Motivic Homotopy Theory

(Draft)

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Contents

Preface

These notes grew out of my participation in Marc Hoyois's Oberseminar on motivic cohomology at the University of Regensburg (SoSe 2026), combined with my own study of the subject. The primary references are the work of Bachmann–Hoyois [BH21] on norms, and the recent paper of Bachmann–Elmanto–Morrow [BEM25]. The goal is to give a self-contained introduction to motivic homotopy theory.

Organization

Chapter 1 constructs the unstable motivic homotopy category $\mathsf{H}(S)$ via sifted cocompletion, Nisnevich sheafification, and $\mathbb{A}^1$-localization, develops the pointed variant $\mathsf{H}_*(S)$, and establishes the bigraded sphere structure together with the Tate object $\mathbb{T}_S$.

Chapter 2 formally inverts $\mathbb{T}_S$ to obtain the stable category $\mathsf{SH}(S)$, briefly introduces the Annala–Iwasa generalization MotSp, and equips SH with a six-functor formalism via the span-category framework of Mann–Scholze and Cnossen–Lenz–Linskens. The chapter culminates in the Drew–Gallauer universal property, characterizing SH as the universal $\mathbb{A}^1$-invariant six-functor formalism on schemes.

Chapter 3 develops the comparisons of $\mathsf{SH}(S)$ with classical structures: the Bachmann–Elmanto–Morrow equivalence between $\mathsf{SH}(S)$ and its naive stabilization $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ (giving the motivic Eilenberg–MacLane functor and the construction of the motivic K-theory spectrum KGL), the comparison between Voevodsky's slice filtration and Postnikov-type filtrations (culminating in the slice conjecture and $\mathbb{A}^1$-motivic cohomology), and the comparison between motivic and topological orientations (Chern classes, formal group laws, and the universal Thom spectrum MGL).

Chapter 4 (in preparation) treats framed correspondences and the motivic recognition principle.

Warning

This document is a preliminary draft. It likely contains errors, incomplete arguments, and missing references. In particular, some parts were drafted with AI assistance, which may have introduced inaccurate or fabricated citations; readers who notice such discrepancies are encouraged to contact the author. Corrections and suggestions are always welcome.

Conventions

Throughout these notes, we work in the framework of ∞-categories. The word "category" means ∞-category unless otherwise specified; we write "ordinary category" or "1-category" for the classical notion.

All schemes are assumed to be quasi-compact and quasi-separated unless otherwise stated.

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Chapter 1.

Unstable Motivic Homotopy Theory

The starting point of motivic homotopy theory is a striking analogy between algebraic geometry and topology. In topology, for any paracompact Hausdorff space X, the projection $X \times I \to X$ (with $I = [0,1]$) induces a bijection

$$\pi_0 \mathsf{Vect}_r^{\mathrm{top}}(X) \xrightarrow{\sim} \pi_0 \mathsf{Vect}_r^{\mathrm{top}}(X \times I).$$

Vector bundles, and many other invariants, are insensitive to the cylinder direction. The classical Quillen–Suslin theorem [Qui76; Sus76] provides the algebraic counterpart: every finitely generated projective module over $k[x_1, \dots, x_n]$ is free; equivalently, the pullback

$$\pi_0 \mathsf{Vect}_r(\mathrm{Spec}(k)) \xrightarrow{\sim} \pi_0 \mathsf{Vect}_r(\mathbb{A}^n_k)$$

is a bijection. Under this analogy, the affine line $\mathbb{A}^1$ plays the role of the interval I: both serve as "cylinder objects" in their respective categories, suggesting a homotopy theory for algebraic geometry in which the projection $\mathbb{A}^1 \times X \to X$ is declared a homotopy equivalence.

Supporting this intuition, many cohomology theories on smooth schemes are $\mathbb{A}^1$-invariant. For a smooth scheme X over a field k:

- **Chow groups:** $\mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CH}^*(X \times \mathbb{A}^1)$;
- **Grothendieck group:** $\mathrm{K}_0(X) \xrightarrow{\sim} \mathrm{K}_0(X \times \mathbb{A}^1)$;
- **Étale cohomology** (for ℓ invertible in k): $\mathrm{H}^*_{\text{ét}}(X, \mu_\ell) \xrightarrow{\sim} \mathrm{H}^*_{\text{ét}}(X \times \mathbb{A}^1, \mu_\ell)$.

***Remark* 1.0.1** ($\mathbb{A}^1$-invariance fails on non-affine schemes). The functor $\pi_0 \mathsf{Vect}_r \colon \mathsf{Sm}_k^{\mathrm{op}} \to \mathsf{Set}$ is *not* $\mathbb{A}^1$-invariant in general. Take the rank-2 bundle E on $\mathbb{P}^1 \times \mathbb{A}^1$ obtained by gluing the trivial bundle on the standard charts $U_0 = \mathrm{Spec}(k[x]) \times \mathbb{A}^1$ and $U_1 = \mathrm{Spec}(k[x^{-1}]) \times \mathbb{A}^1$ via the transition matrix

$$g_{01}(x,t) = \begin{pmatrix} x & t \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{GL}_2\big(k[x, x^{-1}, t]\big).$$

At $t = 0$, the matrix is diagonal, giving $E|_{\mathbb{P}^1 \times \{0\}} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$. At $t = 1$, the matrix defines a non-trivial extension

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow E|_{\mathbb{P}^1 \times \{1\}} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

whose class in $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k$ is non-zero; by Grothendieck's splitting theorem, this extension is isomorphic as a vector bundle to $\mathcal{O} \oplus \mathcal{O}$. Since $E|_{\mathbb{P}^1 \times \{0\}} \not\simeq E|_{\mathbb{P}^1 \times \{1\}}$ on $\mathbb{P}^1$, the bundle E cannot lie in the image of $\pi_0 \mathsf{Vect}_2(\mathbb{P}^1) \to \pi_0 \mathsf{Vect}_2(\mathbb{P}^1 \times \mathbb{A}^1)$.

Nevertheless—and this is the deep theorem of Morel [Mor12], simplified by Asok–Hoyois–Wendt [AHW17]—if we forcibly $\mathbb{A}^1$-localize Vect_r, the resulting functor still computes the correct values on *smooth affine* schemes. This affine representability theorem is one of the cornerstone applications of motivic homotopy theory.

Overview of the construction. Fix a quasi-compact quasi-separated scheme S, and let Sm_S denote the category of smooth S-schemes of finite presentation. Our goal is to construct the **motivic homotopy category** $\mathsf{H}(S)$: an enlargement of Sm_S in which the projections $\mathbb{A}^1 \times_S X \to X$ become equivalences. The construction proceeds in three universal steps, each with a clear geometric meaning:

(i) **Sifted cocompletion.** Embed Sm_S into $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$, the category of presheaves of anima that convert finite coproducts to finite products.

(ii) **Nisnevich descent.** Force a chosen class of étale-like covers (the *Nisnevich covers*) to become equivalences, yielding $\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S)$.

(iii) **$\mathbb{A}^1$-localization.** Force the projections $\mathbb{A}^1 \times X \to X$ to become equivalences, yielding the motivic homotopy category $\mathsf{H}(S)$.

***Remark* 1.0.2** (Why $\mathcal{P}_\Sigma$ rather than PShv?). One could equally well start from the full presheaf category $\mathsf{PShv}(\mathsf{Sm}_S)$. The advantage of $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$ is threefold:

(i) Since Sm_S is extensive, $\mathcal{P}_\Sigma(\mathsf{Sm}_S) \simeq \mathsf{Shv}_\sqcup(\mathsf{Sm}_S)$ [BH21, Lemma 2.4], which is already a topos for the coproduct topology.

(ii) The Yoneda embedding $\mathsf{Sm}_S \hookrightarrow \mathcal{P}_\Sigma(\mathsf{Sm}_S)$ preserves finite coproducts, and every object is a sifted colimit of representables.

(iii) The universal property of $\mathcal{P}_\Sigma$ as the free sifted cocompletion is essential to the construction of norm functors in Appendix A; see [BH21, §3].

Since finite coproduct decompositions are themselves Nisnevich covers, we have natural inclusions

$$\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S) \subset \mathcal{P}_\Sigma(\mathsf{Sm}_S) \subset \mathsf{PShv}(\mathsf{Sm}_S),$$

and we write $\mathrm{L}_{\mathrm{Nis}} : \mathcal{P}_\Sigma(\mathsf{Sm}_S) \to \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ for the Nisnevich sheafification functor.

1.1. The Nisnevich Topology

The Nisnevich topology, also called the completely decomposed topology, sits strictly between Zariski and étale. In motivic homotopy theory, this choice is forced by two complementary optimality conditions: it is the coarsest topology for which the purity theorem holds, and the finest for which algebraic K-theory is representable (Remark 1.1.1).

1.1.1. Three topologies

For S qcqs, there are three commonly used Grothendieck topologies on Sm_S:

$$\mathrm{Zar} \subset \mathrm{Nis} \subset \text{ét}.$$

- **Zariski:** covers are jointly surjective open immersions.
- **Étale:** covers are jointly surjective étale morphisms.
- **Nisnevich:** covers are jointly surjective étale morphisms with the additional requirement that no residue field extensions occur.

The functor-of-points perspective makes the Nisnevich condition transparent. For a smooth k-scheme X, an étale cover $\{U_i \to X\}$ need not satisfy $\coprod_i U_i(k) \twoheadrightarrow X(k)$: the preimage of a k-rational point might only exist over a finite separable extension L/k. A Nisnevich cover excludes precisely this phenomenon—every k-rational point lifts without field extension.

***Remark* 1.1.1** (Why Nisnevich?). The Nisnevich topology is characterized by two complementary optimality conditions:

- **Coarsest topology for which purity holds.** The purity theorem asserts that for a closed immersion $Z \hookrightarrow X$ in Sm_S with normal bundle ν_Z, there is an $\mathbb{A}^1$-local equivalence $X/(X \setminus Z) \simeq \mathrm{Th}(\nu_Z)$. The Zariski topology is too coarse for this.
- **Finest topology for which K-theory is representable.** Non-connective K-theory $\mathrm{K}\colon \mathsf{Sm}_S^{\mathrm{op}} \to \mathsf{Sp}$ satisfies Nisnevich descent but not étale descent.

1.1.2. Nisnevich covers

Definition 1.1.2 (Nisnevich cover). A family of étale morphisms $(f_i\colon U_i \to X)_{i\in I}$ in Sm_S is a **Nisnevich cover** if for every $x \in X$, there exists $i \in I$ and $u \in U_i$ with $f_i(u) = x$ and $\kappa(u) \xrightarrow{\sim} \kappa(x)$.

***Remark* 1.1.3.** The term "completely decomposed" originates in arithmetic: a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is "completely decomposed" in an étale cover precisely when its residue field does not extend.

Theorem 1.1.4 (Equivalent characterizations). *Let X be qcqs and $(f_i\colon U_i \to X)_{i\in I}$ a family of étale morphisms. The following are equivalent:*

(i) ***(Rational points)*** *For every $x \in X$, there exists i and $u \in U_i$ with $f_i(u) = x$ and $\kappa(u) \xrightarrow{\sim} \kappa(x)$.*

(ii) ***(Finite stratification)*** *There is a chain of finitely presented closed subschemes*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_n = X$$

such that some f_i admits a section over each stratum $Y_j = Z_j \setminus Z_{j-1}$.

(iii) ***(Refinement by Nisnevich squares)*** *There is a stratification as above and a finitely presented étale family $(g_j\colon V_j \to X)_{j=1}^n$ refining (f_i), with each g_j restricting to an isomorphism over Y_j.*

Proof. The implications (iii) ⇒ (ii) ⇒ (i) are immediate. We prove (i) ⇒ (iii).

Replacing each U_i by an affine open cover, we may assume all U_i are quasi-compact and separated, hence all f_i are of finite presentation. Consider the collection of closed subschemes $Y \subseteq X$ for which (iii) fails; we show this collection is empty by Zorn's lemma.

Let $(Y_\alpha)_{\alpha\in A}$ be a descending chain and $Y = \bigcap_\alpha Y_\alpha$. If (iii) holds for Y, then by finite presentation it holds for some Y_α (limits of finitely presented data along cofiltered systems). Hence if (iii) fails, there exists a minimal closed $Y \subseteq X$ for which it fails. Replacing X by Y, we may assume (iii) fails for X but holds for all proper closed subschemes.

Clearly $X \neq \varnothing$. Take a generic point $x \in X$. By (i), for some i there is a morphism $\mathrm{Spec}(\kappa(x)) \to U_i$ lifting $\mathrm{Spec}(\kappa(x)) \to X$. Since $\mathcal{O}_{X,x}^{\mathrm{red}} = \kappa(x)$, the nilpotent lifting property of étale morphisms yields a lift $\mathrm{Spec}(\mathcal{O}_{X,x}) \to U_i$. Passing to a limit, we obtain a quasi-compact open neighborhood $V \subseteq X$ of x and a section $s\colon V \to U_i$.

Take a finitely presented closed $Z \subset X$ with $V = X \setminus Z$. By minimality, (iii) holds for Z: there is a stratification $\varnothing = Z_0 \subset \cdots \subset Z_n = Z$ and a refining family $(h_j\colon W_j \to Z)$ with the required properties. For each j, lift W_j to a finitely presented open $V_j \subseteq U_{i_j}$ with $g_j\colon V_j \to X$ étale and an isomorphism over Y_j. Extending the chain to $\varnothing = Z_0 \subset \cdots \subset Z_n \subset Z_{n+1} = X$ with $V_{n+1} = V$ and $g_{n+1} = f_i \circ s$ realizes (iii) for X, contradicting minimality. □

For a qcqs scheme X, the **small Nisnevich topos** X_{Nis} of X is the topos of Nisnevich sheaves of anima on the category $\mathsf{Et}_X^{\mathrm{fp}}$ of finitely presented étale X-schemes.

Proposition 1.1.5 ([BH21, A.3]). *Let X be a qcqs scheme.*

(i) For every $Y \in \mathrm{Et}_X^{\mathrm{fp}}$ and every point $y \in Y$, the functor

$$F \longmapsto F\big(\operatorname{Spec} O_{Y,y}^h\big),$$

where $O_{Y,y}^h$ is the henselization of Y at y, is the inverse image functor of a point of the hypercompletion $X_{\mathrm{Nis}}^{\wedge}$. These points form a conservative family.

(ii) X_{Nis} is coherent in the sense of [Lur18, Definition A.2.1.6].

1.1.3. Nisnevich squares

The key feature of the Nisnevich topology in practice is that it is generated by a particularly simple class of covers.

Definition 1.1.6 (Nisnevich square). A pullback square in Sm_S

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & \lrcorner & \big\downarrow p \\ U & \overset{j}{\hookrightarrow} & X \end{array}$$

is a **Nisnevich square** if

- j is an open immersion;
- p is étale;
- p induces an isomorphism $p^{-1}(X \setminus U)_{\mathrm{red}} \xrightarrow{\sim} (X \setminus U)_{\mathrm{red}}$.

From the functor-of-points perspective, a Nisnevich square decomposes X's R-points into two classes: those landing in the open U, handled by j; and those in the closed complement $X \setminus U$, covered by p without any residue field extension.

Lemma 1.1.7. *The family $\{j : U \hookrightarrow X,\ p : V \to X\}$ is a Nisnevich cover of X.*

Proof. For $x \in U$, j has x in its image with trivial residue extension. For $x \in X \setminus U$, the assumption gives $v \in p^{-1}(x)$ with $\kappa(v) \simeq \kappa(x)$. □

The deeper result is that Nisnevich squares *generate* the Nisnevich topology:

Proposition 1.1.8 (Brown–Gersten). *A ($\mathcal{D}$-valued) presheaf $\mathcal{F}$ on Sm_S is a Nisnevich sheaf if and only if $\mathcal{F}(\varnothing)$ is terminal and $\mathcal{F}$ sends every Nisnevich square to a pullback square:*

$$\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(V) \times_{\mathcal{F}(U \times_X V)} \mathcal{F}(U).$$

Proof. ($\Rightarrow$) By Lemma 1.1.7, $\{U \hookrightarrow X, V \to X\}$ is a Nisnevich cover, and the sheaf condition directly yields the claimed pullback square. Applying the sheaf condition to the empty cover of $\varnothing$ shows that $\mathcal{F}(\varnothing)$ is terminal.

