using uncurrying, one can identify $X_\bullet$ by the functor

$$X\colon \mathcal{I} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{U}.$$

We admit the following fact:

(*) There exists a functor $\overline{X} \in \mathsf{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I} \times \mathbb{\Delta}_+^{\mathrm{op}}, \mathcal{U})$ which is a p-left Kan extension of X.

Let $\overline{X}$ as in (*), so we can identify $\overline{X}$ with an augmented simplicial object of $\mathsf{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I}, \mathcal{U})$. It follows from [Lurie, 2017, Lemma 3.2.2.9] that this augmented object is colimit diagram which G obviously preserves.

□

So, let $\mathcal{I} = \mathbb{\Delta}$ be the simplex category, so that $q\colon \mathcal{I} \to \mathsf{Cat}$ can be identified with a cosimplicial object $\mathcal{C}^\bullet$. If $\mathcal{C}^\bullet$ satisfy the conditions of Proposition 4.2.1, then we have $\varprojlim \mathcal{C}^\bullet \simeq \mathsf{LMod}_T(\mathcal{C}^0)$. Now, we want to make some mild assumptions on $\mathcal{C}^\bullet$ to obtain an explicit description of the monad T.

Theorem 4.2.2. (Lurie–Beck–Chevalley) *Let $\mathcal{C}^\bullet\colon \mathbb{\Delta} \to \mathsf{Cat}$ be a cosimplicial object of categories which satisfies the following property:*

(*) Let d^i denote the i-th coface morphism of the cosimplicial object. For every $\alpha\colon [m] \to [n]$ in $\mathbb{\Delta}$, the induced diagram

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "\mathrm{d}^{0*}"] \arrow[d] & \mathcal{C}^{m+1} \arrow[d] \\ \mathcal{C}^n \arrow[r, "\mathrm{d}^{0*}"] & \mathcal{C}^{n+1} \end{tikzcd}$$

is left adjointable. In particular, each coface morphism $\mathrm{d}^{0*}\colon \mathcal{C}^n \to \mathcal{C}^{n+1}$ admits a left adjoint $d^0_\sharp\colon \mathcal{C}^{n+1} \to \mathcal{C}^n$.

Let $\mathcal{C} = \varprojlim \mathcal{C}^\bullet$ be a totalization of $\mathcal{C}^\bullet$. Then:

1. *The forgetful functor $G\colon \mathcal{C} \to \mathcal{C}^0$ admits a left adjoint F.*

2. *The diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "G"] & \mathcal{C}^0 \arrow[d, "\mathrm{d}^{1*}"] \\ \mathcal{C}^0 \arrow[r, "\mathrm{d}^{0*}"] & \mathcal{C}^1 \end{tikzcd}$$

is left adjointable. That is, the canonical natural transformation $\mathrm{d}^0_\sharp \circ \mathrm{d}^{1*} \xRightarrow{\sim} G \circ F$ is an equivalence of functors from $\mathcal{C}^0$ to itself.

3. *The adjunction $\begin{tikzcd} \mathcal{C}^0 \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "G"] \end{tikzcd}$ satisfy conditions 3a) and 3b) of Theorem 3.4.7.*

Proof. Now, let $\overline{\mathcal{C}}^\bullet\colon \mathbb{\Delta}_+ \to \mathsf{Cat}$ be the limit of the diagram $\mathcal{C}^\bullet$. Let $T\colon \mathbb{\Delta}_+ \times [1] \to \mathbb{\Delta}_+$ be the functor given by the formula

$$T([n], i) = \begin{cases} [n], & \text{if} \quad i = 0 \\ [0] \star [n] \simeq [n+1], & \text{if} \quad i = 1. \end{cases}$$

Let's regard the composite functor

$$H\colon \mathbb{\Delta}_+ \times [1] \xrightarrow{T} \mathbb{\Delta}_+ \xrightarrow{\overline{\mathcal{C}}^\bullet} \mathsf{Cat}$$

as an augmented cosimplicial object $\overline{X}^\bullet$ of the category $\mathsf{Fun}([1], \mathsf{Cat})$. This is equivalent to the requirement that for $i \in \{0,1\}$, the restriction $H\mid_{\mathbb{\Delta}_+ \times \{i\}}$ is a limit diagram. For $i = 0$, this follows from our construction of $\overline{\mathcal{C}}^\bullet$, and for $i = 1$, it follows from the observation that $H\mid_{\mathbb{\Delta}_+ \times \{1\}}$ is split.

Let $X^\bullet = \overline{X}^\bullet$ be the underlying cosimplicial object of $\mathsf{Fun}([1], \mathsf{Cat})$. Condition (*) is equivalent to the requirement that $X^\bullet$ is a cosimplcial object of $\mathsf{Fun}^{\mathrm{LAd}}([1], \mathsf{Cat})$. It follows from Proposition 4.1.4 that $\overline{X}^\bullet$ is an augmented cosimplicial object of $\mathsf{Fun}^{\mathrm{LAd}}([1], \mathsf{Cat})$. Since $\overline{X}^{-1}$ is an object of $\mathsf{Fun}^{\mathrm{LAd}}([1], \mathsf{Cat})$, we deduce that the forgetful functor $G\colon \mathcal{C} \simeq \overline{\mathcal{C}}^{-1} \to \mathcal{C}^0$ admits a left adjoint. This shows 1. One can find that 2 is equivalent to the requirement that the morphism $X^{-1} \to X^0$ is a morphism of $\mathsf{Fun}^{\mathrm{LAd}}([1], \mathsf{Cat})$. And 3 follws immediately from Proposition 4.2.1. $\square$

Now, we can give a simple criterion for a functor $\mathcal{C}^{-1} \to \varprojlim \mathcal{C}^\bullet$ to be an equivalence.

Corollary 4.2.3. (Lurie–Beck–Chevalley) *Let $\mathcal{C}^\bullet\colon \Delta_+ \to \mathsf{Cat}$ be an augmented cosimplical object of Cat, and set $\mathcal{C} = \mathcal{C}^{-1}$. Let $G\colon \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that:*

1. *$\mathcal{C}^{-1}$ admits geometric realizations of G-split simplical objects, and those geometric realizations are preserved by G.*

2. *For every morphism $\alpha\colon [m] \to [n]$ in Δ_+, the diagram*

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "\mathrm{d}^{0*}"] \arrow[d] & \mathcal{C}^{m+1} \arrow[d] \\ \mathcal{C}^n \arrow[r, "\mathrm{d}^{0*}"] & \mathcal{C}^{n+1} \end{tikzcd}$$

is left adjointable.

Then, the canonical functor $\theta\colon \mathcal{C} \to \varprojlim_{n\in\Delta} \mathcal{C}^n$ admits a fully faithful left adjoint. If G is conservative, then θ is an equivalence.

Remark 4.2.4. Applying Corollary 3.4.11, i. If we replace all the "left" in the above statement with "right", "simplicial" with "cosimplicial" and "geometric reailzation" with "totalization", we get the dual version of Corollary 4.2.3.

Proof. Let $\mathcal{D}^\bullet\colon \Delta_+ \to \mathsf{Cat}$ be the limit of $\mathcal{C}^\bullet\,|_\Delta$, so that we have a morphsim $\alpha\colon \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ in $\mathsf{Fun}(\Delta, \mathsf{Cat})$ which induces the identity morphism from $\mathcal{C}^n = \mathcal{D}^n$ to itself for $n \geq 0$. Using condition 2 of Theorem 4.2.2, we conclude that the canonical morphism $G'\colon \mathcal{D}^{-1} \to \mathcal{D}^0 \simeq \mathcal{C}^0$ admits a left adjoint F' which is satisfy the conditions 3a) and 3b) of Theorem 3.4.7.

Now, applying Corollary 3.6.3 to the diagram

$$\begin{tikzcd} \mathcal{C}^{-1} \arrow[rr, "U"] \arrow[dr, "G"'] & & \mathcal{D}^{-1} \arrow[dl, "G'"] \\ & \mathcal{C}^0, & \end{tikzcd}$$

we are reduced to showing that the canonical transformation $G'F' \Rightarrow GF$ is an equivalence. It is clear from Theorem 4.2.2 allows us to identify both functors with the composition

$$\mathcal{C}^0 \xrightarrow{\mathrm{d}^{1*}} \mathcal{C}^1 \xrightarrow{\mathrm{d}^0_\sharp} \mathcal{C}^0.$$

$\square$

4.3 Application: Derived Categories Satisfy Descent

In this section, we will provide an application of the Corollary 4.2.3 on derived categories of Grothendieck abelian categories.

Notation 4.3.1. Let $\mathcal{A}$ be an additive category. We let $\mathrm{C}^+(\mathcal{A})$ denote the full subcategory of $\mathrm{C}(\mathcal{A})$ spanned by those bouded below complexes M^*, i.e. $M^n \simeq 0$ for $n \ll 0$.

Definition 4.3.2. Let $\mathcal{A}$ be an abelian category with enough injective objects, let $\mathcal{A}_{\mathrm{inj}}$ denote the full subcategory spanned by injective objects. Then the *(bounded below) derived category* of $\mathcal{A}$ is the localization

$$\mathsf{D}^+(\mathcal{A}) \coloneqq \mathrm{K}(\mathcal{A}_{\mathrm{inj}}) = \mathrm{C}^+(\mathcal{A}_{\mathrm{inj}})[\mathrm{HEq}^{-1}],$$

where HEq is the collection of homotopy equivalences of complexes in $\mathrm{Ch}^+(\mathcal{A})$.

Remark 4.3.3. Our approach to define the derived category is called *derived category via injective resolution*. One can also use the approach [Cnossen, 2025, Proposition 4.5.6] to define the derived category $\mathsf{D}(\mathcal{A})$ as the localization at quasi-isomorphisms in $\mathrm{C}(\mathcal{A})$.

In fact, the quasi-isomorphisms are precisely the homotopy equivalences of complexes in $\mathrm{C}^+(\mathcal{A}_{\mathrm{inj}})$. $\mathsf{D}^+(\mathcal{A})$ in Definition 4.3.2 can be identified as the full subcategory of $\mathsf{D}(\mathcal{A})$ consisting of complexes X such that $\mathrm{H}^n(X) = 0$ for $n \ll 0$.

To show that derived categories satisfy descent, we need to show that the construction $\mathcal{A} \mapsto \mathsf{D}^+(\mathcal{A})$ is functorial in an appropriate sense at first. Recall that we denote the category of 1-categories by $\mathsf{Cat}_{(1)}$.

Lemma 4.3.4. *Let $\mathsf{Cat}_{(1)}^{\mathrm{Ab\,I,ex}} \subseteq \mathsf{Cat}_{(1)}$ be the subcategory spanned by those abelian categories with enough injective objects and exact functors between them. Then the construction $\mathcal{A} \mapsto \mathsf{D}^+(\mathcal{A})$ defines a functor*

$$\mathsf{D}^+ \colon \mathsf{Cat}_{(1)}^{\mathrm{Ab\,I,ex}} \to \mathsf{Cat}, \quad \mathcal{A} \mapsto \mathsf{D}^+(\mathcal{A}).$$

Proof. Let us denote $\mathsf{Cat}^{\mathrm{st,tex}}$ the category of stable categories equipped with the t-structures whose heart has enough injective objects, where the morphisms are t-exact functors.

Therefore, there exists a canonical functor

$$(-)^{\heartsuit} \colon \mathsf{Cat}^{\mathrm{st,tex}} \to \mathsf{Cat}_{(1)}^{\mathrm{Ab\,I,ex}}, \quad \mathcal{C} \mapsto \mathcal{C}^{\heartsuit}.$$

Now, let $\mathcal{X} \subseteq \mathsf{Cat}^{\mathrm{st,tex}}$ be the full subcategory spanned by those categories which are equivalent to $\mathsf{D}^+(\mathcal{A})$ for some $\mathcal{A} \in \mathsf{Cat}_{(1)}^{\mathrm{Ab\,I,ex}}$. By the universal property of derived categories([Lurie, 2017, Theorem 1.3.3.2]), the restriction of the functor $(-)^{\heartsuit}$ to $\mathcal{X}$ is fully faithful, hence an equivalence. One can find that the inverse functor is precisely D^+ we need. ☐

Recall that for every abelian category $\mathcal{A}$ with enough injective objects, [Lurie, 2017, Proposition 1.3.2.19] construct the canonical t-structure $(\mathsf{D}^+(\mathcal{A})^{\leq 0}, \mathsf{D}^+(\mathcal{A})^{\geq 0})$ on $\mathsf{D}^+(\mathcal{A})$, where

- the category $\mathsf{D}^+(\mathcal{A})^{\leq 0}$ is the full subcategory spanned by those complexes M^* with $\mathrm{H}^n(M) \simeq 0$ for $n > 0$.
- the category $\mathsf{D}^+(\mathcal{A})^{\geq 0}$ is the full subcategory spanned by those complexes M^* with $\mathrm{H}^n(M) \simeq 0$ for $n < 0$.

Now, we will introduce our main theorem in this section.

Theorem 4.3.5. (Mann) *Let $\mathcal{A}^\bullet$ be a cosimplicial object of* Cat *with $\mathcal{A} = \varprojlim_{n \in \Delta} \mathcal{A}^n$ in* Cat. *Assume that the following conditions hold:*

1. *All $\mathcal{A}^n$ are Grothendieck abelian categories.*
2. *For every $\alpha \colon [n] \to [m]$ in Δ, the transition functor $\alpha^* \colon \mathcal{A}^n \to \mathcal{A}^m$ is exact and admits a right adjoint α_*.*

3. *For all* $\alpha\colon [n] \to [m]$ *in* Δ *the diagram*

$$\begin{tikzcd} \mathsf{D}(\mathcal{A}^m) \arrow[r, "\mathrm{R}\,\mathrm{d}^{0*}"] \arrow[d, "\alpha^*"] & \mathsf{D}(\mathcal{A}^{m+1}) \arrow[d, "\alpha'^*"] \\ \mathsf{D}(\mathcal{A}^n) \arrow[r, "\mathrm{R}\,\mathrm{d}^{0*}"] & \mathsf{D}(\mathcal{A}^{n+1}) \end{tikzcd}$$

is right adjointable, where $\alpha' = \mathrm{id}_{[0]} \star \alpha$.

Then:

a) *The category* $\mathcal{A}$ *is Grothendieck abelian and admits the following description: An object of* $\mathcal{A}$ *is a pair* (X, α)*, where* X *is an object of* $\mathcal{A}^0$ *and* $\alpha\colon \mathrm{d}^{0*} X \xrightarrow{\sim} \mathrm{d}^{1*} X$ *is an isomorphism in* $\mathcal{A}^1$ *such that the following diagram in* $\mathcal{A}^2$ *commutes:*

$$\begin{tikzcd} & \mathrm{d}^{0*}\,\mathrm{d}^{1*} X \arrow[r, equal] & \mathrm{d}^{2*}\,\mathrm{d}^{0*} X \arrow[dr, "\mathrm{d}^{2*}\alpha"] & \\ \mathrm{d}^{0*}\,\mathrm{d}^{0*} X \arrow[ur, "\mathrm{d}^{0*}\alpha"] \arrow[dr, equal] & & & \mathrm{d}^{2*}\,\mathrm{d}^{1*} X \\ & \mathrm{d}^{1*}\,\mathrm{d}^{0*} X \arrow[r, "\mathrm{d}^{1*}\alpha"] & \mathrm{d}^{1*}\,\mathrm{d}^{1*} X \arrow[ur, equal] & \end{tikzcd}$$

A morphism $(X_1, \alpha_1) \to (X_2, \alpha_2)$ *in* $\mathcal{A}$ *is a morphism* $f\colon X_1 \to X_2$ *in* $\mathcal{A}^0$ *such that* $\mathrm{d}^{1*} f \circ \alpha_1 \simeq \alpha_2 \circ \mathrm{d}^{0*} f$

b) *Suppose that* $\mathrm{d}^{1*}\colon \mathcal{A}^0 \to \mathcal{A}^1$ *sends injective object to* d^0_**-acyclic objects. Then there is a natural equivalence of categories*

$$\mathsf{D}^+(\mathcal{A}) \xrightarrow{\sim} \varprojlim_{n\in\Delta} \mathsf{D}^+(\mathcal{A}^n).$$

Moreover, if the categories $\mathsf{D}(\mathcal{A})$ *and* $\mathsf{D}(\mathcal{A}^n)$ *are left-complete, then the above equivalence can extend to an equivalence*

$$\mathsf{D}(\mathcal{A}) \xrightarrow{\sim} \varprojlim_{n\in\Delta} \mathsf{D}(\mathcal{A}^n)$$

by [Lurie, 2017, Remark 1.2.1.18].