($\Leftarrow$) Assume $\mathcal{F}$ sends Nisnevich squares to pullbacks. We show that $\mathcal{F}$ satisfies the sheaf condition for any Nisnevich cover of X. By Theorem 1.1.4(iii), there is a refining family $(g_i : V_i \to X)_{i=1}^n$ together with a stratification $\varnothing = Z_0 \subset \cdots \subset Z_n = X$ such that each g_i is an isomorphism over Y_i. We proceed by induction on n.

For $n = 0$, we have $X = \varnothing$, and the hypothesis on $\mathcal{F}(\varnothing)$ gives the result. For $n \geq 1$, set $U = X \setminus Z_1$, so that the inclusion $\iota : U \hookrightarrow X$ is an open immersion with complement Z_1.

Since g_1 is an isomorphism over Z_1, the pair $\{U \hookrightarrow X,\ V_1 \to X\}$ forms a Nisnevich square, giving

$$\mathcal{F}(X) \simeq \mathcal{F}(V_1) \times_{\mathcal{F}(U \times_X V_1)} \mathcal{F}(U).$$

The induced stratification on U has length $n-1$, so by the inductive hypothesis $\mathcal{F}$ satisfies descent for the induced covers of U and of $U \times_X V_1$. Pasting pullbacks finishes the proof. □

***Warning* 1.1.9** (Non-subcanonicality). The Nisnevich topology is *not* subcanonical: representable presheaves $y(X) = \mathrm{Map}_{\mathsf{Sm}_S}(-, X)$ are in general not Nisnevich sheaves. Concretely, Nisnevich descent for h_X would require X-points to glue along Nisnevich covers, but this already fails for henselizations: the henselian local ring $\mathcal{O}^h_{Y,y}$ is the colimit of étale neighborhoods of y, yet $X(\mathcal{O}^h_{Y,y})$ generally does *not* agree with $\operatorname{colim} X(U)$ over those neighborhoods unless X is sufficiently rigid.

Consequently, throughout the construction of $\mathsf{H}(S)$ we must Nisnevich-sheafify representables before working with them. The same warning applies, a fortiori, to the cdh topology of §2.3.1.

1.1.4. Examples

Example 1.1.10 (Zariski covers are Nisnevich). Open immersions induce identity maps on residue fields, so Zariski covers are automatically Nisnevich. The standard cover $\{\mathbb{A}^1, \mathbb{A}^1\} \to \mathbb{P}^1$ corresponds to the Nisnevich square

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

Example 1.1.11 (The m-th power map). Let k be a field with $\operatorname{char}(k) \nmid m$ and $a \in k^\times$. Consider

$$\begin{aligned} f : (\mathbb{A}^1 \setminus \{a\}) \sqcup (\mathbb{A}^1 \setminus \{0\}) &\longrightarrow \mathbb{A}^1, \\ \text{first component:} \quad x &\longmapsto x, \\ \text{second component:} \quad y &\longmapsto y^m. \end{aligned}$$

The first component is an open immersion; the second is étale. This f is a Nisnevich cover if and only if a has an m-th root in k.

Example 1.1.12 (Algebraic K-theory is a Nisnevich sheaf). By Brown–Gersten it suffices to check Nisnevich excision (the empty case is trivial). For a Nisnevich square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \xrightarrow[j]{} & Y, \end{array}$$

perfect complexes satisfy flat descent, so we obtain a pullback square of stable categories

$$\begin{array}{ccc} \mathsf{Perf}(W) & \longleftarrow & \mathsf{Perf}(V) \\ \uparrow & & \uparrow \\ \mathsf{Perf}(U) & \longleftarrow & \mathsf{Perf}(Y). \end{array}$$

Setting $Z = Y \setminus U = V \setminus W$ and taking horizontal fibers gives a diagram of Karoubi sequences (using full faithfulness of $j_*\colon \mathsf{D}(U) \to \mathsf{D}(Y)$). Since K-theory sends Karoubi sequences to fiber sequences of spectra, the K-theory square is also a pullback.

1.1.5. Milnor excision

Restricting to affine schemes, Nisnevich squares take a particularly clean ring-theoretic form.

Definition 1.1.13 (Milnor square). A **Milnor square** for a commutative ring R is a pullback

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & \lrcorner & \downarrow \\ R_f & \longrightarrow & R_f/IR_f \end{array}$$

where $f \in R$, $I \subset R$ is an ideal, and $R/I \xrightarrow{\sim} R_f/IR_f$ (equivalently, f is a unit in R/I).

A Milnor square is not literally a Nisnevich square: in $\mathrm{Spec}(R) = X$ the open is $U = \mathrm{Spec}(R_f)$ and the would-be "cover" $V = \mathrm{Spec}(R/I) \to X$ is a closed immersion, not an étale morphism. Nevertheless every Nisnevich sheaf still satisfies the corresponding excision property; one can deduce this from Nisnevich descent via henselization along I.

Proposition 1.1.14 (Milnor excision). *For every Nisnevich sheaf $\mathcal{F}$ and every Milnor square,*

$$\mathcal{F}(\mathrm{Spec}(R)) \xrightarrow{\sim} \mathcal{F}(\mathrm{Spec}(R_f)) \times_{\mathcal{F}(\mathrm{Spec}(R_f/IR_f))} \mathcal{F}(\mathrm{Spec}(R/I)).$$

1.2. $\mathbb{A}^1$-LOCALIZATION

Write $\mathsf{PShv}_{\mathbb{A}^1}(\mathsf{Sm}_S) \subset \mathsf{PShv}(\mathsf{Sm}_S)$ for the full subcategory of $\mathbb{A}^1$**-invariant** presheaves: those $\mathcal{F}$ with $\mathcal{F}(U) \xrightarrow{\sim} \mathcal{F}(U \times_S \mathbb{A}^1)$ for all $U \in \mathsf{Sm}_S$. The inclusion admits a left adjoint

$$\mathrm{L}_{\mathbb{A}^1}\colon \mathsf{PShv}(\mathsf{Sm}_S) \longrightarrow \mathsf{PShv}_{\mathbb{A}^1}(\mathsf{Sm}_S),$$

called the $\mathbb{A}^1$**-localization functor**.

Definition 1.2.1 ($\mathbb{A}^1$-local equivalence and $\mathbb{A}^1$-homotopy). Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathsf{PShv}(\mathsf{Sm}_S)$.

- f is an $\mathbb{A}^1$**-local equivalence** if $\mathrm{L}_{\mathbb{A}^1}(f)$ is an isomorphism.
- Two maps $f, g\colon \mathcal{F} \to \mathcal{G}$ are $\mathbb{A}^1$**-homotopic** if there exists $H\colon \mathcal{F} \times \mathbb{A}^1 \to \mathcal{G}$ with $H \circ i_0 \simeq f$ and $H \circ i_1 \simeq g$, where $i_s\colon * \to \mathbb{A}^1$ is the inclusion of $s \in \{0, 1\}$.
- f is an $\mathbb{A}^1$**-homotopy equivalence** if there exists g with fg and gf each $\mathbb{A}^1$-homotopic to the appropriate identity.

Proposition 1.2.2. *Every $\mathbb{A}^1$-homotopy equivalence is an $\mathbb{A}^1$-local equivalence. In particular, the projection $p\colon \mathcal{F} \times \mathbb{A}^1 \to \mathcal{F}$ is an $\mathbb{A}^1$-local equivalence for every $\mathcal{F}$.*

Proof. We first show $p\colon X \times \mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-homotopy equivalence for every $X \in \mathsf{Sm}_S$ (and hence, by universal colimits in PShv, for every presheaf). The zero section $s\colon X \to X \times \mathbb{A}^1$ satisfies $p \circ s = \mathrm{id}_X$. The reverse composite $s \circ p$ is $\mathbb{A}^1$-homotopic to $\mathrm{id}_{X \times \mathbb{A}^1}$ via

$$(X \times \mathbb{A}^1) \times \mathbb{A}^1 \xrightarrow{\mathrm{id}_X \times \mu} X \times \mathbb{A}^1,$$

where $\mu\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the ring multiplication: at $t = 1$ this gives $\mathrm{id}_{X\times\mathbb{A}^1}$, at $t = 0$ it gives $s \circ p$.

Now suppose $a, b\colon \mathcal{F} \to \mathcal{G}$ are $\mathbb{A}^1$-homotopic via H. Since $p \circ i_0 = p \circ i_1 = \mathrm{id}$ and $\mathrm{L}_{\mathbb{A}^1}(p)$ is an isomorphism, applying $\mathrm{L}_{\mathbb{A}^1}$ gives $\mathrm{L}_{\mathbb{A}^1}(i_0) = \mathrm{L}_{\mathbb{A}^1}(i_1)$. Hence

$$\mathrm{L}_{\mathbb{A}^1}(a) = \mathrm{L}_{\mathbb{A}^1}(H) \circ \mathrm{L}_{\mathbb{A}^1}(i_0) = \mathrm{L}_{\mathbb{A}^1}(H) \circ \mathrm{L}_{\mathbb{A}^1}(i_1) = \mathrm{L}_{\mathbb{A}^1}(b).$$

The result follows from the definition of homotopy equivalence. □

1.2.1. *Interval objects*

So far we have introduced two notions of equivalence on $\mathsf{PShv}(\mathsf{Sm}_S)$: $\mathbb{A}^1$-local equivalence and $\mathbb{A}^1$-homotopy equivalence. A third one, the singular-complex construction via the algebraic simplices $\Delta^\bullet_S$, will appear in Theorem 1.2.11 below. Do all three coincide? Following Hoyois [Hoy], the answer is yes, and this is a formal consequence of $\mathbb{A}^1$ being an **interval object**.

Definition 1.2.3 (Bipointed object)**.** Let C be a category with finite products. A **bipointed object** is an object $I \in C$ with two global sections $0, 1\colon * \rightrightarrows I$.

Definition 1.2.4 (Three homotopy categories)**.** Let C have finite products and I be a bipointed object.

(i) I**-localization** (nullification): formally invert all projections $X \times I \to X$:

$$C/I \;\coloneqq\; C\left[\{X \times I \to X\}^{-1}\right].$$

(ii) **Localization at I-homotopy equivalences**:

$$C\left[\mathrm{hoeq}_I^{-1}\right].$$

(iii) I**-homotopy category**, via simplicial enrichment:

$$\mathrm{Ho}(C, I), \qquad \mathrm{Map}_{\mathrm{Ho}(C,I)}(X, Y) = \mathrm{colim}_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Map}_C(X \times I^n, Y).$$

In general, there are canonical functors

$$C\left[\mathrm{hoeq}_I^{-1}\right] \longrightarrow \mathrm{Ho}(C, I) \longrightarrow C/I,$$

neither of which is generally an equivalence.

Definition 1.2.5 (Interval object)**.** A bipointed object $I \in C$ is an **interval object** if

(i) $X \times I$ exists for all $X \in C$;

(ii) I is I-contractible: $I \to *$ is an I-homotopy equivalence.

Example 1.2.6.

- The unit interval $[0, 1] \in \mathsf{Top}$ with multiplication $\mu(x, y) = xy$.
- The affine line $\mathbb{A}^1 \in \mathsf{Sm}_S$ with ring multiplication.
- The real line $\mathbb{R}$ and complex plane $\mathbb{C}$ in Top, both with multiplication.

Theorem 1.2.7 (Morel–Voevodsky [MV99])**.** *For any category C and interval object $I \in C$,*

$$C\left[\mathrm{hoeq}_I^{-1}\right] \xrightarrow{\sim} \mathrm{Ho}(C, I) \xrightarrow{\sim} C/I.$$

Proof sketch. The key observation is that I-contractibility makes every projection $\mathrm{pr}_1 : X \times I \to X$ an I-homotopy equivalence: the zero section $(\mathrm{id}_X, 0)$ is its I-homotopy inverse, the required I-homotopy being induced from the contraction $I \to *$ by taking the product with id_X. In particular, projections $X \times I \to X$ already lie in hoeq_I, so any localization that inverts I-homotopy equivalences automatically inverts them, and conversely. The three universal properties defining the categories therefore coincide, and the natural functors in the chain are equivalences. The mapping-anima identification $\mathrm{Ho}(C, I) \xrightarrow{\sim} C/I$ follows from the same I-contraction, which induces a simplicial homotopy on $\mathrm{Map}_C(X \times I^\bullet, Y)$ that becomes invertible after $\mathrm{colim}_{\Delta^{\mathrm{op}}}$. □

Corollary 1.2.8. *The affine line $\mathbb{A}^1$ with sections $0, 1$ and ring multiplication is an interval object in Sm_S and in $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$. Hence the three a priori distinct notions in §1.2 coincide on $\mathcal{P}_\Sigma(\mathsf{Sm}_S)$:*

(i) $\mathbb{A}^1$-localization (inverting $\mathcal{F} \times \mathbb{A}^1 \to \mathcal{F}$);

(ii) localization at $\mathbb{A}^1$-homotopy equivalences;

(iii) the singular complex $\mathrm{L}_{\mathbb{A}^1}$.

***Remark* 1.2.9.** The multiplication $\mu(x, y) = xy$ on $\mathbb{A}^1$ provides the $\mathbb{A}^1$-homotopy from $\mathrm{id}_{\mathbb{A}^1}$ (at $y = 1$) to the constant map 0 (at $y = 0$). This is why 1, not 0, plays the role of the basepoint: the unit of the multiplication is the structure that makes $\mathbb{A}^1$ an interval object.

1.2.2. *Explicit formula for* $\mathrm{L}_{\mathbb{A}^1}$

Corollary 1.2.8 identifies $\mathbb{A}^1$-localization abstractly with the $\mathbb{A}^1$-homotopy category $\mathrm{Ho}(\mathsf{PShv}(\mathsf{Sm}_S), \mathbb{A}^1)$, whose mapping anima are given by $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Map}(- \times \mathbb{A}^\bullet, -)$. We now convert this into an explicit "singular complex" formula, modeled on its topological namesake.

Definition 1.2.10 (Algebraic simplex)**.** The standard n-simplex over S is

$$\Delta^n_S \ \coloneqq\ \mathrm{Spec}\big(\mathcal{O}_S[T_0, \dots, T_n]/(T_0 + \cdots + T_n - 1)\big) \ \subset\ \mathbb{A}^{n+1}_S.$$

Face and degeneracy maps are defined as in topology, yielding a cosimplicial scheme $\Delta^\bullet_S : \mathbf{\Delta} \to \mathsf{Sch}_S$. Note $\Delta^n_S \simeq \mathbb{A}^n_S$ as S-schemes (but not as cosimplicial objects).

Theorem 1.2.11 (Explicit formula for $\mathbb{A}^1$-localization)**.** *For $\mathcal{F} \in \mathsf{PShv}(\mathsf{Sm}_S)$ and $X \in \mathsf{Sm}_S$,*

$$(\mathrm{L}_{\mathbb{A}^1}\mathcal{F})(X) \ \simeq\ \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathcal{F}(X \times \Delta^n_S).$$

Proof. Write $\mathcal{F}'(X) \coloneqq \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathcal{F}(X \times \Delta^n_S)$.

$\mathbb{A}^1$-invariance. The projection $p : X \times \mathbb{A}^1 \to X$ and zero section s induce maps on the cosimplicial objects $X \times \Delta^\bullet_S$ and $X \times \mathbb{A}^1 \times \Delta^\bullet_S$. We have $s^* \circ p^* = \mathrm{id}$, and the multiplication $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ defines an $\mathbb{A}^1$-homotopy $p^* s^* \simeq \mathrm{id}$. Since $\Delta^1_S \simeq \mathbb{A}^1_S$, this $\mathbb{A}^1$-homotopy translates into a simplicial homotopy on $\mathcal{F}(X \times \Delta^\bullet_S)$ between $p^* s^*$ and id, which becomes an equivalence after $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}$. Hence $p^* : \mathcal{F}'(X) \xrightarrow{\sim} \mathcal{F}'(X \times \mathbb{A}^1)$.