Proof. a) Since $\mathsf{Cat}_{(1)}$ is a $(2,1)$-category. By [Lurie, 2017, Lemma 1.3.3.10], we have an equivalence

$$\mathcal{A} \simeq \varprojlim_{n\in\Delta^{\leq 2}} \mathcal{A}^n.$$

So, we get the explicit description of $\mathcal{A}$ and $\mathcal{A}$ is an abelian category. Since the transition maps are exact and commute with colimits, $\mathcal{A}$ satisfies the AB5(colimits exist and filtered colimits are exact). By [BEKE, 2000, Proposition 3.10], we know that Grothendieck abelian categories are precisely the presentable abelian AB5 categories, and Pr^{L} admits small limits and $\mathsf{Pr}^{\mathrm{L}} \subseteq \mathsf{Cat}$ preserves small limits by [Lurie, 2009, Proposition 5.5.3.13]. Therefore, $\mathcal{A}$ is presentable and hence Grothendieck.

b) By Lemma 4.3.4 we can obtain an augmented cosimplicial object $\mathsf{D}(\mathcal{A}^\bullet)\colon \Delta_+ = \Delta^{\triangleleft} \to \mathsf{Cat}$, where we write $\mathcal{A}^{-1} \coloneqq \mathcal{A}$. We will denote the coface morphism d^{0*} in $\mathsf{D}(\mathcal{A}^\bullet)$ by F.

Applying Theorem 4.2.2 to $(\mathcal{A}^\bullet)^{\mathrm{op}}$, we know that the coface morphism $\mathrm{d}^{0*}\colon \mathcal{A} \to \mathcal{A}^0$ admits a right adjoint d^0_* such that the natural transformation $\mathrm{d}^{0*}\,\mathrm{d}^0_* \xRightarrow{\sim} \mathrm{d}^0_*\,\mathrm{d}^{1*}$ is an isomorphism of functors $\mathcal{A}^0 \to \mathcal{A}^0$.

By condition 2. d^{1*} sends injective object to F-acylic objects, so we can lift the natural isomorphism to derived categories, that is $\mathrm{d}^{0*}\,\mathrm{R}\,\mathrm{d}^0_* \xRightarrow{\sim} \mathrm{R}\,\mathrm{d}^0_*\,\mathrm{d}^{1*}$ is an equivalence of functors from $\mathsf{D}^+(\mathcal{A}^0)$ to itself.

By Remark 4.2.4, it suffices to show that $\mathsf{D}^+(\mathcal{A}^\bullet)$ satisfy the dual conditions in Corollary 4.2.3.

a) Let $X^\bullet$ be an F-split cosimplicial object in $\mathsf{D}(\mathcal{A})$. We need to show that $\mathsf{D}(\mathcal{A})$ admits the totalization of $X^\bullet$. Since $\Delta_{\mathrm{inj}} \subseteq \Delta$ is initial by [Lurie, 2018a, Tag 04RE], one can reduce to the semicosimplicial object underlying $X^\bullet$. Now, we want to use [Lurie, 2017, Corollary 1.2.4.12][1] to show the totalization is exist. For any $n \in \mathbb{Z}$, we can consider the unnormalized complex

$$0 \to K(n) \to \mathrm{H}\, X^0 \xrightarrow{\partial^1} \mathrm{H}^n\, X^1 \to \mathrm{H}^n\, X^2 \to \cdots \tag{4.1}$$

associated with $\mathrm{H}^n\, X^\bullet$, where we put $K(n) \coloneqq \ker \partial^1$. Since F is exact and $FX^\bullet$ is split by assumption, it follows that applying F to (4.1) yields an acyclic complex((The exactness is used to identify $F(K(n))$ with the kernel of $\mathrm{H}^n\, FX^0 \to \mathrm{H}^n\, FX^1$). As F is conservative, we deduce that (4.1) is acyclic for all $n \in \mathbb{Z}$. Therefore, use the dual version of [Lurie, 2017, Corollary 1.2.4.12] to $\mathcal{C} = \mathsf{D}(\mathcal{A})$, we know that the totalization $\mathrm{Tot}(X^\bullet) = \varprojlim_{n\in\Delta} X^n$ is exist in $\mathsf{D}(\mathcal{A})$, and $X \to X^0$ induces an isomorphism $\mathrm{H}^n\, X \xrightarrow{\sim} K(n)$, for all n. One can also find that $FX^\bullet$ admits a totalization and $\mathrm{H}^n\, \mathrm{Tot}(FX^\bullet) \xrightarrow{\sim} F(K(n))$ is an isomorphism for all n.

b) By condition 3, for every $\alpha\colon [n] \to [m]$ in Δ, the diagram

$$\begin{tikzcd} \mathsf{D}(\mathcal{A}^m) \arrow[r, "\mathrm{R}\,\mathrm{d}^0"] \arrow[d, "\alpha^*"] & \mathsf{D}(\mathcal{A}^{m+1}) \arrow[d, "\alpha'^*"] \\ \mathsf{D}(\mathcal{A}^n) \arrow[r, "\mathrm{R}\,\mathrm{d}^0"] & \mathsf{D}(\mathcal{A}^{n+1}) \end{tikzcd}$$

is right adjointable.

Therefore, the canonical functor

$$\theta\colon \mathsf{D}^+(\mathcal{A}) \to \varprojlim_{n\in\Delta} \mathsf{D}(\mathcal{A}^n)$$

admits a fully faithful left adjoint by Corollary 4.2.3. Now, we only need to show that the forgetful functor $F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}^0)$ is conservative. It follows from we can check everything after applying H^n, and $\mathrm{d}^{0*}\colon \mathcal{A} \to \mathcal{A}^0$ is conservative.

□

[1] We point out the following typo in [Lurie, 2017, Corollary 1.2.4.12]: The conditions in the statement need to be verified for all $q \in \mathbb{Z}$, not just $q \geq 0$.

Chapter 5

Descent Theory For Commutative Ring Spectra

In Chapter 4, we introduce the Lurie–Beck–Chevalley Descent(Corollary 4.2.3). And we show that, in a suitable case, the derived category of Grothendieck abelian categories satisfies descent(Theorem 4.3.5).

In this Chapter, we will first recall the notion of $\mathbb{E}_k$-ring (spectra) and then we will find that the category of static commutative rings $\mathsf{CRing}^{\mathrm{static}}$ can be identified as a full subcategory of the category of commutative ring spectra CAlg. We will also see that for a static commutative ring R, we have a canonical category equivalence

$$\mathsf{D}(R) \simeq \mathsf{Mod}(R),$$

where $\mathsf{Mod}(R)$ is the category of R-module spectra(we will use Throrem 5.1.5 to regard R as a commutative ring spectrum).

Let $f\colon A \to B$ be a homomorphism between static commutative rings, then the cosimplicial object $\mathsf{Mod}^{\mathrm{static}}(B^\bullet)$(where B^n denotes the n-th tensor power of B over A) in $\mathsf{Cat}_{(1)}$ satisfies the conditions of Theorem 4.3.5. Therefore one can know that $\mathsf{Mod}(A)$ can be recoverd as the totalization of cosimplicial object $\mathsf{D}(B^\bullet) \simeq \mathsf{Mod}(B^\bullet)$ of Cat.

In practice, we want to construct descent theory for the morphism of commutative ring spectra $A \to B$. These commutative ring spectra do not necessarily originate from the static commutative rings.

5.1 $\mathbb{E}_k$-Ring Spectra

Remark 5.1.1. We will not provide all the proofs of this section, and we refer the readers to [Lurie, 2017, § 7.1] for details.

Definition 5.1.2. Let Sp be the category of spectra equipped with the tensor product of spectra(§ 2.8.3), then an $\mathbb{E}_k$*-ring spectrum* is an $\mathbb{E}_k$-algebra in Sp. We let $\mathsf{Alg}^{(k)}$ denote the category $\mathsf{Alg}_{\mathbb{E}_k}(\mathsf{Sp})$ of $\mathbb{E}_k$-ring spectra.

By Remark 2.7.4, in the special case $k = \infty$, we will agree that $\mathbb{E}_k$ denotes the commutative pattern Comm and we let CAlg to denote the category $\mathsf{CAlg}(\mathsf{Sp})$, we will also refer to $\mathbb{E}_\infty$-ring spectrum as *commutative ring spectra*.

Remark 5.1.3. By Remark 2.7.4, the sequence of morphisms

$$\mathbb{E}_0 \to \mathbb{E}_1 \to \cdots \to \mathbb{E}_\infty = \mathsf{Comm}$$

induces forgetful functors

$$\mathsf{CAlg} \to \cdots \to \mathsf{Alg}^{(2)} \to \mathsf{Alg}^{(1)} \to \mathsf{Alg}^{(0)}.$$

In this chapter, we will focus on commutative ring spectra.

Definition 5.1.4. Let R be a $\mathbb{E}_1$-ring spectrum. We let $\mathsf{LMod}(R)$ denote the category $\mathsf{LMod}_R(\mathsf{Sp})$(in the sense of Definition 2.4.16). We will refer to $\mathsf{LMod}(R)$ as the category of *left R-module spectra*.

By Proposition B.2.1, we know that if furthermore R is commutative, then there exists a canonical equivalence

$$\mathsf{LMod}(R) \simeq \mathsf{Mod}(R).$$

Where $\mathsf{Mod}(R)$ is the pullback

$$\begin{tikzcd} \mathsf{Mod}(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Mod}(\mathsf{Sp}) \arrow[d] \\ * \arrow[r, "R"] & \mathsf{CAlg} \end{tikzcd}$$

5.1.1 From Static Commutative Ring To Commutative Ring Spectra

Now, we will introduce how to regard a static commutative ring as a commutative ring spectrum.

Recall that the Dold–Kan correspondence([Lurie, 2018a, Tag 00QQ]) induces the Eilenberg–MacLane functor([Lurie, 2018a, Tag 00QX])

$$\mathsf{D}_{\geq 0}(\mathbb{Z}) \to \mathsf{Ani}.$$

It is a right adjoint, it preserves limits. Therefore, one can upgrade it to a functor

$$\begin{tikzcd} \mathsf{Sp}(\mathsf{D}_{\geq 0}(\mathbb{Z})) \arrow[r, "\mathsf{Sp}(K)"] \arrow[d, equal] & \mathsf{Sp}(\mathsf{Ani}) \arrow[d, equal] \\ \mathsf{D}(\mathbb{Z}) \arrow[r, "H"] & \mathsf{Sp}. \end{tikzcd}$$

We will refer to H as the *Eilenberg–MacLane spectrum functor*.

Theorem 5.1.5. *The Eilenberg–MacLane spectrum functor $H\colon \mathsf{D}(\mathbb{Z}) \to \mathsf{Sp}$ refines to a lax symmetric monoidal functor(i.e. lax Comm-monoidal functor in Definition 2.4.5)*

$$H\colon \mathsf{D}(\mathbb{Z})^{\otimes^{\mathrm{L}}_{\mathbb{Z}}} \to \mathsf{Sp}.$$

More generally, for arbitrary static commutative rings R, the functor $C_*(-) \otimes^{\mathrm{L}}_{\mathbb{Z}} R\colon \mathsf{Sp} \to \mathsf{D}(R)$ has a left adjoint $H\colon \mathsf{D}(R) \to \mathsf{Sp}$, which refines to a lax symmetric monoidal functor

$$H\colon \mathsf{D}(R)^{\otimes^{\mathrm{L}}_{R}} \to \mathsf{Sp}.$$

Corollary 5.1.6. *The above induces a fully faithful functor $\mathsf{CRing}^{\mathrm{static}} \to \mathsf{CAlg}$ with essential image those $R \in \mathsf{CAlg}$ for which $\pi_i(R) = 0$ for all $i \neq 0$.*

5.1.2 Morita Theory

In this section, we will introduce the Morita theory in higher algebra, which will provide a really useful criterion for recognizing stable categories.

Construction 5.1.7. Let $\mathcal{C}$ be a stable category. For $X, Y \in \mathcal{C}$, define

$$\mathrm{map}(X,Y)_n \coloneqq \mathrm{Hom}_{\mathsf{Sp}}(X, Y[n]).$$

Note that for $n \geq 0$, we have an equivalence

$$\mathrm{Hom}_{\mathsf{Sp}}(X, Y[n]) \simeq \mathrm{Hom}_{\mathsf{Sp}}(X, \Omega Y[n+1]) \simeq \Omega\,\mathrm{Hom}_{\mathsf{Sp}}(X, Y[n+1])$$

we have $\mathrm{map}(X,Y) \in \mathsf{Sp}$, we will refer to it as the *mapping spectrum*.

Definition 5.1.8. Let $\mathcal{C}$ be a stable category. For $X, Y \in \mathcal{C}$, define

$$\mathrm{Ext}^n(X,Y) \coloneqq \pi_0\,\mathrm{map}(X,Y)_n \simeq \pi_0\,\mathrm{Hom}_{\mathsf{Sp}}(X, Y[n]),$$

and call it the n-th *extension group* between X and Y.

Theorem 5.1.9. (Schwede–Shipley) *Let $\mathcal{C}$ be a stable category. Then $\mathcal{C}$ is equivalent to $\mathsf{LMod}(R)$, for some $\mathbb{E}_1$-ring spectrum R, if and only if $\mathcal{C}$ is presentable and there exists a compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$ in the following sense: if $D \in \mathcal{C}$ an object having the property that $\mathrm{Ext}^n(C,D) \simeq 0$ for all $n \in \mathbb{Z}$, then $D \simeq 0$.*

Proof.

- First, suppose that $\mathcal{C} \simeq \mathsf{LMod}(R)$, and let $C = R$ (regarded as a left module of itself). Then $\mathcal{C}$ is presentable, C is a compact object of $\mathcal{C}$, and $\mathrm{Ext}^n(C,D) \simeq \mathrm{H}^n\,D$ for every $D \in \mathcal{C}$. Therefore, $D \simeq 0$ if and only if $\mathrm{Ext}^n(C,D) \simeq 0$ for all integers n, so that C generates $\mathcal{C}$.

- For the converse, suppose that $\mathcal{C}$ is presentable and $C \in \mathcal{C}$ is a compact generator. Therefore, $\mathcal{C} \in \mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and hence by Proposition 2.8.11, $\mathcal{C}$ can be regarded as a module over Sp. Then, one can find that $\mathcal{C}$ is linearly over Sp. Now, we want to apply Proposition B.1.4 to show the equivalence. It suffices to verify that the conditions of Proposition B.1.4 are satisfied:

  1. Since $\mathcal{C}$ is presentable, $\mathcal{C}$ admits geometric realizations of simpllicial objects.
  2. By the definition of the Lurie tensor product, the action functor $\mathsf{Sp} \times \mathcal{C} \to \mathcal{C}$ preserves colimit, then it preserves the geometric realization of simplical objects.
  3. By adjoint functor theorem([Lurie, 2009, Corollary 5.5.2.9]) and the argument above, the functor $-\otimes C\colon \mathsf{Sp} \to \mathcal{C}$ admits a right adjoint given by the mapping sepctra functor $\mathrm{map}(C,-)\colon \mathcal{C} \to \mathsf{Sp}$.
  4. Since both Sp and $\mathcal{C}$ are stable, $\mathrm{map}(C,-)$ is exact. And since C is a compact object, by Construction 5.1.7, $\mathrm{map}(C,-)$ commute with filtered colimit, and hence $\mathrm{map}(C,-)$ preserves small colimits.
  5. Let $\alpha\colon D \to D'$ be a morphism in $\mathcal{C}$ such that $\mathrm{map}(C, D \xrightarrow{\alpha} D')$ is an equivalence. Let D'' to be the fiber of α, we have $\mathrm{map}(C, D'') \simeq 0$, and hence $\mathrm{Ext}^n(C, D'') \simeq \mathrm{H}^n\,G(D'')$ vanishes for every integer n. Then $D'' \simeq 0$, and hence α is an equivalence.
  6. It suffices to show that for every $X \in \mathsf{Sp}$ and $D \in \mathcal{C}$, we have an equivalence $X \otimes \mathrm{map}(C,N) \simeq \mathrm{map}(C, X \otimes N)$.