Universal property. Let $\mathcal{G}$ be $\mathbb{A}^1$-invariant. Since $\Delta^n_S \simeq \mathbb{A}^n_S$, $\mathbb{A}^1$-invariance forces $\mathcal{G}(X \times \Delta^\bullet_S)$ to have all simplicial structure maps acting as equivalences; hence $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}(X \times \Delta^\bullet_S) \simeq \mathcal{G}(X)$. Using the cartesian-product / internal-hom adjunction $\mathrm{Map}(\mathcal{F}(- \times \Delta^n_S), \mathcal{G}) \simeq \mathrm{Map}(\mathcal{F}, \mathcal{G}(- \times \Delta^n_S))$ for each n, we obtain

$$\mathrm{Map}(\mathcal{F}', \mathcal{G}) \simeq \lim_{\mathbf{\Delta}} \mathrm{Map}(\mathcal{F}(- \times \Delta^n_S), \mathcal{G}) \simeq \mathrm{Map}\big(\mathcal{F}, \lim_{\mathbf{\Delta}} \mathcal{G}(- \times \Delta^n_S)\big) \simeq \mathrm{Map}(\mathcal{F}, \mathcal{G}). \qquad \square$$

***Remark* 1.2.12** (Geometric meaning). Theorem 1.2.11 is the algebro-geometric analogue of the topological singular complex $\mathrm{Sing}_\bullet$. The simplicial anima $\mathcal{F}(X \times \Delta^\bullet_S)$ encodes algebraic homotopy information: 0-simplices are sections, 1-simplices parametrize $\mathbb{A}^1$-paths, 2-simplices parametrize higher homotopies. Taking the colimit over $\boldsymbol{\Delta}^{\mathrm{op}}$ quotients out all $\mathbb{A}^1$-deformations.

Caveat: $\mathrm{L}_{\mathbb{A}^1}$ does not generally preserve Nisnevich sheaves, so the motivic localization $\mathrm{L}_{\mathrm{mot}}$ is *not* simply $\mathrm{L}_{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{Nis}}$; see Remark 1.3.3.

1.3. Motivic Spaces

We now assemble the construction. Let S be qcqs. Since Sm_S is extensive, $\mathcal{P}_\Sigma(\mathsf{Sm}_S) \simeq \mathsf{Shv}_\sqcup(\mathsf{Sm}_S)$, and finite coproduct decompositions are Nisnevich covers, so $\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S) \subset \mathcal{P}_\Sigma(\mathsf{Sm}_S)$.

Definition 1.3.1 (Motivic spaces). The category of **motivic spaces** over S is

$$\mathsf{H}(S) \;:=\; \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S) \cap \mathsf{PShv}_{\mathbb{A}^1}(\mathsf{Sm}_S) \;\subset\; \mathsf{PShv}(\mathsf{Sm}_S),$$

consisting of presheaves of anima satisfying both Nisnevich descent and $\mathbb{A}^1$-invariance. The corresponding localization functor is

$$\mathrm{L}_{\mathrm{mot}} \colon \mathsf{PShv}(\mathsf{Sm}_S) \longrightarrow \mathsf{H}(S).$$

A morphism in $\mathsf{PShv}(\mathsf{Sm}_S)$ is a **motivic equivalence** if $\mathrm{L}_{\mathrm{mot}}$ sends it to an equivalence.

Fact 1.3.2. *(i)* $\mathrm{L}_{\mathrm{Nis}}$ *is left exact. Both* $\mathrm{L}_{\mathbb{A}^1}$ *and* $\mathrm{L}_{\mathrm{mot}}$ *preserve finite products [Hoy14, Proposition C.6].*

(ii) Motivic equivalences form the smallest strongly saturated class in $\mathsf{PShv}(\mathsf{Sm}_S)$ *containing both Nisnevich-local and* $\mathbb{A}^1$*-local equivalences.*

***Remark* 1.3.3** (Non-commutativity of localizations). In general, neither $\mathrm{L}_{\mathrm{mot}} = \mathrm{L}_{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{Nis}}$ nor $\mathrm{L}_{\mathrm{mot}} = \mathrm{L}_{\mathrm{Nis}} \circ \mathrm{L}_{\mathbb{A}^1}$ holds: $\mathrm{L}_{\mathbb{A}^1}$ can destroy Nisnevich descent, while $\mathrm{L}_{\mathrm{Nis}}$ can destroy $\mathbb{A}^1$-invariance. Nevertheless, $\mathsf{H}(S)$ is defined as an intersection, and is therefore independent of the order of localization. Concretely, the motivic localization is computed by ping-ponging:

$$\mathrm{L}_{\mathrm{mot}} \;\simeq\; (\mathrm{L}_{\mathrm{Nis}} \circ \mathrm{L}_{\mathbb{A}^1})^{\circ \mathbb{N}},$$

i.e., by alternating the two localizations transfinitely until the process stabilizes (c.f. [AE16, Theorem 4.27]).

In the spectra-valued setting, however, $\mathrm{L}_{\mathbb{A}^1}$ preserves Nisnevich descent: since Sp is stable, the pullback square in Proposition 1.1.8 is also a pushout, and the Brown–Gersten condition self-dualizes. One thus obtains a functor

$$\mathrm{L}_{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{Nis}} \colon \mathsf{PShv}(\mathsf{Sm}_S, \mathsf{Sp}) \to \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}).$$

1.3.1. Motivic spaces and K-theory

Convention 1.3.4 (Connective vs. non-connective K-theory). Throughout these notes we distinguish two versions of algebraic K-theory by case:

- $\mathrm{K} \colon \mathsf{Sch}^{\mathrm{qcqs,op}} \to \mathsf{Sp}$ denotes **non-connective** algebraic K-theory (Bass–Thomason–Trobaugh); we always use uppercase K for this version.

- $\mathrm{k}\colon \mathsf{Sch}^{\mathrm{qcqs,op}} \to \mathsf{Sp}$ denotes **connective** algebraic K-theory; we always use lowercase k for this version. On smooth schemes over a regular base, $\mathrm{k}(X) \simeq \tau_{\geq 0}\mathrm{K}(X)$ is the connective cover.

Their $\mathbb{A}^1$-localized, homotopy-invariant versions KH and kh inherit the same case convention, as do the motivic spectra KGL and kgl introduced in the stable chapter.

Recall from Remark 1.0.1 that BGL_r, after motivic localization, classifies rank-r vector bundles on smooth affine schemes. Stabilizing with respect to rank via the direct-sum maps $E \mapsto E \oplus \mathcal{O}$ yields

$$\mathrm{BGL}_\infty \;\coloneqq\; \mathrm{colim}_n\, \mathrm{BGL}_n \;\in\; \mathsf{H}(S),$$

which classifies stable vector bundles up to addition of trivial summands; the $\mathbb{Z}$ factor in what follows records the virtual rank. This is the motivic analogue of the classical topological identification $\mathbb{Z}\times\mathrm{BU} \simeq \mathrm{ku}$ for the connective complex K-theory spectrum, and the special case $r = 1$ gives $\mathrm{BGL}_1 = \mathrm{B}\mathbb{G}_m$, classifying line bundles.

Concretely, k satisfies Nisnevich descent (Example 1.1.12) and is $\mathbb{A}^1$-invariant when the base is regular, hence defines an object of $\mathsf{H}(S)$.

Theorem 1.3.5 (Representability of K-theory). *For S regular Noetherian of finite Krull dimension,*

$$\mathrm{L}_{\mathrm{mot}}(\mathbb{Z}\times\mathrm{BGL}_\infty) \;\simeq\; \mathrm{k}.$$

In particular, for affine $U \in \mathsf{Sm}_S$, $[U, \mathbb{Z}\times\mathrm{BGL}_\infty]_{\mathsf{H}(S)} \simeq \mathrm{K}_0(U)$.

Remark **1.3.6** (Affine representability). Although Vect_r is not $\mathbb{A}^1$-invariant on all smooth schemes (Remark 1.0.1), motivic localization still yields the correct values on smooth affine schemes: Asok–Hoyois–Wendt [AHW17] proved that if X is a presheaf on Sm_S such that $\pi_0(X)$ is $\mathbb{A}^1$-invariant on affine schemes and X satisfies affine Nisnevich excision, then $\pi_0(X)(U) \xrightarrow{\sim} [U, X]_{\mathsf{H}(S)}$ for all smooth affine U. Taking $X = \mathrm{BGL}_r$ gives $\pi_0\mathsf{Vect}_r(U) \simeq [U, \mathrm{BGL}_r]_{\mathsf{H}(S)}$ for affine U.

Definition 1.3.7 (Higher K-groups). For $X \in \mathsf{Sm}_S$ and $n \in \mathbb{Z}$, the n-th algebraic K-group of X is

$$\mathrm{K}_n(X) \;\coloneqq\; \pi_n\mathrm{K}(X).$$

In particular, $\mathrm{K}_0(X)$ is the Grothendieck group of the symmetric monoidal category of perfect complexes on X, and $\mathrm{K}_1(X)$ contains $\mathcal{O}_X(X)^\times$ via the determinant map.

Example 1.3.8 (K-theory of a field). For k a field, $\mathrm{K}_0(k) = \mathbb{Z}$ and $\mathrm{K}_1(k) = k^\times$. Matsumoto's theorem identifies $\mathrm{K}_2(k)$ with the Milnor K-group $\mathrm{K}_2^{\mathrm{M}}(k)$, namely the quotient of $k^\times \otimes_{\mathbb{Z}} k^\times$ by the Steinberg relation $\{a, 1-a\} = 0$ for $a \neq 0, 1$. Closed-form descriptions of $\mathrm{K}_n(k)$ for $n \geq 3$ are generally unavailable.

Remark **1.3.9** (The role of regularity). Nisnevich descent of K-theory holds without regularity (Example 1.1.12); $\mathbb{A}^1$-invariance does not. For regular Noetherian X, Quillen proved $\mathrm{K}_n(X) \xrightarrow{\sim} \mathrm{K}_n(X\times\mathbb{A}^1)$ for all $n \geq 0$. For singular schemes this fails: with $R = k[\varepsilon]/(\varepsilon^2)$,

$$\mathrm{K}_1(R[t]) \neq \mathrm{K}_1(R),$$

since $(1+\varepsilon t) \in R[t]^\times$ contributes a class in $\mathrm{K}_1(R[t])$ not coming from $\mathrm{K}_1(R)$.

To restore $\mathbb{A}^1$-invariance on non-regular schemes, Weibel introduced **homotopy K-theory**

$$\mathrm{KH}(X) \;\coloneqq\; \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}\, \mathrm{K}\big(X\times\Delta^n\big),$$

which is precisely $\mathrm{L}_{\mathbb{A}^1}$ applied to K.

Proposition 1.3.10 (Vanishing of negative K-theory on regular schemes). *For X regular Noetherian, $\mathrm{K}_n(X) = 0$ for all $n < 0$. In particular, $\mathrm{k}(X) \simeq \mathrm{K}(X)$ on regular schemes.*

Proof. $\mathsf{Perf}(X) \simeq \mathsf{D}^b(\mathsf{Coh}(X))$ by regularity, and the latter carries a bounded t-structure with Noetherian heart; the claim follows from Antieau–Gepner–Heller [AGH19, Theorem 1.2]. □

***Remark* 1.3.11** (Bass splitting as a precursor to Bott periodicity). Classically, Bass's fundamental theorem yields a natural splitting

$$\mathrm{K}_n(X \times \mathbb{G}_m) \;\simeq\; \mathrm{K}_n(X) \oplus \mathrm{K}_{n-1}(X)$$

for regular X, which is the unstable shadow of a much stronger periodicity: after passing to the stable motivic category and enforcing $\mathbb{P}^1$-periodicity, k is delooped to a motivic ring spectrum $\mathrm{KGL}_S \in \mathsf{CAlg}(\mathsf{SH}(S))$ satisfying Bott periodicity

$$\mathbb{T}_S \otimes \mathrm{KGL}_S \;\simeq\; \mathrm{KGL}_S;$$

see Definition 3.1.10 for the construction. The $\mathbb{G}_m$-direction of Bass's fundamental theorem becomes the $(1,1)$-summand of Bott periodicity; the remaining simplicial S^1-direction comes from the $\mathbb{P}^1$-bundle formula for K-theory, which is itself unstable, but whose combination with Bass's splitting into the single statement $\mathbb{T}_S \otimes \mathrm{KGL} \simeq \mathrm{KGL}$ is genuinely stable in nature.

1.4. Pointed Motivic Spaces

1.4.1. *Basic constructions*

Definition 1.4.1 (Pointed motivic spaces). The category $\mathsf{H}_*(S)$ of **pointed motivic spaces** is the category of pointed objects in $\mathsf{H}(S)$.

The functor $(-)_+ \colon \mathsf{H}(S) \to \mathsf{H}_*(S)$ adds a disjoint basepoint, $X_+ = X \sqcup *$, and is left adjoint to the forgetful functor.

The **smash product** on $\mathsf{H}_*(S)$ is the pushout

$$X \wedge Y \;:=\; \mathrm{cofib}(X \vee Y \to X \times Y),$$

making $\mathsf{H}_*(S)$ symmetric monoidal with unit S_+.

We adopt the basepoint conventions:

- $\mathbb{A}^1$ is pointed at 1;
- $\mathbb{G}_m$ is pointed at 1.

For $\mathbb{P}^1_S$, rather than fixing a basepoint we work throughout with the following object.

Definition 1.4.2 (Tate object). The **Tate object** is

$$\mathbb{T}_S \;:=\; \mathrm{cofib}\big(\infty \colon S \hookrightarrow \mathbb{P}^1_S\big) \;\in\; \mathsf{H}_*(S),$$

i.e. $\mathbb{P}^1_S$ with the section $\infty = [1:0]$ collapsed to the basepoint. We use $\mathbb{T}_S$ throughout in place of $(\mathbb{P}^1_S, \infty)$, both in this chapter and in the stable chapter.

Definition 1.4.3 (Suspension and loops). For $X \in \mathsf{H}_*(S)$,

- the **suspension** $\Sigma X := S^1 \wedge X$, where $S^1 = [1]/\partial[1]$ is the simplicial circle (as a constant sheaf);
- the **loop space** $\Omega X := \mathrm{fib}(* \to X)$.

These form an adjunction $\Sigma \dashv \Omega$.

1.4.2. Bigraded spheres

In $\mathsf{H}_*(S)$ there are two essentially different "circles".

Definition 1.4.4 (Motivic spheres).

- The **simplicial circle** is $S^{1,0} := S^1$ (constant sheaf).
- The **Tate circle** is $S^{1,1} := \mathbb{G}_m$ (pointed at 1).
- The **motivic** (p,q)**-sphere**, for $p \geq q \geq 0$, is

$$S^{p,q} \;:=\; (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge q}.$$

The key computation:

Proposition 1.4.5 (Tate object as a motivic sphere). *In* $\mathsf{H}_*(S)$, $\mathbb{T}_S \simeq S^{2,1} = S^1 \wedge \mathbb{G}_m$.

Proof. Consider the standard Nisnevich square for $\mathbb{P}^1$:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

By Brown–Gersten in the pointed setting, this is a pushout square in $\mathsf{H}_*(S)$; the corner $\mathbb{P}^1$ inherits its basepoint from the cocone (the image of $1 \in \mathbb{G}_m$), and any two rational points of $\mathbb{P}^1$ give equivalent pointed objects in $\mathsf{H}_*(S)$, so the pushout is $\mathbb{T}_S$. Since $\mathbb{A}^1 \simeq *$ in $\mathsf{H}(S)$, both copies of $\mathbb{A}^1$ collapse to the basepoint, and the pushout becomes

$$\mathbb{T}_S \;\simeq\; * \sqcup_{\mathbb{G}_m} * \;\simeq\; \Sigma\mathbb{G}_m \;=\; S^1 \wedge \mathbb{G}_m \;=\; S^{2,1}. \qquad \square$$

***Remark* 1.4.6.** This makes precise the observation that $\mathbb{G}_m$ realizes a kind of "algebraic circle": under Betti realization $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$ is homotopy equivalent to the topological S^1. The motivic sphere $\mathbb{G}_m$ plays the role of $S^{1,1}$, while the simplicial circle S^1 is $S^{1,0}$. Their smash product $\mathbb{T}_S \simeq S^{2,1}$ is the natural "algebraic 2-sphere".

This bigraded structure is what makes motivic homotopy theory richer than classical homotopy theory: the weight direction q encodes Tate twists specific to algebraic geometry.