     Now, let $\mathcal{X} \subseteq \mathsf{Sp}$ be the full subcategory spanned by those spectra X such that $X \otimes \mathrm{map}(C,N) \simeq \mathrm{map}(C, X\otimes N)$. Since Sp is generated by the sphere spectrum $\mathbb{S}$ under colimits. It suffices to show that $\mathcal{X}$ is stable under colimits and $\mathbb{S} \in \mathcal{X}$. Since $\mathrm{map}(C,-)$ preserves colimits, $\mathcal{X}$ is stable under colimits and $\mathbb{S} \in \mathcal{X}$ follows from $\mathbb{S}$ is the unit object with respect to the tensor product of spectra.

□

In fact, one can get a powerful result.

Corollary 5.1.10. *Let $1 \leq k \leq \infty$. The contruction $R \mapsto \mathsf{LMod}(R)^{\otimes}$ determines a fully faithful embedding*

$$\mathsf{Alg}^{(k)} \hookrightarrow \mathsf{Alg}_{\mathbb{E}_k}(\mathsf{Pr}^{\mathrm{L}})$$

of $\mathbb{E}_{k-1}$-monoidal presentable categories. An $\mathbb{E}_{k-1}$-monoidal category $\mathcal{C}^{\otimes} \to \mathbb{E}_{k-1}$ belongs to the essential image of this embedding if and only if the following conditions are satisfied:

- *$\mathcal{C}$ is stable and presentable, and if $k \geq 1$, then the tensor product functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.*
- *The unit object $\mathbb{1}_{\mathcal{C}}$ is compact.*
- *The object $\mathbb{1}_{\mathcal{C}}$ generates $\mathcal{C}$ in the following sense: if $C \in \mathcal{C}$ is an object such that $\mathrm{Ext}^i(\mathbb{1}, \mathcal{C}) \simeq 0$ for all $i \in \mathbb{Z}$, then $C \simeq 0$.*

Now, we can identify the category of left module spectra over a static commutative ring R as the derived category of R.

Corollary 5.1.11. *Let R be a static commutative ring. Then the canonical functor*

$$\mathsf{D}(R) \simeq \mathsf{LMod}_{R[0]}(\mathsf{D}(R)) \xrightarrow{H} \mathsf{LMod}(R) \simeq \mathsf{Mod}(R)$$

is an equivalence.

5.1.3 Algebras Over Commutative Rings

Let $\mathcal{C}$ be a symmetric monoidal category and $R \in \mathsf{CAlg}(\mathcal{C})$ be a commutative ring algebra of $\mathcal{C}$. Under some mild assumptions, $\mathsf{Mod}_R(\mathcal{C})$ of R-modules of $\mathcal{C}$ inherits the structure of a symmetric monoidal category. Moreover, the forgetful functor $\mathsf{Mod}_R(\mathcal{C}) \to \mathcal{C}$ is lax symmetric monoidal. It follows that every associative algebra $A \in \mathsf{Alg}(\mathsf{Mod}_R(\mathcal{C}))$ determines an $A' \in \mathsf{Alg}(\mathcal{C})$, and we have that this forgetful functor is a categorical equivalence. In fact, we have a more general result.

Theorem 5.1.12. *Let $k \geq 1$ and $\mathcal{C}$ be an $\mathbb{E}_{k+1}$-monoidal category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and the tensor product functor $- \otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations in each variable. Let $R \in \mathsf{Alg}_{/\mathbb{E}_{k+1}}(\mathcal{C}) \simeq \mathsf{Alg}_{/\mathbb{E}_1}(\mathsf{Alg}_{/\mathbb{E}_k}(\mathcal{C}))$, so that $\mathsf{LMod}_R(\mathcal{C})$ inherits $\mathbb{E}_1$-monoidal structure. For every associative algebra $A \in \mathsf{Alg}_{/\mathbb{E}_k}(\mathsf{LMod}_R(\mathcal{C}))$ having image $A' \in \mathsf{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})$, the forgetful functor*

$$\theta\colon \mathsf{LMod}_A(\mathsf{LMod}_R(\mathcal{C})) \to \mathsf{LMod}_{A'}(\mathcal{C})$$

is an equivalence of $\mathbb{E}_{k+1}$-monoidal categories.

Moreover, let

$$\theta_R\colon \mathsf{RMod}_A(\mathsf{LMod}_R(\mathcal{C})) \to \mathsf{RMod}_{A'}(\mathcal{C}) \, and \, \theta_L\colon \mathsf{LMod}_A(\mathsf{RMod}_R(\mathcal{C})) \to \mathsf{LMod}_{A'}(\mathcal{C})$$

be the forgetful functor induced by θ. And $M \in \mathsf{RMod}_A(\mathsf{LMod}_R(\mathcal{C}))$, $N \in \mathsf{LMod}_A(\mathsf{RMod}_R(\mathcal{C}))$, then the canonical morphism

$$\theta_R(M) \otimes_{A'} \theta_L(N) \to \theta(M \otimes_A N)$$

is an equivalence in $\mathcal{C}$.

Proof. [Lurie, 2017, Theorem 7.1.3.1 and Proposition 7.1.3.3] ∎

5.2 Descendable Objects

In this section, we will construct the descent theory for morphisms of commutative rings[1]. We refer the readers to [Mathew, 2016] for further discussion.

Unless otherwise specified, the category $\mathcal{C}$ used in this section is a presentable symmetric monoidal category, and the tensor product functor $-\otimes-\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves colimit in each variable.

Definition 5.2.1. For every object $A\in\mathcal{C}$, we denote by $\langle A\rangle\subseteq\mathcal{C}$ the smallest full subcategory containing A which is stable under finite (co)limits, retracts, and tensor products. We say that A is *descendable* if $\langle A\rangle$ contains the unit $\mathbb{1}_{\mathcal{C}}$.

By Corollary 5.1.10, such $\mathcal{C}$ can be identified as $\mathsf{Mod}(R)$ for some commutative ring spectrum R. Now, let $A\in\mathsf{CAlg}(\mathcal{C})$ be a commutative algebra in $\mathcal{C}$. Then A can be naturally regarded as a commutative R-algebra. And hence, $\mathsf{Mod}_A(\mathcal{C})$ can be recognized by $\mathsf{Mod}(R')$ for some $R'\in\mathsf{CAlg}$ by Theorem 5.1.12.

Proposition 5.2.2. *If $A\in\mathsf{CAlg}(\mathcal{C})$ is a descendable commutative algebra, and M is an object in $\mathcal{C}$ such that $M\otimes A\simeq 0$, then we have $M\simeq 0$.*

Proof. Let $\mathcal{X}\subseteq\mathcal{C}$ be a subcategory spanned by those $N\in\mathcal{C}$ such that $M\otimes N\simeq 0$. Then we have $\mathcal{X}$ is closed under finite (co)limits, retracts, and tensor products. Since $A\in\mathcal{X}$, we have $\langle A\rangle\subseteq\mathcal{D}$. Therefore, $\mathcal{X}=\mathcal{C}$. □

Our goal in this section is to show that for descendable commutative algebra $A\in\mathsf{CAlg}(\mathcal{C})$, the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "-\otimes A"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}_A(\mathcal{C}) \arrow[l, shift left, "\text{Forget}"] \end{tikzcd}$$

will exhibit $\mathcal{C}$ comonadic over $\mathsf{Mod}_A(\mathcal{C})$.

For associative algebra $A\in\mathsf{Alg}(\mathcal{C})$, one can construct a cosimplicial object

$$A \rightrightarrows A\otimes A \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \cdots$$

in $\mathsf{CAlg}(\mathcal{C})$, and refer to it as the *cobar construction* of A, denoted by $\mathrm{CB}^\bullet(A)$.

One can extend $\mathrm{CB}^\bullet(A)$ to an agumented cosimplicial object

$$\mathbb{1}_{\mathcal{C}} \longrightarrow A \rightrightarrows A\otimes A \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \cdots$$

and denote it by $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$.