Proposition 1.4.7 (Punctured affine spaces). *For* $n \geq 1$, $\mathbb{A}^n \setminus \{0\} \simeq S^{2n-1,n}$ *in* $\mathsf{H}_*(S)$ *(with any choice of basepoint, e.g.* $(1,0,\ldots,0)$*).*

Proof. By induction on n. The case $n = 1$ is $\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m \simeq S^{1,1}$. For $n \geq 2$, the Nisnevich square

$$\begin{array}{ccc} (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^{n-1} \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^n \setminus \{0\} \end{array}$$

is a pushout in $\mathsf{H}_*(S)$. Since $\mathbb{A}^1 \simeq *$, the right two terms collapse, yielding

$$\mathbb{A}^n \setminus \{0\} \;\simeq\; \Sigma\big((\mathbb{A}^{n-1} \setminus \{0\}) \wedge \mathbb{G}_m\big) \;=\; S^{1,0} \wedge S^{2n-3,n-1} \wedge S^{1,1} \;=\; S^{2n-1,n}. \qquad \square$$

Corollary 1.4.8 (Thom spaces). *In* $\mathsf{H}_*(S)$,

$$\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}) \;\simeq\; S^{2n,n} \quad \textit{and} \quad \mathbb{P}^n/\mathbb{P}^{n-1} \;\simeq\; S^{2n,n}.$$

Proof. The cofiber sequence $\mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \to \mathbb{A}^n/(\mathbb{A}^n \setminus \{0\})$ together with $\mathbb{A}^n \simeq *$ gives

$$\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}) \;\simeq\; \Sigma(\mathbb{A}^n \setminus \{0\}) \;\simeq\; \Sigma S^{2n-1,n} \;=\; S^{2n,n}.$$

For the second, write $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus \{0\})$ via the standard chart $\mathbb{A}^n \subset \mathbb{P}^n$ and the open complement of the origin of that chart; their intersection is $\mathbb{A}^n \setminus \{0\}$. The resulting Zariski (hence Nisnevich) square

$$\begin{array}{ccc} \mathbb{A}^n \setminus \{0\} & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \mathbb{P}^n \setminus \{0\} & \longrightarrow & \mathbb{P}^n \end{array}$$

is a pushout in $\mathsf{H}_*(S)$. The radial projection from $0 \in \mathbb{A}^n$ realizes $\mathbb{P}^n \setminus \{0\}$ as the total space of an $\mathbb{A}^1$-bundle over the hyperplane at infinity $\mathbb{P}^{n-1}$, hence $\mathbb{P}^n \setminus \{0\} \simeq \mathbb{P}^{n-1}$ in $\mathsf{H}_*(S)$. Collapsing $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ in the pushout therefore identifies $\mathbb{P}^n/\mathbb{P}^{n-1}$ with $\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}) \simeq S^{2n,n}$. □

Chapter 2.

Stable Motivic Homotopy Theory I — Construction and Six Functors

Grothendieck envisioned, long before stable categories were available, that every cohomology theory on schemes should factor through a single universal "category of motives" $\mathrm{Mot}(k)$. In the modern dictionary "studying cohomology" is increasingly understood as "studying six-functor formalisms", and Grothendieck's program becomes the search for a universal such formalism. The motivic stable homotopy category $\mathsf{SH}(S)$ of Morel–Voevodsky is the answer for $\mathbb{A}^1$-invariant cohomology theories, and constructing it together with its six-functor structure is the goal of this chapter.

The starting point is the unstable category $\mathsf{H}_*(S)$ from the previous chapter — pointed objects in $\mathsf{H}(S) = \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S) \cap \mathsf{PShv}_{\mathbb{A}^1}(\mathsf{Sm}_S)$ — which is already rich enough to host a great deal of algebraic geometry. What it lacks is the formal machinery of a stable homotopy theory: fiber and cofiber sequences that coincide, the suspension–loop equivalence $\Sigma \simeq \Omega^{-1}$, and the six-functor formalism that underlies modern thinking about cohomology. All of these become available after one further universal step: formally inverting the Tate object $\mathbb{T}_S \simeq S^1 \wedge \mathbb{G}_m$ under the smash product. The result is $\mathsf{SH}(S)$, the **motivic stable homotopy category**, the central object of this chapter.

Drew–Gallauer [DG22] supplied a precise form of Grothendieck's search: there is a universal presentable six-functor formalism on finite-type schemes which is cohomologically smooth, satisfies excision, and kills the $\mathbb{A}^1$-homology. This universal object is precisely the motivic formalism SH of Morel–Voevodsky. Their proof is constructive and recovers the construction we shall give: the passage to $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_-, \mathsf{Sp})$ is forced by excision, and $\otimes$-inverting $\mathbb{T}_S$ is forced by the triviality of $\mathbb{A}^1$-homology.

It bears emphasizing that SH is not the final word: the axioms above — especially $\mathbb{A}^1$-invariance — exclude important arithmetic-geometric cohomology theories. Prismatic cohomology does not factor through SH, and more generally p-adic cohomology theories in characteristic p require fundamentally different frameworks. Recently, Annala–Iwasa [AI25] have proposed a non-$\mathbb{A}^1$-invariant refinement called *motivic spectra* MotSp, of which SH is the $\mathbb{A}^1$-localization $\mathsf{SH} \simeq \mathrm{L}_{\mathbb{A}^1}\mathsf{MotSp}$; this broader framework recovers algebraic K-theory more directly via a Conner–Floyd-style isomorphism [AHI24a] and is the home of Atiyah duality for arbitrary qcqs derived schemes [AHI24b]. We should therefore think of $\mathsf{SH}(S)$ as the universal coefficient system for $\mathbb{A}^1$-invariant cohomology, not as the definitive answer to Grothendieck's question; the broader MotSp is briefly introduced in Section 2.1.5, but SH remains the protagonist of these notes.

The chapter is organized as follows.

- Section 2.1 constructs $\mathsf{SH}(S)$ as the formal inversion of $\mathbb{T}_S$ in $\mathsf{H}_*(S)$, developing the symmetric-spectrum and telescope models on which the construction rests. We close the section with the Annala–Iwasa generalization MotSp and its relation to SH.
- Section 2.2 sets up the language of six-functor formalisms via the span-category framework of Mann–Scholze, in the 2-categorical form due to Cnossen–Lenz–Linskens.

- Section 2.3 applies this framework to SH via the cdh topology, equipping it with its full six-functor structure (cdh descent as a corollary) and developing the relative duality theory of cohomologically smooth, étale, and proper morphisms à la Heyer–Mann.
- Section 2.4 establishes the Drew–Gallauer universal property — SH is initial among presentable six-functor formalisms which are cohomologically smooth, satisfy excision, and kill $\mathbb{A}^1$-homology — and surveys the principal realizations (ℓ-adic étale, Betti, de Rham, Berkovich, solid).

The next chapter (Chapter 3) takes up the comparisons of $\mathsf{SH}(S)$ with classical structures: with the naive stabilization $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$, with Postnikov-type filtrations, and with the topological theory of orientations and formal group laws.

2.1. Formal inversion

This section gives a careful construction of the stable motivic category $\mathsf{SH}(S)$, together with the technology—symmetric sequences, free algebras, Bousfield localization—needed both for this construction and for the construction of specific motivic spectra in subsequent sections.

The goal of this section is to invert the Tate object $\mathbb{T}_S$ of Definition 1.4.2 under the smash product, producing a presentably symmetric monoidal category $\mathsf{SH}(S)$ that extends the construction chain

$$\mathsf{Sm}_S \xrightarrow{\text{よ}} \mathsf{PShv}(\mathsf{Sm}_S) \xrightarrow{\mathrm{L}_{\mathrm{mot}}} \mathsf{H}(S) \xrightarrow{(-)_+} \mathsf{H}_*(S) \xrightarrow{\sigma^\infty_{\mathbb{T}_S}} \mathsf{SH}(S),$$

from Chapter 1 by one further universal step.

2.1.1. *The universal ⊗-inversion problem*

Let $\mathcal{C} \in \mathsf{CAlg}(\mathsf{Pr}^L)$ be a presentably symmetric monoidal category and let $T \in \mathcal{C}$. We seek the initial presentably symmetric monoidal $\mathcal{C}$-algebra in which the image of T becomes ⊗-invertible: that is, the initial object of $\mathsf{CAlg}(\mathsf{Pr}^L)_{\mathcal{C}/}$ whose structure functor sends T to an invertible object.

Definition 2.1.1. Let $\mathcal{C} \in \mathsf{CAlg}(\mathsf{Pr}^L)$ and $\mathcal{D} \in \mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L)$. Given a Bousfield localization $L\colon \mathcal{D} \to \mathcal{D}'$, we say L is **symmetric monoidal** if the composition $\mathcal{D} \xrightarrow{L} \mathcal{D}' \hookrightarrow \mathcal{D}$ is of the form $A \otimes -$ for some $A \in \mathsf{CAlg}(\mathcal{C})$.

***Remark* 2.1.2.** Symmetric monoidal localizations are also called **⊗-Bousfield localizations** in some references, reflecting the fact that they are precisely the Bousfield localizations whose class of local equivalences is closed under $- \otimes X$ for every $X \in \mathcal{D}$.

Proposition 2.1.3 ([Rob15, Proposition 2.9],[AI25, p. 1.2.2])**.** *There exists a symmetric monoidal Bousfield localization*

$$(-)[T^{-1}]\colon \mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L) \to \mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L).$$

For a $\mathcal{C}$-module $\mathcal{D}$, we refer to $\mathcal{D}[T^{-1}]$ as the **formal inversion** of T in $\mathcal{D}$.

***Remark* 2.1.4.** Since the localization $(-)[T^{-1}]$ is symmetric monoidal, the unit map $u\colon \mathcal{C} \to \mathcal{C}[T^{-1}]$ exhibits $\mathcal{C}[T^{-1}]$ as an idempotent in $\mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L)$, and thus u induces a symmetric monoidal structure on $\mathcal{C}[T^{-1}]$, which in turn induces an equivalence

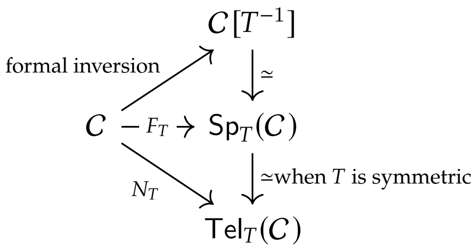
$$\mathsf{Mod}_{\mathcal{C}[T^{-1}]}(\mathsf{Pr}^L) \xrightarrow{\sim} \mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L)[T^{-1}].$$

This motivates the following abstract definition.

Definition 2.1.5. The **motivic stable homotopy category** is the formal inversion of $\mathbb{T}_S$ in $\mathsf{H}_*(S)$:

$$\mathsf{SH}(S) \;:=\; \mathsf{H}_*(S)\big[\mathbb{T}_S^{-1}\big].$$

This definition is too abstract to work with geometrically; we need a concrete model of the localization $(-)[\mathbb{T}_S^{-1}]$.



Here F_T is the free T-spectrum functor built from symmetric sequences (Section 2.1.3), and N_T is the telescope construction; the two agree when T is symmetric, see Proposition 2.1.16.

2.1.2. *Symmetric sequences and free algebras*

Definition 2.1.6 (Symmetric sequences)**.** Let $\mathbb{F} = \mathsf{Fin}^{\simeq}$ denote the groupoid of finite sets and bijections. For $C \in \mathsf{CAlg}(\mathsf{Pr}^L)$, the category of **symmetric sequences** in C is

$$\mathsf{SSeq}(C) := \mathsf{Fun}(\mathbb{F}, C),$$

equipped with the Day convolution symmetric monoidal structure induced by the disjoint-union monoidal structure on $\mathbb{F}$.

In fact, an object in $\mathsf{SSeq}(C)$ is given by a sequence $E_\bullet = (E_0, E_1, \dots)$ in C with a Σ_n-action on E_n. For $X_\bullet, Y_\bullet \in \mathsf{SSeq}(C)$, we have a formula

$$(X_\bullet \otimes Y_\bullet)_n = \bigoplus_{p+q=n} \Sigma_{p+q} \otimes_{\Sigma_p \times \Sigma_q} (X_p \otimes Y_q).$$

From this perspective, one obtains several canonical functors:

- $U\colon \mathsf{SSeq}(C) \to C$, carrying $X_\bullet$ to X_0.
- $F\colon C \to \mathsf{SSeq}(C)$, carrying C to $(C, \varnothing, \varnothing, \dots)$, and left adjoint to U.
- A shift functor $s_-\colon \mathsf{SSeq}(C) \to \mathsf{SSeq}(C)$, given by

$$(s_- Y)_\bullet := (Y_1, Y_2, \dots).$$

- s_- admits a left adjoint s_+, given by

$$(s_+ Y)_\bullet := \big(\varnothing,\, Y_0,\, \dots,\, \Sigma_n \otimes_{\Sigma_{n-1}} Y_{n-1},\, \dots\big).$$

***Remark* 2.1.7** (Notation)**.** The notation s_+ (left adjoint, "shift up") and s_- (right adjoint, "shift down") is borrowed from [AI25, p. 1.3.1]. We will see in Lemma 2.1.12 that, after passing to $\mathsf{Sp}_T(C)$, the localized s_- becomes equivalent to $T \otimes -$ and s_+ becomes equivalent to $\mathrm{Map}(T, -)$. Despite this, we will retain the traditional notation σ^∞ and ω^∞ for the resulting suspension spectrum / underlying object adjunction, in line with the analogy with classical spectra; see the discussion after Definition 2.1.11.

Definition 2.1.8 (Symmetric algebra). For $X \in \mathcal{C}$, write

$$\mathrm{Sym}(X) \;\coloneqq\; \bigoplus_{n\geq 0} \big(X^{\otimes n}\big)_{h\Sigma_n} \;\in\; \mathsf{CAlg}(\mathcal{C})$$

for the symmetric algebra generated by X.

2.1.3. *Lax T-spectra, T-spectra, and T-telescopes*

For $T \in \mathcal{C}$, consider the symmetric sequence

$$T_{\mathrm{sym}} \coloneqq s_+F(T) = (\varnothing, T, \varnothing, \varnothing, \dots) \;\in\; \mathsf{SSeq}(\mathcal{C}).$$

We may form its symmetric algebra $\mathrm{Sym}(T_{\mathrm{sym}}) \in \mathsf{CAlg}(\mathsf{SSeq}(\mathcal{C}))$.

Definition 2.1.9 (Lax T-spectra). The category of **lax T-spectra** in $\mathcal{C}$ is

$$\mathsf{Sp}_T^{\mathrm{lax}}(\mathcal{C}) \;\coloneqq\; \mathsf{Mod}_{\mathrm{Sym}(T_{\mathrm{sym}})}\big(\mathsf{SSeq}(\mathcal{C})\big) \;\in\; \mathsf{CAlg}(\mathsf{Pr}^L)_{\mathcal{C}/}.$$

The adjunction (F, U), together with the free-module functor $\mathrm{Sym}(T_{\mathrm{sym}}) \otimes -$, induces an adjunction

$$\mathcal{C} \;\underset{U}{\overset{F_T}{\rightleftarrows}}\; \mathsf{Sp}_T^{\mathrm{lax}}(\mathcal{C})$$

We may therefore use U to study the structure of $\mathsf{Sp}_T^{\mathrm{lax}}(\mathcal{C})$.

Lemma 2.1.10. *For each $n \geq 0$, $U_n\big(\mathrm{Sym}(T_{\mathrm{sym}})\big) \simeq T^{\otimes n}$.*

Proof. Since $T_{\mathrm{sym}} = s_+F(T)$ is concentrated at level 1, the Day convolution formula forces $T_{\mathrm{sym}}^{\otimes m}$ to be concentrated at level m; hence only the summand $\mathrm{Sym}^n(T_{\mathrm{sym}})$ of $\mathrm{Sym}(T_{\mathrm{sym}})$ contributes at level n, giving

$$U_n\big(\mathrm{Sym}(T_{\mathrm{sym}})\big) \;\simeq\; U_n\big(\mathrm{Sym}^n(s_+F(T))\big) \;\simeq\; \big(\Sigma_n \otimes T^{\otimes n}\big)_{h\Sigma_n}.$$

The Σ_n-action on $\Sigma_n \otimes T^{\otimes n}$ is diagonal, so the homotopy orbit is equivalent to $T^{\otimes n}$. □

This lemma shows that lax T-spectra "see" $T^{\otimes n}$ at level n; consequently, by Bousfield localizing at the relevant bonding maps, we will obtain a concrete model of T-spectra from $\mathsf{Sp}_T^{\mathrm{lax}}(\mathcal{C})$.

Definition 2.1.11 (T-spectra). The category of **T-spectra** in $\mathcal{C}$ is the accessible localization

$$\mathrm{L}_{\mathsf{Sp}}\colon \mathsf{Sp}_T^{\mathrm{lax}}(\mathcal{C}) \longrightarrow \mathsf{Sp}_T(\mathcal{C}),$$

spanned by lax T-spectra E for which the structure map $E \to \underline{\mathrm{Map}}(T, s_-E)$ is an equivalence.

More concretely, an object of $\mathsf{Sp}_T(\mathcal{C})$ is a sequence $E_\bullet = (E_0, E_1, \dots)$ in $\mathcal{C}$ with Σ_n-action on E_n, together with bonding equivalences

$$E_n \xrightarrow{\sim} \underline{\mathrm{Map}}(T, E_{n+1}) \qquad (n \geq 0).$$

By [Lur17, Proposition 4.1.7.4], $\mathsf{Sp}_T(\mathcal{C})$ admits a unique presentably symmetric monoidal structure for which L_{Sp} is symmetric monoidal.

Lemma 2.1.12 ([AI25, p. 1.3.13]). *On $\mathsf{Sp}_T(\mathcal{C})$, there are natural equivalences*

$$s_- \simeq T \otimes -, \qquad \mathrm{L}_{\mathsf{Sp}}\, s_+ \simeq \underline{\mathrm{Map}}(T, -),$$

and T acts as an equivalence on $\mathsf{Sp}_T(\mathcal{C})$. In particular, $\mathrm{L}_{\mathsf{Sp}}\, s_+$ and s_- are mutually inverse equivalences of $\mathsf{Sp}_T(\mathcal{C})$.

The adjunction $F_T \dashv U$ can be derived to the following adjunction:

$$\mathcal{C} \underset{\omega^\infty}{\overset{\sigma^\infty}{\rightleftarrows}} \mathsf{Sp}_T(\mathcal{C})$$

where $\omega^\infty E = E_0$ on the underlying symmetric sequence. The notation is consistent with the analogy with classical spectra: σ^∞ sends an object of $\mathcal{C}$ to its T-suspension spectrum, while ω^∞ extracts the underlying 0-th level. Since T becomes invertible in $\mathsf{Sp}_T(\mathcal{C})$, this adjunction extends naturally: for each $j \in \mathbb{Z}$ one has an adjunction $\sigma^{\infty-j} \dashv \omega^{\infty-j}$, with $\omega^{\infty-j}E$ the "j-th level" of the spectrum.

Corollary 2.1.13 (Explicit description of T-spectra). *The data of an object $E \in \mathsf{Sp}_T(\mathcal{C})$ is equivalent to:*

- *a sequence of objects $\big(E(j)\big)_{j\in\mathbb{Z}}$ in $\mathcal{C}$, with $E(j) := \omega^{\infty-j}E$;*
- *equivalences $E(j) \xrightarrow{\sim} \underline{\mathrm{Map}}(T, E(j+1))$ for every $j \in \mathbb{Z}$.*

A commutative algebra structure on E corresponds further to compatible graded multiplications $E(i) \otimes E(j) \to E(i+j)$.

Proof. The first two clauses follow from Lemma 2.1.12; the multiplicative statement is dealt with in §3.1.1. □

Proposition 2.1.14 (Formal inversion and T-spectra, [AI25, p. 1.3.14]). *There is a categorical equivalence*

$$\mathcal{C}[T^{-1}] \simeq \mathsf{Sp}_T(\mathcal{C}).$$

Recall that in stable homotopy theory, stabilization is realized as a colimit. We now apply the same idea to give a more direct construction of $\mathsf{SH}(S)$: namely, by writing $\mathcal{C}[T^{-1}]$ as a colimit along $T \otimes -$ in $\mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L)$. This construction, however, requires an additional symmetry condition on T; see Proposition 2.1.16.

Definition 2.1.15. The category of T**-telescopes** in $\mathcal{C}$ is given by the following colimit in $\mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L)$:

$$\mathsf{Tel}_T(\mathcal{C}) := \operatorname{colim}\big(\mathcal{C} \xrightarrow{T\otimes -} \mathcal{C} \xrightarrow{T\otimes -} \cdots\big).$$

Proposition 2.1.16 (Telescope comparison, [AI25, p. 1.6.3]). *If T is **symmetric** in the sense that the cyclic permutation*

$$T^{\otimes 3} \longrightarrow T^{\otimes 3}, \qquad a \otimes b \otimes c \longmapsto b \otimes c \otimes a,$$

is homotopic to the identity in $\mathcal{C}$, then there is an equivalence

$$\mathcal{C}[T^{-1}] \simeq \mathsf{Sp}_T(\mathcal{C}) \simeq \mathsf{Tel}_T(\mathcal{C})$$

in $\mathsf{Mod}_{\mathcal{C}}(\mathsf{Pr}^L)$.

2.1.4. *Specialization to motivic homotopy theory*

We now apply the preceding machinery to motivic homotopy theory. The input is the pointed motivic homotopy category $\mathsf{H}_*(S)$ from Chapter 1, equipped with the smash product, and the Tate object $\mathbb{T}_S$.

Proposition 2.1.17 ($\mathbb{T}_S$ is symmetric). *The cyclic permutation $\sigma\colon \mathbb{T}_S^{\wedge 3} \to \mathbb{T}_S^{\wedge 3}$ is $\mathbb{A}^1$-homotopic to the identity in $\mathsf{H}_*(S)$.*

Proof. By Proposition 1.4.5, $\mathbb{T}_S \simeq \mathbb{P}^1_S$. It suffices to show that a transposition on $\mathbb{P}^1 \wedge \mathbb{P}^1$ is $\mathbb{A}^1$-homotopic to $-\mathrm{id}$, since a 3-cycle in Σ_3 is a composite of two transpositions and acts as $(-\mathrm{id})^2 = \mathrm{id}$.

Consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1 \wedge \mathbb{P}^1$ via its action on the two factors. Each elementary unipotent $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$) is $\mathbb{A}^1$-homotopic to the identity via the family parametrized by $t \in \mathbb{A}^1$, and $\mathrm{SL}_2(\mathbb{Z})$ is generated by such matrices; hence every element of $\mathrm{SL}_2(\mathbb{Z})$ acts $\mathbb{A}^1$-homotopically as the identity. In particular, taking

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

we have $A \simeq \mathrm{id}$. The algebraic factorization

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore gives

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \simeq \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence the transposition τ on $\mathbb{P}^1 \wedge \mathbb{P}^1$ is $\mathbb{A}^1$-homotopic to $(-\mathrm{id}) \wedge \mathrm{id} = -\mathrm{id}$. □

Corollary 2.1.18 (Telescope model of $\mathsf{SH}(S)$). *Applying Proposition 2.1.16 to $(\mathsf{H}_*(S), \mathbb{T}_S)$ yields*

$$\mathsf{SH}(S) \simeq \operatorname{colim}\Big(\mathsf{H}_*(S) \xrightarrow{-\wedge\mathbb{T}_S} \mathsf{H}_*(S) \xrightarrow{-\wedge\mathbb{T}_S} \cdots\Big)$$

in $\mathsf{Mod}_{\mathsf{H}_*(S)}(\mathsf{Pr}^L)$.

Terminology 2.1.19. Objects of $\mathsf{SH}(S)$ are called **($\mathbb{A}^1$-invariant) motivic spectra**.

Proposition 2.1.20 (Functoriality of SH). *The assignments $S \mapsto \mathsf{SH}(S)$ and $(f\colon T \to S) \mapsto (f^*\colon \mathsf{SH}(S) \to \mathsf{SH}(T))$, where f^* is the symmetric monoidal pullback, define a functor*

$$\mathsf{SH}\colon (\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \longrightarrow \mathsf{CAlg}(\mathsf{Pr}^L_{\mathrm{st}}).$$

Stability follows from $\mathbb{T}_S \simeq S^1 \wedge \mathbb{G}_m$: inverting $\mathbb{T}_S$ inverts the simplicial S^1, hence $\mathsf{SH}(S)$ is stable.

Putting together all the constructions from Chapter 1 and this section, the motivic spectrum functor $M_S\colon \mathsf{Sm}_S \to \mathsf{SH}(S)$ factors as the composite

$$\mathsf{Sm}_S \xrightarrow{\text{よ}} \mathsf{PShv}(\mathsf{Sm}_S) \xrightarrow{\mathrm{L_{mot}}} \mathsf{H}(S) \xrightarrow{(-)_+} \mathsf{H}_*(S) \xrightarrow{\sigma^\infty_{\mathbb{T}_S}} \mathsf{SH}(S),$$

where each step realizes a universal property:

- よ: free cocompletion;
- $\mathrm{L}_{\mathrm{mot}}$: free imposition of Nisnevich descent and $\mathbb{A}^1$-invariance;
- $(-)_+$: free adjunction of a basepoint;
- $\sigma^{\infty}_{\mathbb{T}_S}$: free $\otimes$-inversion of $\mathbb{T}_S$ (Definition 2.1.5).

Concretely, $M_S(X) = \sigma^{\infty}_{\mathbb{T}_S}(\mathrm{L}_{\mathrm{mot}}(X)_+)$ for $X \in \mathsf{Sm}_S$. The subscript $\mathbb{T}_S$ distinguishes this one-step passage from $\mathsf{H}_*(S)$ to $\mathsf{SH}(S)$ from the generic formal-inversion functor σ^{∞} of Section 2.1.3 and from the simplicial-suspension functor Σ^{∞} introduced below.

***Remark* 2.1.21.** Since every step can be upgraded to a symmetric monoidal functor, we have $M_S(S) \simeq \mathbb{1}_S \in \mathsf{SH}(S)$ and $M_S(X \times_S Y) \simeq M_S(X) \otimes M_S(Y)$.

One also has the following properties:

- the map $M_S(\mathbb{A}^1) \to M_S(S)$ induced by $\mathbb{A}^1 \to S$ is an equivalence;
- the Tate motive $\mathbb{T}_S$ is the cofiber of $M_S(S) \xrightarrow{\infty} M_S(\mathbb{P}^1_S)$, and is invertible by construction;
- M_S carries Nisnevich distinguished squares in Sm_S to cartesian squares in $\mathsf{SH}(S)$.

From the construction above, $\mathsf{SH}(S)$ is the initial presentably symmetric monoidal stable category satisfying the properties above.

Recall from Definition 1.4.4 that there are two types of "circles" in $\mathsf{H}_*(S)$. After stabilization the simplicial circle becomes the categorical suspension [1], computed by the pushout

$$\begin{array}{ccc} E & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & E[1]. \end{array}$$

Combined with the unstable identity $\mathbb{T}_S \simeq S^1 \wedge \mathbb{G}_m$ in $\mathsf{H}_*(S)$, the bigraded sphere $S^{i,j} \in \mathsf{SH}(S)$ can therefore be expressed as

$$S^{i,j} \;\simeq\; \mathbb{T}_S^{\otimes j}[i-2j],$$

and for any $E \in \mathsf{SH}(S)$ we write $\Sigma^{i,j}E := E \otimes S^{i,j}$.

Instead of inverting $\mathbb{T}_S$ in a single step, one may first stabilize $\mathsf{H}_*(S)$ with respect to the simplicial circle S^1. This produces the category of $\mathbb{A}^1$-invariant Nisnevich sheaves of spectra

$$\mathsf{Sp}(\mathsf{H}_*(S)) \;\simeq\; \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}),$$

together with the simplicial suspension functor $\Sigma^{\infty} \colon \mathsf{H}_*(S) \to \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$. Since S^1 is already invertible in $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$, inverting $\mathbb{T}_S$ there is equivalent to inverting $\mathbb{G}_m$; write

$$\sigma^{\infty} \colon \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \longrightarrow \mathsf{SH}(S)$$

for the resulting formal-inversion functor, and ω^{∞} for its right adjoint. We then have the factorization

$$\sigma^{\infty}_{\mathbb{T}_S} \;\simeq\; \sigma^{\infty} \circ \Sigma^{\infty} \colon \mathsf{H}_*(S) \longrightarrow \mathsf{SH}(S),$$

and the identification $\mathsf{SH}(S) \simeq \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})[\mathbb{G}_m^{-1}]$ [ABH26, p. 2.2.3]. The lax symmetric monoidal right adjoint

$$\omega^{\infty} \colon \mathsf{SH}(S) \longrightarrow \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$$

is explicitly described by $\omega^{\infty}(E) = \mathrm{map}_{\mathsf{SH}(S)}(M_S(-), E)$.

***Remark* 2.1.22.** Objects of $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ are also known as **motivic S^1-spectra**, and this category is sometimes denoted $\mathsf{SH}^{S^1}(S)$; we will not use this alternative notation.

2.1.5. *The Annala–Iwasa motivic spectra category* MotSp

The construction of $\mathsf{SH}(S)$ given above incorporates $\mathbb{A}^1$-invariance from the start: we $\otimes$-invert $\mathbb{T}_S$ *within* the $\mathbb{A}^1$-localized world $\mathsf{H}_*(S)$. A more recent perspective due to Annala–Iwasa [AI25] reverses this priority: one performs the $\mathbb{P}^1$-inversion in a much larger, non-$\mathbb{A}^1$-invariant base, and recovers SH at the end as the $\mathbb{A}^1$-localization. The resulting non-$\mathbb{A}^1$-invariant category MotSp is the natural home of recent advances in motivic Atiyah duality [AHI24b] and the algebraic Conner–Floyd theorem [AHI24a], and we record its definition here for use later in these notes.

The base category is the elementary-blowup-excisive Zariski sheaf category. Recall that an **abstract blowup square** (Definition 2.3.1) consists of a closed immersion $Z \hookrightarrow X$ together with a proper map $\tilde{X} \to X$ that is an isomorphism over $X \setminus Z$, and that an **elementary blowup square** is the special case where $\tilde{X} \to X$ is the blowup along a regularly immersed Z.

Definition 2.1.23 (Elementary blowup excision, [AHI24a, Definition 2.1])**.** A presheaf $\mathcal{F} : (\mathsf{Sm}_S)^{\mathrm{op}} \to \mathsf{Sp}$ **satisfies elementary blowup excision** (**ebu**) if it sends every elementary blowup square in Sm_S to a pullback square. We write

$$\mathcal{P}_{\mathrm{Zar,ebu}}(\mathsf{Sm}_S, \mathsf{Sp}) \subseteq \mathsf{PShv}(\mathsf{Sm}_S, \mathsf{Sp})$$

for the full subcategory of Zariski sheaves of spectra satisfying elementary blowup excision.

Definition 2.1.24 (Annala–Iwasa–Hoyois motivic spectra, [AI25, §3.2], [AHI24a, §2])**.** The **(non-$\mathbb{A}^1$-invariant) motivic spectra over** S are

$$\mathsf{MotSp}(S) := \mathcal{P}_{\mathrm{Zar,ebu}}(\mathsf{Sm}_S, \mathsf{Sp})_*\big[(\mathbb{P}^1_S)^{-1}\big],$$

the formal inversion of the pointed projective line $(\mathbb{P}^1_S, \infty)$ in pointed Zariski sheaves of spectra satisfying elementary blowup excision. (Annala–Iwasa originally defined MotSp in [AI25, §3.2] via Zariski sheaves alone; the additional ebu enforcement, which is the form we use, is the variant adopted in [AHI24a, §2].)

***Remark* 2.1.25** (Three differences from $\mathsf{SH}(S)$)**.** Comparing Definition 2.1.24 with the construction of $\mathsf{SH}(S)$ above, the differences are:

- *Topology.* The base topology is Zariski, not Nisnevich, but the Nisnevich descent is recovered (only generically) from elementary blowup excision plus base-change of pullbacks.
- *Excision.* Elementary blowup excision replaces the full proper-excision condition of the cdh topology, and is strictly weaker.
- *No $\mathbb{A}^1$-invariance.* Crucially, $\mathsf{MotSp}(S)$ is *not* required to be $\mathbb{A}^1$-invariant. This is the main novelty.