For further discussion, we need to use Pro-completion; we refer the reader to [Lurie, 2018a, Tag 063J] for details.

Definition 5.2.3. Let $\mathcal{C}$ be a category with finite limits.

- A cofiltered diagram $F\colon\mathcal{I}\to\mathcal{C}$ is called *pro-constant* if, when regarded as an object in $\mathsf{Pro}(\mathcal{C})$, it lies in the essential image of the canonical functor $\mathcal{C}\to\mathsf{Pro}(\mathcal{C})$.
- A cosimplicial object $X^\bullet\colon\mathbb{\Delta}\to\mathcal{C}$ is called *pro-constant* if the associated diagram $\mathbb{Z}^{\mathrm{op}}_{\geq 0}\to\mathcal{C}$ is *pro-constant*.

[1] In fact, we can construct the descent theory for the morphisms of $\mathbb{E}_2$-rings.

Remark 5.2.4. By our setting of $\mathcal{C}$, the dual version of [Lurie, 2017, Theorem 1.2.4.1], we can describe the associated diagram of $X^\bullet\colon \mathbb{\Delta} \to \mathcal{C}$ by

$$\mathrm{Tot}(X^\bullet\mid_{\mathbb{\Delta}_{\leq n}}) \to \mathrm{Tot}(X^\bullet\mid_{\mathbb{\Delta}_{\leq n-1}}) \to \cdots \to \mathrm{Tot}(X^\bullet\mid_{\mathbb{\Delta}_{\leq 1}}) \to \mathrm{Tot}(X^\bullet\mid_{\mathbb{\Delta}_{\leq 0}}),$$

and we will simply denote $\mathrm{Tot}(X^\bullet\mid_{\mathbb{\Delta}_{\leq n}})$ by $\mathrm{Tot}_n(X^\bullet)$.

One can easily find a criterion of pro-constant objects.

Proposition 5.2.5. *Let $\mathcal{C}$ be a category with finite limits. A cofiltered diagram $F\colon \mathcal{I} \to \mathcal{C}$ defines a pro-constant object in $\mathcal{C}$ if and only if the following two conditions are satisfied:*

1. *F admits a limit in $\mathcal{C}$.*

2. *Given any functor $G\colon \mathcal{C} \to \mathcal{D}$ preserving finite limits, then the limit of F is also preserved by G.*

Proof. One direction follows from taking $\mathcal{D} = \mathsf{Pro}(\mathcal{C})$, and another direction follows from the universal property of Pro-completion. □

Let's provide some useful examples.

Example 5.2.6. By the dual version of Remark 3.2.4, the split cosimplicial object $X^\bullet$ is pro-constant.

Example 5.2.7. Let $X \in \mathcal{C}$ and $e\colon X \to X$ be an *idempotent* self-morphism, that is, $e^2 \simeq e$ in a homotopy-coherent way, which can be expressed by the condition that one has an action of the monoid $\{1, x\}$ with two elements on X. In this case, the tower

$$\cdots \to X \xrightarrow{e} X \xrightarrow{e} X,$$

is pro-constant if it admits a limit. This holds for the image of an idempotent is always a universal limit.

Remark 5.2.8. Let $F\colon \mathcal{I} \to \mathcal{C}$ be a pro-constant cofiltered diagram, then for any $X \in \mathcal{C}$, the natural morphism

$$\left(\varprojlim_{\mathcal{I}} F(i)\right) \otimes X \to \varprojlim_{\mathcal{I}}(F(i) \otimes X)$$

is an equivalence.

We can use the pro-constant object to describe descendable commutative algebra $A \in \mathsf{CAlg}(\mathcal{C})$.

Proposition 5.2.9. *Let $A \in \mathcal{C}$ be a commutative algebra, then A is descendable if and only if $\mathrm{CB}^\bullet(A)$ is a pro-constant object and $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ is a limit diagram.*

Proof.

- Suppose that A is descendable, then let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory spanned by those $M \in \mathcal{C}$ such that $\mathrm{CB}^\bullet_{\mathrm{aug}}(A) \otimes M$ is a limit diagram and the associated diagram converging to M defines a pro-constant object. It suffices to show that $\mathbb{1}_{\mathcal{C}} \in \mathcal{X}$. Note that $A \in \mathcal{X}$. And one can verify that $\mathcal{X}$ is closed under finite (co)limits, retracts and tensor products by Remark 5.2.8. Therefore $\langle A \rangle \subseteq \mathcal{X}$.

- For the converse, if $\mathrm{CB}^\bullet_{\mathrm{aug}}(A)$ is a limit diagram, and $\mathrm{CB}^\bullet(A)$ is an pro-constant objet, it follows that $\mathbb{1}_{\mathcal{C}}$ is a retract of $\mathrm{Tot}_n \mathrm{CB}^\bullet(A)$ for $n \gg 0$. However, $\mathrm{Tot}_n \mathrm{CB}^\bullet(A) \subseteq \langle A \rangle$, therefore, $\mathbb{1}_{\mathcal{C}} \in \langle A \rangle$.

□

Proposition 5.2.9 shows that A admits descent if and only if $\mathbb{1}$ can be obtained as a retract of a finite colimit of a diagram in $\mathcal{C}$ consisting of objects, each of which admits the structure of a module over A.

Corollary 5.2.10. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a symmetric monoidal functor. Given $A \in \mathsf{CAlg}(\mathcal{C})$, if A is descendable, then $F(A)$ does as well.*

Now, we can prove our main theorem of this section.

Theorem 5.2.11. *Let $A \in \mathsf{CAlg}(\mathcal{C})$ be a descendable commutative algebra. Then the adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "- \otimes A"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}_A(\mathcal{C}) \arrow[l, shift left, "\text{Forget}"] \end{tikzcd}$$

will exhibit $\mathcal{C}$ comonadic over $\mathsf{Mod}_A(\mathcal{C})$. In particular, the canonical functor

$$\mathcal{C} \to \mathrm{Tot}\left(\mathsf{Mod}_{\mathcal{C}}(\mathrm{CB}^{\bullet}(A))\right)$$

is an equivalence.

Proof. It suffices to verify the conditions in Corollary 3.4.11:

1. The conservativity follows from Proposition 5.2.2.

2. Let $X^{\bullet} \colon \mathbb{\Delta} \to \mathcal{C}$ be a $- \otimes A$-split cosimplicial object, then we need to show that

$$A \otimes \mathrm{Tot}(X^{\bullet}) \to \mathrm{Tot}(A \otimes X^{\bullet})$$

is an equivalence. By Remark 5.2.8, it suffices to show that $X^{\bullet}$ is pro-constant. Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory spanned by those $M \in \mathcal{C}$ such that $M \otimes X^{\bullet}$ is pro-constant. And the result comes from the same argument of the proof of Proposition 5.2.9.

□

From the relative viewpoint, we can define descendable morphisms.

Definition 5.2.12. Let $A, B \in \mathsf{CAlg}(\mathcal{C})$ be commutative algebra, and $A \to B$ be a morphism in $\mathsf{CAlg}(\mathcal{C})$. Then we can regard B as a commutative algebra in $\mathsf{Mod}_A(\mathcal{C})$. We say that $A \to B$ is *descendable* if B is descendable in $\mathsf{Mod}_A(\mathcal{C})$.

At the end of this section, we will discuss some properties of descendable morphisms.

Proposition 5.2.13. *Let $A \to B \to C$ be a sequence in $\mathsf{CAlg}(\mathcal{C})$.*

1. *If $A \to B$ and $B \to C$ are descendable, so does $A \to C$.*

2. *If $A \to C$ is descendable, so dose $A \to B$.*

Proof.

1. It suffices to show that in $\mathsf{Mod}_A(\mathcal{C})$, we have $A \in \langle C \rangle$. Since $A \to B$ is descendable, $A \in \langle B \rangle$, and since $B \to C$ is descendable, in $\mathsf{Mod}_B(\mathcal{C})$, we have $B \in \langle C \rangle$, and hence in $\mathsf{Mod}_A(\mathcal{C})$, $B \in \langle C \rangle$.

2. Since a C-module is in particular a B-module and $\langle B \rangle$ contains every B-module, therefore $C \in \langle B \rangle$.