Proposition 2.1.26 (SH is the $\mathbb{A}^1$-localization of MotSp, [AI25, §3.2])**.** *There is a canonical $\mathbb{A}^1$-localization functor*

$$\mathrm{L}_{\mathbb{A}^1} : \mathsf{MotSp}(S) \longrightarrow \mathsf{SH}(S)$$

realising $\mathsf{SH}(S)$ *as the full subcategory of $\mathbb{A}^1$-invariant objects in* $\mathsf{MotSp}(S)$*. Equivalently,*

$$\mathsf{SH}(S) \simeq \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_*\big[(\mathbb{P}^1_S)^{-1}\big] \simeq \mathrm{L}_{\mathbb{A}^1}\mathsf{MotSp}(S).$$

The advantage of MotSp over SH is that genuine algebraic K-theory $\mathrm{K}(-)$, which is *not* $\mathbb{A}^1$-invariant on singular schemes, lives natively in MotSp rather than only in its $\mathbb{A}^1$-localized homotopy variant KH. The Annala–Hoyois–Iwasa Conner–Floyd theorem [AHI24a, Theorem 1.2] states that, on qcqs derived schemes,

$$\mathrm{MGL}^{**}(X) \otimes_{\mathrm{L}} \mathbb{Z}[\beta^{\pm 1}] \simeq \mathrm{K}^{**}(X),$$

where L is the Lazard ring; the equivalence sees the full K-theory rather than KH precisely because the universal oriented motivic ring spectrum MGL is now constructed inside MotSp.

***Warning* 2.1.27** (Scope of these notes). Outside this subsection, the notes work entirely with SH, since most structural results we develop — the Bachmann–Elmanto–Morrow comparison, the slice conjecture, the homotopy t-structure — assume $\mathbb{A}^1$-invariance. Refining these to MotSp is an active area of research; we point the reader to [AI25; AHI24b; AHI24a] for the current state of the art.

2.2. Six-functor formalisms

Having constructed SH as a functor $(\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathsf{CAlg}(\mathsf{Pr}^L_{\mathrm{st}})$, we now extend it to a full six-functor formalism. The present section sets up the abstract framework: general definition (Section 2.2.1), recollement and base change (Section 2.2.2), and the span-category encoding via the Cnossen–Lenz–Linskens universal property (Section 2.2.3). The application of this framework to SH, including cdh descent and the Heyer–Mann theory of cohomologically smooth, étale, and proper morphisms, is the subject of Section 2.3; the universal property of SH in its precise Drew–Gallauer form, together with the principal realizations, is taken up in Section 2.4.

2.2.1. *Six-functor formalisms*

A modern viewpoint, originating in Grothendieck's school, organizes "cohomology with coefficients" on a class of geometric objects C as a single coherent package: an assignment $X \mapsto \mathsf{D}(X)$ of a category of "coefficient systems" to each object, together with six functors relating these categories under morphisms.

An informal definition. A **six-functor formalism** on C assigns to each $X \in C$ a presentable symmetric monoidal stable category $\mathsf{D}(X)$, and to each morphism $f\colon Y \to X$ in C four functors

$$f^* \dashv f_*\colon \mathsf{D}(Y) \rightleftarrows \mathsf{D}(X), \qquad f_! \dashv f^!\colon \mathsf{D}(X) \rightleftarrows \mathsf{D}(Y),$$

together with the symmetric monoidal product $\otimes$ and internal hom $\underline{\mathrm{Map}}$ on each $\mathsf{D}(X)$, such that:

- f^* and $f_!$ are colimit-preserving and symmetric monoidal (resp. projection-formula compatible);
- base change: for every pullback square in C,

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

  the canonical maps $g^* f_! \to f'_! g'^*$ and $f^! g_* \to g'_* f'^!$ are equivalences;
- projection formula: $f_!(A \otimes f^* B) \simeq f_!(A) \otimes B$ for $A \in \mathsf{D}(Y)$, $B \in \mathsf{D}(X)$.

Coherent versions of these data live as a 2-functor out of a span 2-category; the span-category formulation is the precise way of encoding all the higher coherences and is the subject of Section 2.2.3.

The six functors carry distinct geometric content:

- f^* is *pullback*, f_* the *right-derived direct image*, $\otimes$ the *symmetric monoidal product* encoding cup products, and $\underline{\mathrm{Map}}$ the right adjoint to $\otimes$.
- $f_!$ is the *exceptional / proper-supported pushforward*: it agrees with f_* when f is proper and with $j_\#$ (extension by zero) when f is an open immersion.
- $f^!$ is the *exceptional pullback*, encoding orientation and duality data; in many examples, for f smooth of relative dimension d, $f^! \simeq f^* \otimes \omega_f$ with ω_f the relative dualizing complex.

The three operations f_*, $f^!$, $\underline{\mathrm{Map}}$ are *right adjoints* of f^*, $f_!$, $\otimes$ respectively, hence determined by the others. Modern formulations therefore typically specify only the three left functors $(f^*, f_!, \otimes)$ (a so-called **three-functor formalism**) and recover the rest by adjunction.

There are many examples: locally compact Hausdorff topological spaces [Sch26, Lecture VII], ℓ-adic étale cohomology of schemes [Sch26, Appendix to Lecture VII], $\mathcal{D}$-modules on smooth varieties in characteristic zero (and the closely related Betti and de Rham ring stacks [Sch26, Lecture X]), arithmetic $\mathcal{D}$-modules [Sch26, Lecture XII], solid quasi-coherent sheaves on analytic stacks [Sch26, Lecture IX][CS23], the p-adic formalism on rigid-analytic varieties [Man22], and the motivic formalism SH [Sch26, Lecture XI] — the universal $\mathbb{A}^1$-invariant one by Drew–Gallauer (Theorem 2.4.1). Coherent cohomology $X \mapsto \mathsf{QCoh}(X)$ on (derived) schemes only comes with the four "easy" functors $(f^*, f_*, \otimes, \underline{\mathrm{Map}})$; on proper maps the exceptional pair reduces to Grothendieck–Serre duality $f_! = f_*$ [Lur18, §6.4], and the full six-functor package is delicate, requiring either restrictions on the class E or the passage to solid coefficients [Sch26, Lecture VIII].

For pedagogical introductions complementing the present chapter, we recommend Scholze's lecture notes [Sch26], Gallauer's mini-course [Gal21] (the source of much of the present subsection's exposition), and the textbook treatment in Cisinski–Déglise [CD19]. The encoding via span 2-categories adopted here is closest to Mann–Scholze [Man22; Sch26] and Cnossen–Lenz–Linskens [CLL25], to which we now turn after recording the recollement and base-change conventions in the next subsection.

2.2.2. *Recollement and base change*

The Cnossen–Lenz–Linskens universal property in the next subsection will be phrased in terms of (I, P)-*biadjointability*, which packages two pieces of data — left base change for I-morphisms and right base change for P-morphisms — together with localization (or recollement). We record these notions here.

Definition 2.2.1 (Recollement, localization axiom; cf. [Aok26, Definition 2.9])**.** A diagram of stable presentable categories

$$\mathcal{C}_U \xleftarrow{j^*} \mathcal{C}_X \xrightarrow{i^*} \mathcal{C}_Z$$

(i.e., a diagram in the 2-category $\mathsf{Pr}^L_{\mathrm{st}}$) is called a **recollement** if the following conditions are satisfied:

(i) The morphism j^* admits a left adjoint $j_!$, and the counit $j_! j^* \to \mathrm{id}$ is an equivalence.

(ii) The morphism i^* admits a right adjoint i_*, and the unit $\mathrm{id} \to i_* i^*$ is an equivalence.

(iii) The composite j^*i_* is zero (equivalently, $i^*j_!$ is zero), and the induced square

$$\begin{array}{ccc} j_!j^* & \longrightarrow & \mathrm{id} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_*i^* \end{array}$$

in $\mathrm{End}(C_X)$ is a pushout.

A functor $\mathsf{D}\colon (\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathsf{Pr}^L_{\mathrm{st}}$ satisfies the **localization axiom** if for every $X \in \mathsf{Sch}^{\mathrm{qcqs}}$, every closed immersion $i\colon Z \hookrightarrow X$ with open complement $j\colon U \hookrightarrow X$ gives rise to a recollement

$$\mathsf{D}(U) \xleftarrow{j^*} \mathsf{D}(X) \xrightarrow{i^*} \mathsf{D}(Z).$$

Definition 2.2.2 (Base change properties)**.** Let $\mathsf{D}\colon (\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathsf{Pr}^L_{\mathrm{st}}$ be a functor.

- A morphism $g\colon C \to D$ satisfies **left base change** if for every pullback square

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

  the functor $k^*\colon \mathsf{D}(B) \to \mathsf{D}(A)$ admits a left adjoint $k_\sharp$, and the canonical morphism $k_\sharp h^* \to f^*g_\sharp$ is an equivalence.
- A morphism $f\colon B \to D$ satisfies **right base change** if for every pullback square as above, $h^*\colon \mathsf{D}(C) \to \mathsf{D}(A)$ admits a right adjoint h_*, and the canonical morphism $g^*f_* \to h_*k^*$ is an equivalence.

The link between recollement, base change, and Nisnevich/cdh descent will be proved in Proposition 2.3.8 after we set up the cdh topology in Section 2.3.1.

2.2.3. *Span categories and suitable decompositions*

The economical encoding of a six-functor formalism, initiated by Liu–Zheng [LZ24] and brought to its present form by Mann [Man22] and Scholze [Sch26], packages all the operations $(f^*, f_!, \otimes)$ and their coherences into a single lax symmetric monoidal functor on a span category. Given a wide subcategory $E \subset C$ closed under composition and base change, the **span category** $\mathsf{Span}(C, E)$ has the same objects as C and morphisms $X \to Y$ given by spans

$$X \leftarrow Z \xrightarrow{e} Y, \qquad e \in E,$$

composed by pullback; when C has finite products, $\mathsf{Span}(C, E)$ inherits a Cartesian symmetric monoidal structure. A **three-functor formalism** on (C, E) is then a lax symmetric monoidal functor

$$\mathsf{D}\colon \mathsf{Span}(C, E) \longrightarrow \mathsf{Cat},$$

encoding f^* (backward legs), $f_!$ (forward legs), and $\otimes$ (monoidal structure) in a single coherent package; the right adjoints f_*, $f^!$, $\underline{\mathrm{Map}}$ are recovered automatically and are therefore *properties* of D rather than additional data.

The challenge is that constructing a lax monoidal functor out of $\mathsf{Span}(C, E)$ directly is difficult. In practice, one starts from a much simpler piece of data, namely a functor

$$\mathsf{D}_0\colon C^{\mathrm{op}} \longrightarrow \mathsf{CAlg}(\mathsf{Cat})$$

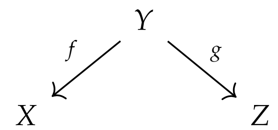
encoding only f^* and $\otimes$, and asks when it extends to $\mathrm{Span}(C, E)$. The key input is a **suitable decomposition** of E in the sense of Mann [Man22, Definition A.5.9]: a pair of wide subcategories $I, P \subset E$ (thought of as "étale-like" and "proper-like" morphisms respectively) such that every $e \in E$ factors as $e = p \circ i$ with $i \in I$ and $p \in P$, subject to mild cancellation and truncation axioms. The extension theorem then reads: if $i^* = \mathrm{D}_0(i)$ admits a left adjoint $i_\sharp$ and $p^* = \mathrm{D}_0(p)$ admits a right adjoint p_* satisfying the expected base change and projection formulae—in particular the mixed Beck–Chevalley equivalence $i_\sharp p'_* \simeq p_* i'_\sharp$—then D_0 admits an essentially unique extension to a three-functor formalism D, in which

$$f_! = p_* i_\sharp \qquad \text{for any factorization } f = p \circ i.$$

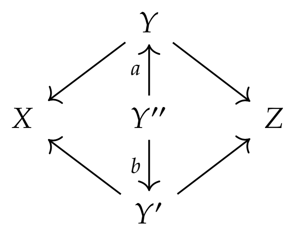
Recently, Cnossen–Lenz–Linskens [CLL25] gave a conceptually streamlined account of this extension theorem. They enhance $\mathrm{Span}(C, E)$ to a 2-category whose 2-morphisms encode the adjunction data of $i_\sharp$ and p_* in a fully coherent way, and they prove that the inclusion $C^{\mathrm{op}} \hookrightarrow \mathrm{Span}_2(C, E)_{I,P}$ is a universal (I, P)-biadjointable functor. In this setup, Mann's extension theorem becomes a formal corollary; as a byproduct, the unit and counit witnessing $i_\sharp \dashv i^*$ are part of the structure of D, rather than non-canonical choices recoverable only after the fact.

Construction 2.2.3 (CLL 2-categorical span category)**.** The 2-category $\mathrm{Span}_2(C, E)_{I,P}$ has:

- *objects:* objects in C;
- 1-*morphisms* $X \to Z$: spans



  with $g \in E$;
- 2-*morphisms* between two such spans: diagrams



  with $a \in P$ and $b \in I$.

Theorem 2.2.4 ([CLL25, Theorem 5.17])**.** *Let C be a category with finite products and let $I, P \subseteq E \subseteq C$ be wide subcategories closed under base change such that*

(i) I and P are left cancellable;

(ii) every map in E factors as a map in I followed by a map in P;

(iii) every map in $I \cap P$ is truncated.

Let $\mathrm{D}_0 \colon C^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat})$ be a functor satisfying:

(i) for every $i \colon a \to b$ in I, the functor $i^* \colon \mathrm{D}_0(b) \to \mathrm{D}_0(a)$ admits a left adjoint $i_\sharp$ satisfying the Beck–Chevalley condition with respect to pullbacks in C and the projection formula $i_\sharp(- \otimes i^*(-)) \simeq i_\sharp(-) \otimes (-)$;

(ii) for every $p \colon a \to b$ in P, the functor $p^* \colon \mathrm{D}_0(b) \to \mathrm{D}_0(a)$ admits a right adjoint p_* satisfying the dual Beck–Chevalley condition and dual projection formula;

(iii) for every pullback square in C with vertical maps in P and horizontal maps in I, the mixed Beck–Chevalley map $j_\sharp p_* \to q_* i_\sharp$ is an equivalence.

Then the lax symmetric monoidal functor $(C^\times)^{\mathrm{op}} \to \mathsf{Cat}^\times$ corresponding to D_0 admits a unique extension to a lax symmetric monoidal 2-functor

$$\mathsf{D}\colon \mathsf{Span}_2(C, E)^{\otimes}_{I,P} \longrightarrow \mathsf{Cat}^\times.$$

Moreover, for any $e \in E$ with factorization $e = p \circ i$ ($i \in I$, $p \in P$), the exceptional pushforward is $e_! \simeq p_* i_\sharp$.

The above is the Mann-style extension formulation; equivalently, the inclusion $C^{\mathrm{op}} \hookrightarrow \mathsf{Span}_2(C, E)_{I,P}$ is the universal (I, P)-biadjointable functor (in the sense of Definition 2.2.2).

Sketch of proof. The argument proceeds in three steps.

Step 1. One verifies directly that the inclusion $C^{\mathrm{op}} \hookrightarrow \mathsf{Span}_2(C, E)_{I,P}$ is (I, P)-biadjointable, so that it is a candidate for the universal functor.

Step 2. Formal 2-categorical arguments produce an abstract initial (I, P)-biadjointable 2-category $\mathbb{B}(C; I, P)$ together with a comparison functor

$$\mathbb{B}(C; I, P) \longrightarrow \mathsf{Span}_2(C, E)_{I,P}.$$

This functor is essentially surjective on objects and 1-morphisms, so one reduces to checking that it induces equivalences on Hom-categories.

Step 3. The Hom-category comparison reduces in turn to identifying the *free* (I, P)-biadjointable functor, which is computed explicitly as

$$Y \longmapsto \mathsf{Span}\big(E_{/Y},\ P,\ I\big),$$

i.e. the span category of E-morphisms into Y, with legs of type P and I respectively. This identification completes the proof.

□

Thus, to construct a six-functor formalism it suffices to verify the three biadjointability conditions: left base change for I, right base change for P, and the mixed Beck–Chevalley equivalence. All coherences come for free from the universal property.

This is the framework in which we place SH. Taking $C = \mathsf{Sch}^{\mathrm{qcqs}}$, E = morphisms locally of finite presentation, I = open immersions, P = proper morphisms, the six-functor formalism on SH is obtained by verifying that $f^*\colon \mathsf{SH}(X) \to \mathsf{SH}(Y)$ and the smash product satisfy precisely the (I, P)-biadjointability hypotheses; the deep geometric input is smooth base change and proper base change for SH, due to Morel–Voevodsky [MV99] and Ayoub [Ayo07] / Cisinski–Déglise [CD19]. We carry out the verification in Section 2.3.2 below.

2.3. CDH DESCENT AND THE SIX-FUNCTOR FORMALISM ON SH

We now apply the framework of Section 2.2 to the motivic stable category SH. The key technical input is the cdh topology, which couples the Nisnevich topology with abstract blowup squares; we set it up in Section 2.3.1, verify the localization axiom and base-change properties for SH to assemble the resulting six-functor formalism in Section 2.3.2, and then develop the relative duality theory of cohomologically smooth, étale, and proper morphisms in Section 2.3.3.

2.3.1. *cdh topology*

Abstract blowup squares

Definition 2.3.1 (Abstract blowup square). A commutative diagram in $\mathsf{Sch}^{\mathrm{qcqs}}$

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Z & \overset{}{\underset{i}{\hookrightarrow}} & X \end{array}$$

is an **abstract blowup square** if

- i is a closed immersion;
- p is proper;
- p induces an isomorphism $p^{-1}(X \setminus Z)_{\mathrm{red}} \xrightarrow{\sim} (X \setminus Z)_{\mathrm{red}}$;
- $E \simeq (\tilde{X} \times_X Z)_{\mathrm{red}}$.

***Remark* 2.3.2** (Comparison with Nisnevich squares). Abstract blowup squares are structurally parallel to Nisnevich squares:

	Nisnevich square	**Abstract blowup square**
Decomposition of X	$U \overset{\text{open}}{\hookrightarrow} X$	$Z \overset{\text{closed}}{\hookrightarrow} X$
Covering morphism	$V \to X$ étale	$\tilde{X} \to X$ proper
Isomorphism away from	$X \setminus U$	$X \setminus Z$
Typical source	étale neighborhoods	Blowups, normalizations

Both decompose X into a known part and a part needing coverage, then handle the latter via the appropriate class of morphisms.

Example 2.3.3 (cdh cover of the nodal curve). For the nodal curve N with $\mathrm{char}(k) \neq 2$:

$$\begin{array}{ccc} E & \longrightarrow & \tilde{N} \simeq \mathbb{A}^1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Z = \{0\} & \hookrightarrow & N, \end{array}$$

where p is the normalization (finite, hence proper) and $E = \mathrm{Spec}(k[c]/(c^2-1)) \simeq \mathrm{Spec}(k) \sqcup \mathrm{Spec}(k)$ encodes the two tangent directions at the node. For a cdh sheaf $\mathcal{F}$,

$$\mathcal{F}(N) \simeq \mathcal{F}(\mathbb{A}^1) \times_{\mathcal{F}(\{c=1\}\sqcup\{c=-1\})} \mathcal{F}(\{0\}).$$

Thus invariants of the singular curve N are completely determined by their values on the smooth model $\mathbb{A}^1$, the singular point $\{0\}$, and the gluing data over E.

Definition of the cdh topology

Definition 2.3.4 (cdh topology). For X qcqs, the **cdh topology** on $\mathsf{Sch}_X^{\mathrm{qcqs}}$ is the Grothendieck topology generated by

(i) Nisnevich squares (Definition 1.1.6), and

(ii) abstract blowup squares (Definition 2.3.1).

A presheaf $\mathcal{F}$ on $\mathsf{Sch}_X^{\mathrm{qcqs}}$ is a **cdh sheaf** if it sends both types of squares to pullback squares—equivalently, satisfies both Nisnevich excision and proper excision.

Cohomological dimension of the Nisnevich and cdh topoi

We pause to record two structural results — bounds on the homotopy dimension of the Nisnevich and cdh topoi — that are not needed for the construction of the six-functor formalism on SH, but enter crucially in later applications such as the homotopy t-structure of Section 3.1.7 and the comparison with cdh-motivic cohomology of Section 3.2.6. Recall that a topos $\mathcal{X}$ has **homotopy dimension** $\leq d$ if every $(d-1)$-connected sheaf on $\mathcal{X}$ admits a global section.

Theorem 2.3.5 (Clausen–Mathew, [CM21]). *For any qcqs scheme X of finite Krull dimension,*

$$\dim \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Et}_X) \;\leq\; \mathrm{Kr.dim}(X),$$

where the left-hand side is the homotopy dimension of the Nisnevich topos on the small étale site of X.

This generalizes the classical vanishing of Nisnevich cohomology above the Krull dimension (Grothendieck, Kato–Saito) by controlling not only cohomology but the truncation behavior of arbitrary sheaves of anima: on a finite-dimensional scheme, every Nisnevich sheaf of spectra is hypercomplete, and every Postnikov tower converges. The bound is sharp and reflects a structural advantage of the Nisnevich topology over its neighbors: the Zariski topos has the same dimensional bound but is too coarse to see Galois descent, while the étale topos has no analogous finite bound (its cohomological dimension can be infinite, e.g. for fields of large absolute Galois group). Geometrically, X has no non-trivial Nisnevich coverings iff X is henselian local, so Nisnevich stalks are computed at henselian local rings $\mathcal{O}^h_{X,x}$.

The cdh topos admits a parallel bound in terms of the *valuation dimension*: $\mathrm{v.dim}(X) \leq d$ iff for every integral affine open $\mathrm{Spec}(A) \subseteq X$ and every strict chain $A \subseteq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subseteq \mathrm{Frac}(A)$ of valuation subrings of the fraction field, one has $n \leq d$. One always has $\mathrm{v.dim}(X) \geq \mathrm{Kr.dim}(X)$, with equality when X is Noetherian.

Theorem 2.3.6 (Elmanto–Hoyois–Iwasa–Kelly, [EHIK21]). *For any qcqs scheme X,*

$$\dim \mathsf{Shv}_{\mathrm{cdh}}(\mathsf{Sch}_X^{\mathrm{fp}}) \;\leq\; \mathrm{v.dim}(X).$$

Geometrically, X has no non-trivial cdh coverings iff every valuation ring of a function field of X is henselian, so cdh stalks are computed at henselian valuation rings. The valuation dimension is the right invariant because it is $\mathbb{A}^1$-stable ($\mathrm{v.dim}(X \times \mathbb{A}^n) = \mathrm{v.dim}(X) + n$) and well-behaved on non-Noetherian schemes, where Krull dimension fails. Theorem 2.3.6 is the technical input behind several recent advances in motivic cohomology over singular and non-Noetherian bases, including the Elmanto–Morrow construction of motivic cohomology of arbitrary qcqs schemes [EM23] and the comparison with prismatic cohomology of Bachmann–Elmanto–Morrow [BEM25].

***Remark* 2.3.7** (Finitarity). The Nisnevich and cdh topologies are both *finitary*: every covering family admits a finite subcover. As a consequence, sheafification commutes with filtered colimits, the topoi $\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S)$ and $\mathsf{Shv}_{\mathrm{cdh}}(\mathsf{Sch}_S^{\mathrm{fp}})$ are compactly generated by the sheafifications of representables, and on finite-dimensional schemes hypercompletion is automatic by Theorems 2.3.5 and 2.3.6.

2.3.2. *The six-functor formalism for* SH

We now apply the framework of Section 2.2.3 to SH. The geometric input is the localization axiom together with base change for étale and proper morphisms; from these the descent properties of SH (Nisnevich and cdh) follow by the general fact below.

Proposition 2.3.8 (Localization plus base change implies descent). *Let* $\mathsf{D}\colon (\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathsf{Pr}^L_{\mathrm{st}}$ *satisfy the localization axiom (Definition 2.2.1).*

(i) If every étale morphism satisfies left base change (Definition 2.2.2), then D *satisfies Nisnevich descent.*

(ii) If every finitely presented proper morphism satisfies right base change, then D *satisfies abstract blowup descent.*

Proof. We treat abstract blowup descent; the Nisnevich case is similar (and dual). Consider an abstract blowup square

$$\begin{array}{ccc} E & \xrightarrow{k} & \tilde{X} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Z & \underset{i}{\hookrightarrow} & X. \end{array}$$

By assumption, p^* and q^* admit right adjoints p_* and q_*. We must show that the comparison functor

$$(p^*, i^*)\colon \mathsf{D}(X) \longrightarrow \mathsf{D}(\tilde{X}) \times_{\mathsf{D}(E)} \mathsf{D}(Z)$$

is an equivalence. Its right adjoint G sends a pair (A, B) with common image $C \in \mathsf{D}(E)$ to the fiber product

$$G(A, B) \;=\; p_*A \times_{(iq)_*C} i_*B.$$

Conservativity of (p^*, i^*). By the localization axiom, (i^*, j^*) is jointly conservative, where $j\colon U = X \setminus Z \hookrightarrow X$. Since p restricts to an isomorphism over U, we have $j^* \simeq h^*p^*$ (with $h\colon U \hookrightarrow \tilde{X}$ the open complement of k), so (p^*, i^*) is also jointly conservative.

Counit is an equivalence. It remains to show that for every (A, B), the counit $(p^*, i^*) \circ G(A, B) \to (A, B)$ is an equivalence. We verify the two components separately.

(i^*-component.) Since i is a closed immersion, i_* is fully faithful by the localization axiom, so $i^*i_* \simeq \mathrm{id}$. Combined with right base change for p along i, which gives $i^*p_*A \simeq q_*k^*A \simeq q_*C$, the fiber product becomes

$$i^*(p_*A \times_{(iq)_*C} i_*B) \;\simeq\; q_*C \times_{q_*C} B \;\simeq\; B.$$

(p^*-component.) We must show $p^*(p_*A \times_{(iq)_*C} i_*B) \xrightarrow{\sim} A$. By the localization axiom, (k^*, h^*) is jointly conservative on $\mathsf{D}(\tilde{X})$, so it suffices to check after applying each.

Applying k^*: using $k^*p^* = q^*i^*$ and the previous case,

$$k^*p^*(p_*A \times_{(iq)_*C} i_*B) \;\simeq\; q^*B \;\simeq\; C \;\simeq\; k^*A.$$

Applying h^*: since i_*B and $(iq)_*C$ are supported on Z, applying $j^* = h^*p^*$ kills them, leaving j^*p_*A. We must show $j^*p_*A \simeq h^*A$. Consider the diagram of fiber sequences supplied by the localization axiom:

$$\begin{array}{ccccc} i_*i^!p_*A & \longrightarrow & p_*A & \longrightarrow & j_*j^*p_*A \\ \downarrow & & \| & & \downarrow \\ p_*k_*k^!A & \longrightarrow & p_*A & \longrightarrow & p_*h_*h^*A. \end{array}$$

The left vertical arrow is an equivalence by right base change for i along p, and the middle arrow is the identity, hence the right vertical arrow is also an equivalence. Since $ph = j$ we have $p_*h_* = j_*$; because j_* is fully faithful, this gives $j^*p_*A \simeq h^*A$. □

We now state the localization axiom for SH.

Theorem 2.3.9 (Morel–Voevodsky, [MV99]; for the general case, [Hoy17; CD19])**.** *The stable motivic functor* $\mathsf{SH}\colon (\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathsf{Pr}^L_{\mathrm{st}}$ *satisfies the localization axiom. (Morel–Voevodsky proved this over a Noetherian base of finite Krull dimension; the extension to qcqs schemes is due to Hoyois [Hoy17] and Cisinski–Déglise [CD19].)*

Theorem 2.3.10 (Existence: Ayoub [Ayo07], Cisinski–Déglise [CD19]; universal property: Cnossen–Lenz–Linskens [CLL25])**.** *There is an essentially unique lax symmetric monoidal extension making the following diagram commute:*

$$\begin{array}{ccc} (\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} & \xrightarrow{\ \mathsf{SH}\ } & \mathsf{CAlg}(\mathsf{Pr}^L_{\mathrm{st}}) \\ \big\downarrow & \nearrow & \\ \mathsf{Span}_2(\mathsf{Sch}^{\mathrm{qcqs}}, E)^{\otimes}_{I,P} & & \end{array}$$

with E = morphisms locally of finite presentation, I = open immersions, and P = proper morphisms. Explicitly, $\mathsf{Span}_2(\mathsf{Sch}^{\mathrm{qcqs}}, E)_{I,P}$ *has:*

- objects: *quasi-compact quasi-separated schemes;*
- 1-morphisms $X \to Z$*: spans*

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ X & & & & Z \end{array}$$

  with g locally of finite presentation;
- 2-morphisms *between two such spans: diagrams*

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow & \uparrow{\scriptstyle a} & \searrow & \\ X & & Y'' & & Z \\ & \nwarrow & \downarrow{\scriptstyle b} & \nearrow & \\ & & Y' & & \end{array}$$

  with a proper and b an open immersion.

In this extension, the ***exceptional pushforward*** *along $f \in E$ is given by $f_! = p_* i_\sharp$ for any factorization $f = p \circ i$ with i an open immersion and p proper.*

***Remark* 2.3.11** (Structure of the proof)**.** The proof decomposes into a formal step and a deep geometric step.

The *formal step* is the universal property of $\mathsf{Span}_2(-, E)^{\otimes}_{I,P}$ (Theorem 2.2.4): given any lax symmetric monoidal functor $\mathsf{D}_0\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{CAlg}(\mathsf{Cat})$ which is (I, P)-biadjointable and satisfies the projection formula, there is an essentially unique extension to $\mathsf{Span}_2(\mathcal{C}, E)^{\otimes}_{I,P}$. This step is purely 2-categorical and contains no algebro-geometric content.

The *geometric step* is to verify the input hypothesis, namely that SH is (I, P)-biadjointable. This breaks further into three ingredients:

- **Open-immersion base change and the projection formula for $i_\sharp$**: these follow essentially from the construction of SH. More generally, for any smooth $i: T \to S$ the functor $i_\sharp$ on $\mathsf{PShv}(\mathsf{Sm}_{-})$ is computed by restriction along $\mathsf{Sm}_T \to \mathsf{Sm}_S$, and is compatible with base change at the presheaf level; the open-immersion case used below is the specialization to i an open immersion.
- **Localization**: the recollement (i^*, j^*) for closed immersions i with open complement j, i.e. Theorem 2.3.9.
- **Proper base change and the projection formula for p_***: this is the deep input. Ayoub [Ayo07] proved proper base change for projective morphisms; Cisinski–Déglise [CD19] extended this to arbitrary proper morphisms using Chow's lemma and Noetherian approximation, with the localization axiom [MV99] providing the bridge from projective to proper.

Once all three ingredients are in hand, the full six-functor package on SH follows by invoking the universal property (Theorem 2.2.4). The mixed Beck–Chevalley equivalence $i_\sharp p'_* \simeq p_* i'_\sharp$ is not an additional input: it follows from proper base change applied to a pullback square one of whose edges is an open immersion.

By Proposition 2.3.8 together with Theorem 2.3.9 and the proper base change ingredient of Remark 2.3.11, we obtain:

Corollary 2.3.12. *The functor* SH *is a cdh sheaf.*

2.3.3. *Cohomological smoothness, étaleness, and properness*

Three classes of morphisms organize the entire content of any six-functor formalism: the cohomologically smooth, étale, and proper ones. The Drew–Gallauer characterization (Theorem 2.4.1 below) is phrased in their language, and they appear throughout applications. We give the definitions and basic properties here, following Heyer–Mann [HM24].

Throughout this subsection, (C, E) denotes a geometric setup and $\mathsf{D}: \mathsf{Span}(C, E) \to \mathsf{Cat}$ a three-functor formalism (or a six-functor formalism, when stated). We write Cat_2 for the 2-category of categories.

Cohomological smoothness

The motivating example is Poincaré duality.

Theorem 2.3.13 (Poincaré duality, classical). *Let $f: X \to Y$ be a proper map of topological manifolds of relative dimension d (i.e. locally of the form $Y \times D^d \to Y$). Then $f_*: \mathsf{D}(X, \mathbb{Z}) \to \mathsf{D}(Y, \mathbb{Z})$ admits a right adjoint $f^!$, and there is a sheaf $\omega_f := f^!(\mathbb{1}_Y) \in \mathsf{D}(X, \mathbb{Z})$, locally isomorphic to $\mathbb{Z}[d]$, with*

$$f^! \simeq f^* \otimes \omega_f.$$

*The sheaf ω_f is the **dualizing complex** of f.*

A morphism in E which satisfies the analogue of Poincaré duality in D will be called **cohomologically smooth** (or **D-smooth**). Two requirements are immediate:

- There is an isomorphism $f^! \simeq f^* \otimes \omega_f$ for some sheaf ω_f;
- $\omega_f := f^!(\mathbb{1}_Y)$ is invertible.