□

Proposition 5.2.14. *Let $\mathcal{I}$ be a category and $p \colon \mathcal{I} \to \mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a diagram. Then for $A \in \mathsf{CAlg}(\varprojlim_{\mathcal{I}} p)$, A is descendable if and only if it image in $\mathsf{CAlg}(p(i))$ is descendable for each $i \in \mathcal{I}$.*

Proof. [Mathew, 2016, Proposition 3.25] □

5.2.1 Using The Adams Tower To Describe Pro-Constant

By [Mathew et al., 2017, § 2.1], we can also use the Adams tower to describe the pro-constant. Let $A \in \mathsf{Alg}(\mathcal{C})$ be an associative algebra of $\mathcal{C}$.

Construction 5.2.15. (Adams Tower) Let $M \in \mathcal{C}$ be an object, then we can form a tower in $\mathcal{C}$

$$\cdots \to T_2(A, M) \to T_1(A, M) \to T_0(A, M) \simeq M$$

as follows:

1. $T_1(A, M)$ is the fiber of the morphism $A \to A \otimes M$ induced by $\mathbb{1}_{\mathcal{C}} \to A$, so that $T_1(A, M)$ admits a natural morphism to M.

2. More generally, $T_i(A, M) \coloneqq T_1(A, T_{i-1}(A, M))$ which a nartual morphism to $T_{i-1}(A, M)$.

Inductively, this defines the functors T_i and the desired tower. We will call this the A-*Adams tower* of M. Observe that the A-Adams tower of M is simply the tensor product of M with the A-Adams tower of $\mathbb{1}_{\mathcal{C}}$.

Remark 5.2.16. We can write the construction of the Adams tower in another way. Let $I = \mathrm{fib}(\mathbb{1}_{\mathcal{C}} \to A)$, so that I is a nonunital associative algebra (Remark 2.2.5) in $\mathcal{C}$ equipped with a morphism $I \to \mathbb{1}_{\mathcal{C}}$. In fact, we can get a tower

$$\cdots \to I^{\otimes n} \to I^{\otimes(n-1)} \to \cdots \to I^{\otimes 2} \to I \to \mathbb{1}_{\mathcal{C}},$$

and this is precisely the A-Adams tower $\{T_i(A, \mathbb{1}_{\mathcal{C}})\}_{i \geq 0}$. The A-Adams tower for M is obtained by tensoring this with M.

Here is a simple example of Adams tower:

Example 5.2.17. Let $\mathcal{C} = \mathsf{Mod}(\mathbb{Z})$ and $A = \mathbb{Z}/p$. Then the Adams tower $\{T_i(\mathbb{Z}/p, M)\}$ for an object $M \in \mathcal{C}$ is given by

$$\cdots \to M \xrightarrow{p} M \xrightarrow{p} M.$$

The A-Adams tower has two basic properties:

Proposition 5.2.18.
- *For each i, $\mathrm{cofib}(T_i(A, M) \to T_{i-1}(A, M))$ admits the structure of A-moudle in $\mathcal{C}$ and aach morphism $T_i(A, M) \to T_{i-1}(A, M)$ becomes nullhomotopic after tensoring with A.*

- *Suppose M is an A-module in $\mathcal{C}$. Then the successive morphisms $T_i(A, M) \to T_{i-1}(A, M)$ in the Adams tower are nullhomotopic.*

Proof.
- Since $T_i(A, M) \simeq T_1(T_{i-1}(A, M))$, everything reduce to the case $i = 1$. The cofiber of $T_1(A, M) \to M$ is precisely $A \otimes M$ from the definition. And $M \to A \otimes M$ admits a section after tensoring with A. Therefore, $T_1(A, M) \to M$ becomes nullhomotopic after tensoring with A.

- Now, let M be an A-module in $\mathcal{C}$. Then we have a canonical morphism $A \otimes M \to M$, so the morphism $M \to A \otimes M$ admits a section, and hence $T_i(A, M) \to T_{i-1}(A, M)$ is nullhomotopic.

□

By the construction, one can find that the A-Adams tower is a $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$-filtered object in $\mathcal{C}$. We want to show that it will correspond to $\mathrm{CB}^{\bullet}(A)$ under the stable Dold–Kan correspondence (Remark 5.2.4). From the proof of stable Dold–Kan correspondence, we can find an approach to show this.

Definition 5.2.19. Let $S \in \mathsf{Fin}$ be a nonempty finite set,

- we will use $\mathcal{P}(S)$ to denote the partially ordered set of nonempty subsets of S ordered by inclusion.
- We wil use $\mathcal{P}^+(S)$ to denote the partially ordered set of *all* subsetes of S ordered by inclusion.

For readers familiar with the proof of the stable Dold–Kan correspondence, it is immediate that $\mathcal{P}([n])$ coincides with $\mathcal{J}^{\leq n}$, and $\mathcal{P}^+([n])$ coincides with $\mathcal{J}^{+,\leq n}$.

Construction 5.2.20. Suppose given morphisms $f_s \colon X_s \to Y_s$ for each $s \in S$. Then we obtain a functor

$$F^+(\{f_s\}) \colon \mathcal{P}^+(S) \to \mathcal{C}$$

whose value on a subset $S' \subset S$ is given by

$$F^+(\{f_s\})(S') = \bigotimes_{s_1 \notin S'} X_{s_1} \otimes \bigotimes_{s_2 \in S'} Y_{s_2}.$$

We will let $F(\{f_s\}) \colon \mathcal{P}(S) \to \mathcal{C}$ denote the restriction of $F^+(\{f_s\})$.

Fact 5.2.21. Let S be a finite nonempty set, then we have an equivalence

$$\varprojlim_{\mathcal{P}(S)} F(\{f_s\}) \simeq \operatorname{cofib}\left(\bigotimes_{s\in S} \operatorname{fib}(X_s \to Y_s) \to \bigotimes_{s\in S} X_s\right).$$

Using this fact, one can get the following result.

Proposition 5.2.22. *Let $A \in \mathcal{C}$ be an associative algebra, then we have an equivalence*

$$\operatorname{Tot}_n(\mathrm{CB}^\bullet(A)) \simeq \operatorname{cofib}\left(I^{\otimes(n+1)} \to \mathbb{1}_{\mathcal{C}}\right).$$

Proof. By [Lurie, 2017, Lemma 1.2.4.17], one can obtain an initial functor

$$\mathcal{P}([n]) \to \mathbb{\Delta}^{\leq n}$$

is initial. Therefore, we obtain $\operatorname{Tot}_n(\mathrm{CB}^\bullet(A))$ is equivalent to the limit of such $F(\{f_s\})$, where $f_s \colon \mathbb{1}_{\mathcal{C}} \to A$ for each $s \in [n]$ is the unit morphism of associative algebra A. By Fact 5.2.21 and Remark 5.2.16, we obtain the result. □

5.3 Descent For R-Linear Category

In this section, we will introduce the descent theory for R-linear categories.

Definition 5.3.1. Let R be a commutative ring spectrum, An *R-linear category* is a presentable category $\mathcal{C}$ linear over $\mathsf{Mod}(R)$, such that the action functor

$$- \otimes_R - \colon \mathsf{Mod}(R) \times \mathcal{C} \to \mathcal{C}$$

preserves colimits in each variable.

Let $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$ be Pr^{L} equipped with the Lurie tensor product in Definition 2.8.3. Then $\mathsf{Mod}(R)$ can be regarded as an commutative algebra of $\left(\mathsf{Pr}^{\mathrm{L}}\right)^{\otimes}$ by Proposition B.2.2. We let $\mathsf{LinCat}_R = \mathsf{Mod}_{\mathsf{Mod}(R)}\left(\mathsf{Pr}^{\mathrm{L}}\right)$ as refer to the category of R-linear categories.

Example 5.3.2. Let $\mathbb{S}$ be the sphere specturm, then Sp is the $\mathbb{S}$-linear category, and $\mathsf{LinCat}_{\mathbb{S}} = \mathsf{Mod}_{\mathsf{Sp}}\left(\mathsf{Pr}^{\mathrm{L}}\right)$, by the discussion in § 2.8.3, we have $\mathsf{Mod}_{\mathsf{Sp}} \simeq \mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Given a commutative A-algebra B, the category $\mathsf{Mod}_B(\mathcal{C})$ of B-modules *internal to* $\mathcal{C}$, that is,

$$\mathsf{Mod}_B(\mathcal{C}) \simeq \mathcal{C} \otimes_{\mathsf{Mod}(A)} \mathsf{Mod}(B).$$

Informally, $\mathsf{Mod}_B(\mathcal{C})$ is the target of the functor

$$- \otimes_A - \colon \mathcal{C} \times \mathsf{Mod}(B) \to \mathsf{Mod}_B(\mathcal{C}), \quad (X, M) \mapsto X \otimes_A M.$$

Our goal in this section is to discuss whether descent holds for an arbitrary stable A-linear category $\mathcal{C}$. By the proof of Theorem 5.2.11, we get:

Corollary 5.3.3. *Let $A \to B$ be a descendable morphism between commutative ring spectra. Then for any stable A-linear category $\mathcal{C}$, the canonical functor*

$$\mathcal{C} \to \mathrm{Tot}\left(\mathcal{C} \otimes_{\mathsf{Mod(A)}} \mathsf{Mod}(B)^{(\bullet+1)}\right)$$

induced by tensoring with B is an equivalence.

Proof. Left to the reader. □

Corollary 5.3.4. *Let $f\colon A \to B$ be a descendable morphism between commutative ring spectra. Then the base-change functor $\mathsf{Mod}(A) \to \mathsf{Mod}(B)$ is comonadic.*

Appendix A

Monads for Algebraic Structure

In the Appendix, we will introduce the relation below:

$$
\begin{tikzcd}
 & & \text{Monads} & & \\
\text{Algbraic Patterns} \arrow[r, equal] & \text{Polynomial Monads} \arrow[ur, phantom, "\subseteq"] & & \text{Algebraic Monads} \arrow[ul, phantom, "\subseteq"] \arrow[r, equal] & \text{Lawvere Theories} \\
 & & \text{Analytic Monads} \arrow[ul, phantom, "\subseteq"] \arrow[ur, phantom, "\subseteq"] \arrow[d, equal] & & \\
 & & \text{Operads} & &
\end{tikzcd}
$$

This chapter is not yet complete, and I currently have no plan to finish it.

A.1 Polynomial Monads

A.2 Analytic Monads

A.3 Algebraic Monads

Appendix B

Some Consequences In Higher Algebra

In this chapter, we will provide some useful consequences that will be applied in the context.

B.1 More On Modules

Lemma B.1.1. *Suppose that we are given a coCartesian fibration $\mathcal{O} \to \mathsf{LM}$ which exhibits $\mathcal{M}^{\otimes}$ linear over the monoidal category $\mathcal{C}^{\otimes}$. Then the associative functor $\mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ is a locally coCartesian fibration.*

Proof. For each $n \geq 0$, the morphism $p_{[n]} \colon \mathcal{M}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{[n]}$ is equivalent to $\mathcal{C}^{\otimes}_{[n]} \times \mathcal{M} \to \mathcal{C}^{\otimes}_{[n]}$. The desired result now follows from [Lurie, 2009, Proposition 2.4.2.11] □

Lemma B.1.2. *Let $\mathcal{C}^{\otimes}$ be a monoidal category and $\mathcal{M}^{\otimes}$ linear over $\mathcal{C}^{\otimes}$, and $\theta \colon \mathsf{LMod}(\mathcal{M}) \to \mathsf{Alg}(\mathcal{C})$ the forgetful functor. Then θ is Cartesian fibration. Moreover, a morphism f in $\mathsf{LMod}(\mathcal{M})$ is θ-Cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Proof. [Lurie, 2017, Corollary 4.2.3.2] □

Proposition B.1.3. *Let $\mathcal{C}$ be a monoidal category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and the tensor product $- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations in each variable. For an associative algebra $A \in \mathsf{Alg}(\mathcal{C})$, there exists a monoidal structure on ${}_A\mathsf{BMod}_A(\mathcal{C})$ given by the relative tensor product over A (Definition 3.2.7).*

Proof. [Lurie, 2017, Proposition 4.4.3.12] □

Proposition B.1.4. *Let $\mathcal{C}$ be a monoidal category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and the tensor product $- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations in each variable. Let $\mathcal{M}$ be a category linear over $\mathcal{C}$ and $M \in \mathcal{M}$ be an object. There exists an associative algebra $A \in \mathsf{Alg}(\mathcal{C})$ and an equivalence $\mathsf{LMod}_A(\mathcal{C}) \simeq \mathcal{M}$ of categories linear over $\mathcal{C}$ which carries A to M if and only if the following conditions are satisfied:*

1. *$\mathcal{M}$ admits geometric realizations of simplicial objects.*
2. *The action functor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations of simplicial objects.*
3. *The functor $F \colon \mathcal{C} \to \mathcal{M}$ given by $F(C) = C \otimes M$ admits a right adjoint G.*

4. *The functor G preserves geometric realizations of simplicial objects.*

5. *The functor G is conservative.*

6. *For every object $N \in \mathcal{M}$ and every objects $C \in \mathcal{C}$, the canonical morphism*

$$F(C \otimes G(N)) \simeq C \otimes G(N) \otimes M \simeq C \otimes FG(N) \to C \otimes N$$

is adjoint to an equivalence $C \otimes G(N) \to G(C \otimes N)$.

Proof. [Lurie, 2017, Proposition 4.8.5.8] □

B.2 Left Modules Over Commutative Algebra

Consider the canonical functor $\Phi\colon \mathsf{LM} \to \mathsf{CM}$ as follows:

- For $n \geq 0$, we have $\Phi([n], 0) = (\mathrm{Cut}([n] \star [0]), n+1)$.
- For $n \geq 0$, we have $\Phi([n], 1) = (\mathrm{Cut}([n]), 0)$.

Proposition B.2.1. *Let $\mathcal{O}$ be an operad. Then the following commutative square is a pullback square*

$$\begin{tikzcd} \mathsf{Mod}(\mathcal{O}) \arrow[r, "\Phi^*"] \arrow[d] & \mathsf{LMod}(\mathcal{O}) \arrow[d] \\ \mathsf{CAlg}(\mathcal{O}) \arrow[r, "\mathrm{Cut}^*"] & \mathsf{Alg}(\mathcal{O}). \end{tikzcd}$$

In particular, passing to vertical fibers over $R \in \mathsf{CAlg}(\mathcal{O})$ induces an equivalence

$$\mathsf{Mod}_R(\mathcal{O}) \xrightarrow{\sim} \mathsf{LMod}_{R'}(\mathcal{O})$$

Proof. [Lurie, 2017, Proposition 4.5.1.4]. □

Now, consider the functor $\mathrm{Cut}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathsf{Fin}_*$ induces a morphism between algebraic patterns

$$f\colon \mathsf{BM} \to \mathsf{CM},$$

for operad $\mathcal{O}$, it will induces a pullback

$$f^*\colon \mathsf{Mod}(\mathcal{O}) \to \mathsf{BMod}(\mathcal{O}).$$

Proposition B.2.2. *Let $\mathcal{C}$ be a symmetric monoidal category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and the tensor product $-\otimes-\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ preserves geometric realizations in each variable. Let R be a commutative algebra in $\mathcal{C}$. Then:*

- $\mathsf{Mod}_R(\mathcal{C})$ *admits a symmetric monoidal structure.*
- *The functor $\mathsf{Mod}_R(\mathcal{C}) \to {}_R\mathsf{BMod}_R$ induced by f^* refines to a monoidal functor.*

In particular, the tensor product on $\mathsf{Mod}_R(\mathcal{C})$ is given by following composite:

$$\mathsf{Mod}_R(\mathcal{C}) \times \mathsf{Mod}_R(\mathcal{C}) \to {}_R\mathsf{BMod}_R \times_R {}_R\mathsf{BMod}_R(\mathcal{C}) \xrightarrow{-\otimes_R-} {}_R\mathsf{BMod}_R(\mathcal{C}) \to \mathsf{LMod}_R(\mathcal{C}) \xrightarrow{\sim} \mathsf{Mod}_R(\mathcal{C}).$$

Proof. [Lurie, 2017, Theorem 4.5.2.1] □

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