But this is not yet enough. Consider a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

We expect f' to be D-smooth as well, and we expect the dualizing complexes to be compatible with restriction:

$$g'^*\omega_f \;\simeq\; \omega_{f'}.$$

This compatibility is not automatic from pure category theory; it must be imposed.

Definition 2.3.14 (Cohomologically smooth morphism)**.** A morphism $f\colon X \to Y$ in E is **cohomologically smooth** (or **D-smooth**) if it satisfies:

(i) The natural transformation

$$f^!(\mathbb{1}_Y) \otimes f^*(-) \;\longrightarrow\; f^!(-)$$

is an equivalence; we write $\omega_f := f^!(\mathbb{1}_Y)$ and call it the **dualizing complex** of f.

(ii) The dualizing complex ω_f is $\otimes$-invertible in $\mathsf{D}(X)$.

(iii) For every pullback square as above with $g \in E$, the morphism f' also satisfies ((i))–((ii)), and the canonical map $g'^* f^!(\mathbb{1}_Y) \to f'^!(\mathbb{1}_{Y'})$ is an equivalence.

***Remark* 2.3.15.** Cohomological smoothness in the sense of Definition 2.3.14 is weaker than the geometric notion of a smooth morphism. For instance, in the étale six-functor formalism $X \mapsto \mathsf{Shv}(X_{\text{ét}}, \mathsf{D}_{\text{pf}}(\mathbb{Z}))$, every universal homeomorphism between schemes (a map of underlying topological spaces that remains a homeomorphism after pullback) is cohomologically smooth: étale topology is insensitive to such maps, so they look like isomorphisms at the level of D, and isomorphisms are trivially D-smooth.

The kernel 2-category and Fourier–Mukai transforms

Verifying Definition 2.3.14 directly is awkward because of the base-change clause ((iii)). The next definitions package D-smoothness into a single 2-categorical statement, after which the base-change clause is automatic.

Fix $Y \in C$, and let $C_E \subseteq C$ be the wide subcategory whose morphisms are those in E. The slice $(C_E)_{/Y}$, paired with the class of *all* morphisms, is itself a geometric setup, and D restricts to a functor

$$\mathsf{D}_Y\colon \mathsf{Span}((C_E)_{/Y}, \text{all})^{\otimes} \to \mathsf{Span}(C, E)^{\otimes} \xrightarrow{\mathsf{D}} \mathsf{Cat}^{\times}.$$

Definition 2.3.16 (Fourier–Mukai transform)**.** Given $X_1, X_2 \in (C_E)_{/Y}$ and $K \in \mathsf{D}(X_1 \times_Y X_2)$, the **Fourier–Mukai transform with kernel** K is the functor

$$\mathsf{F}_K\colon \mathsf{D}(X_1) \to \mathsf{D}(X_2), \qquad A \;\longmapsto\; (\pi_2)_!\big(\pi_1^* A \otimes K\big).$$

A span $X_1 \xleftarrow{h} Z \xrightarrow{k} X_2$ over Y induces a Fourier–Mukai transform whose kernel is $K = (h,k)_! \mathbb{1}_Z$.

***Remark* 2.3.17.** The terminology is borrowed from the classical analytic transform $T\colon C^\infty(X) \to C^\infty(Y)$, $f \mapsto \int_X K(x,y) f(x)\,dx$, determined by its kernel K. Definition 2.3.16 categorifies this construction: $\pi_1^* A$ pulls A back to a "function of two variables", $\otimes K$ multiplies pointwise by the kernel, and $(\pi_2)_!$ is integration along the first variable.

Specializing to $f: X \to Y$ in E:

- Setting $X_1 = X$, $X_2 = Y$, and $K_1 = \mathbb{1}_X \in \mathsf{D}(X \times_Y Y) = \mathsf{D}(X)$, the Fourier–Mukai transform is $A \mapsto f_! A$.
- Setting $X_1 = Y$, $X_2 = X$, and $K_2 = \omega_f = f^!(\mathbb{1}_Y) \in \mathsf{D}(Y \times_Y X) = \mathsf{D}(X)$, the transform is $B \mapsto f^* B \otimes \omega_f$.

Idea. The desired identity $f^! \simeq f^* \otimes \omega_f$ should arise by exhibiting the kernels K_1 and K_2 as an adjoint pair in a 2-category whose objects are schemes over Y and whose 1-morphisms are kernels. Since 2-functors preserve adjunctions, the resulting adjunction is automatically compatible with arbitrary base change $g: Y' \to Y$, taking care of clause ((iii)) of Definition 2.3.14 for free.

The invertibility of ω_f is a separate, relatively independent condition; this is what motivates the notion of suaveness below.

Definition 2.3.18 (2-category of kernels)**.** Let $Y \in C$. The **2-category of kernels** $\mathbb{K}_{\mathsf{D},Y}$ is the 2-category whose:

- *Objects* are objects of $(C_E)_{/Y}$, i.e., morphisms $X \to Y$ in E.
- 1-*morphism category* from X_1 to X_2 is $\mathsf{Fun}_Y(X_1, X_2) := \mathsf{D}(X_1 \times_Y X_2)$.
- *Composition* of $M \in \mathsf{Fun}_Y(X_1, X_2)$ and $N \in \mathsf{Fun}_Y(X_2, X_3)$ is

$$N \circ M \;:=\; (\pi_{13})_!\big(\pi_{12}^* M \otimes \pi_{23}^* N\big) \;\in\; \mathsf{D}(X_1 \times_Y X_3),$$

  where the π_{ij} are the projections from $X_1 \times_Y X_2 \times_Y X_3$.

The identity 1-morphism on X is $\Delta_{f!}(\mathbb{1}_X) \in \mathsf{D}(X \times_Y X)$, where $\Delta_f: X \to X \times_Y X$ is the relative diagonal.

Conceptually, $\mathbb{K}_{\mathsf{D},Y}$ is obtained from $\mathsf{Span}((C_E)_{/Y}, \mathrm{all})$, which is closed (being a span category over a category with finite limits) and hence self-enriched, by base-changing the self-enrichment along D_Y; see [Aok26, Definition 2.4].

***Remark* 2.3.19** (The presentable 2-category of kernels)**.** When D takes values in Pr, Aoki [Aok26, Definition 2.5] gives a closely related variant that bypasses enriched category theory entirely. Regarding D_Y as a commutative algebra in $\mathsf{Fun}(\mathsf{Span}((C_E)_{/Y}, \mathrm{all}), \mathsf{Pr})$ via Day convolution, the **presentable 2-category of kernels** is

$$\mathbb{K}^{\mathrm{pres}}_{\mathsf{D},Y} \;:=\; \mathsf{Mod}_{\mathsf{D}_Y}\big(\mathsf{Fun}(\mathsf{Span}((C_E)_{/Y}, \mathrm{all}), \mathsf{Pr})\big).$$

This is the Pr-valued presheaf cocompletion of the enriched kernel 2-category from Definition 2.3.18: by [Aok26, Theorem 2.6],

$$\mathbb{K}^{\mathrm{pres}}_{\mathsf{D},Y} \;\simeq\; \mathsf{PShv}^{\mathsf{Pr}}(\mathbb{K}^{\mathrm{op}}_{\mathsf{D},Y}),$$

so $\mathbb{K}_{\mathsf{D},Y}$ embeds fully faithfully into $\mathbb{K}^{\mathrm{pres}}_{\mathsf{D},Y}$ via the enriched Yoneda embedding. The two formulations are interchangeable for the developments below: properties such as "being a left adjoint" are detected on the Yoneda image, so the kernel-level adjunctions defining suaveness, primness, and smoothness in Definitions 2.3.14 and 2.3.21 can equivalently be formulated in either $\mathbb{K}_{\mathsf{D},Y}$ or $\mathbb{K}^{\mathrm{pres}}_{\mathsf{D},Y}$.

There is a canonical 2-functor

$$\Psi_{\mathsf{D},Y} \;=\; \mathsf{Fun}_Y(Y, -) \;:\; \mathbb{K}_{\mathsf{D},Y} \longrightarrow \mathsf{Cat}_2,$$

sending an object X to $\mathsf{D}(X)$ and a 1-morphism $M \in \mathsf{Fun}_Y(X_1, X_2)$ to the Fourier–Mukai transform $\pi_{1!}\big(M \otimes \pi_2^*(-)\big)$.

For the reader's convenience, we record three facts about adjunctions in 2-categories that we will use freely (proofs are standard, see [HM24, Appendix D]):

- Any 2-functor $F\colon \mathbb{C} \to \mathbb{D}$ preserves adjunctions; in fact, it preserves the entire mate correspondence [HM24, Lemma D.2.7].
- Adjunctions are unique up to a contractible space of choices: any two right adjoints to a given 1-morphism are canonically equivalent.
- A pair (f, g) with candidate unit η and counit ε such that the triangle composites are isomorphisms (not necessarily identities) can be corrected to a strict adjunction by modifying η.

Theorem 2.3.20 (Smoothness via kernels). *A morphism $f : X \to Y$ in E is* D*-smooth if and only if*

(i) The 1-morphism $\mathbb{1}_X \in \mathsf{Fun}_Y(X, Y) = \mathsf{D}(X)$ is a left adjoint in $\mathbb{K}_{\mathsf{D},Y}$. Denote its right adjoint by $\omega_f \in \mathsf{Fun}_Y(Y, X) = \mathsf{D}(X)$.

(ii) ω_f is $\otimes$-invertible in $\mathsf{D}(X)$.

When these hold, $\omega_f \simeq f^!(\mathbb{1}_Y)$.

Proof sketch. ($\Rightarrow$) Suppose f is D-smooth. By Definition 2.3.14((i)), $f^!(-) \simeq f^*(-) \otimes \omega_f$. Combined with the adjunction $f_! \dashv f^!$ this becomes $f_! \dashv f^*(-) \otimes \omega_f$ at the level of functors. Translating to $\mathbb{K}_{\mathsf{D},Y}$, the kernel $\mathbb{1}_X$ corresponds to $f_!$ and ω_f to $f^*(-) \otimes \omega_f$, so $\mathbb{1}_X \dashv \omega_f$ in $\mathbb{K}_{\mathsf{D},Y}$. Invertibility is clause ((ii)).

($\Leftarrow$) Apply $\Psi_{\mathsf{D},Y}$ to the adjunction $\mathbb{1}_X \dashv \omega_f$. Since 2-functors preserve adjunctions, the resulting functors $\Psi_{\mathsf{D},Y}(\mathbb{1}_X) \simeq f_!(-)$ and $\Psi_{\mathsf{D},Y}(\omega_f) \simeq f^*(-) \otimes \omega_f$ form an adjunction in Cat_2. By uniqueness of right adjoints, $f^*(-) \otimes \omega_f \simeq f^!(-)$, which is clause ((i)) of Definition 2.3.14; setting $- = \mathbb{1}_Y$ gives $\omega_f \simeq f^!(\mathbb{1}_Y)$. The base-change clause ((iii)) follows from the fact that $S \mapsto \mathbb{K}_{\mathsf{D},S}$ assembles into a 2-functor $\mathsf{Span}(C, E)^{\otimes} \to \mathsf{Cat}_2^{\times}$ [HM24, Theorem 4.2.4], so the adjunction $\mathbb{1}_X \dashv \omega_f$ is preserved by base change. □

Suaveness and primness

Theorem 2.3.20 decomposes cohomological smoothness as "left-adjointness in $\mathbb{K}_{\mathsf{D},Y}$" plus "invertibility of the dualizing complex". Dropping the invertibility yields broader classes of objects.

Definition 2.3.21 (Suave and prim objects). Let $f : X \to Y$ be in E and $A \in \mathsf{D}(X)$.

(i) A is **f-suave** if, viewed as a 1-morphism $X \to Y$ in $\mathbb{K}_{\mathsf{D},Y}$, it is a left adjoint. Its right adjoint is the **f-suave dual** $\mathsf{SD}_f(A) \in \mathsf{D}(X)$. Write $\mathsf{Suave}_f(X) \subseteq \mathsf{D}(X)$ for the full subcategory of f-suave objects.

(ii) A is **f-prim** if, viewed as a 1-morphism $Y \to X$ in $\mathbb{K}_{\mathsf{D},Y}$, it is a right adjoint. Its left adjoint is the **f-prim dual** $\mathsf{PD}_f(A) \in \mathsf{D}(X)$. Write $\mathsf{Prim}_f(X) \subseteq \mathsf{D}(X)$ for the full subcategory of f-prim objects.

***Remark* 2.3.22** (Origin of the terminology). The terms are due to Scholze (suave) and Hansen (prim); applications typically require D to be a six-functor formalism, although the definition makes sense for three-functor formalisms.

- **Suave** (*relaxed smoothness*): when $A = \mathbb{1}_X$ is f-suave, applying $\Psi_{\mathsf{D},Y}$ to the adjunction $\mathbb{1}_X \dashv \mathsf{SD}_f(\mathbb{1}_X)$ gives $f^*(-) \otimes \mathsf{SD}_f(\mathbb{1}_X) \simeq f^!(-)$. More generally, every f-suave A produces a relation between f^* and $f^!$.
- **Prim** (*primitive properness*): when $A = \mathbb{1}_X$ is f-prim, the dual statement gives $f_!(\mathsf{PD}_f(\mathbb{1}_X) \otimes -) \simeq f_*(-)$. Primness is the phenomenon that f_* is recovered from $f_!$ by twisting; this is strictly weaker than the proper-base-change identity $f_! \simeq f_*$, so it is "primitive" in this sense.

***Remark* 2.3.23** (Stack-theoretic perspective). Aoki [Aok26, Definition 4.5] reformulates these notions at the stack level: a morphism $[X] \to [Y]$ in the presentable kernel 2-category $\mathbb{K}^{\mathrm{pres}}_{\mathsf{D},Y}$ of Remark 2.3.19 is called *weakly suave* if it admits an $[X]$-linear left adjoint, *prim* if it admits an $[X]$-linear right adjoint, and *suave* if it is weakly suave and $[Y]$ is dualizable over $[X]$. Modulo passage between the enriched and presentable kernel 2-categories, Aoki's *weakly suave* matches Definition 2.3.21 verbatim, while his *suave* corresponds to our notion together with dualizability of the suave dual — which by Observation 2.3.31 is exactly the additional input promoting suaveness to cohomological smoothness.

In a precise sense, A is f-suave (or f-prim) iff it is "finite-dimensional relative to f": it admits a reflexive duality.

Proposition 2.3.24 (Reflexive duality). *Let D be a six-functor formalism on (C, E) and $f : X \to Y$ in E.*

(i) If $A \in \mathsf{D}(X)$ is f-suave, then $\mathsf{SD}_f(A)$ is also f-suave (with right adjoint A in $\mathbb{K}_{\mathsf{D},Y}$), and there is a natural isomorphism

$$\mathsf{SD}_f(A) \otimes f^*(-) \;\simeq\; \underline{\mathrm{Map}}\big(A, f^!(-)\big) \colon \mathsf{D}(Y) \to \mathsf{D}(X).$$

Setting $- = \mathbb{1}_Y$ yields $\mathsf{SD}_f(A) \simeq \underline{\mathrm{Map}}(A, f^!(\mathbb{1}_Y))$, exhibiting $\mathsf{SD}_f(A)$ as a relative Verdier dual of A. The bidual identity $\mathsf{SD}_f(\mathsf{SD}_f(A)) \simeq A$ then holds, and SD_f defines a contravariant equivalence

$$\mathsf{SD}_f \colon \mathsf{Suave}_f(X)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Suave}_f(X).$$

(ii) Dually, if $A \in \mathsf{D}(X)$ is f-prim, then $\mathsf{PD}_f(A)$ is also f-prim, with

$$f_!\big(\mathsf{PD}_f(A) \otimes -\big) \;\simeq\; f_*\underline{\mathrm{Map}}(A, -) \colon \mathsf{D}(X) \to \mathsf{D}(Y),$$

and the bidual $\mathsf{PD}_f(\mathsf{PD}_f(A)) \simeq A$ holds. Consequently PD_f defines a contravariant equivalence

$$\mathsf{PD}_f \colon \mathsf{Prim}_f(X)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Prim}_f(X).$$

Proof sketch. The proofs are exercises in the 2-categorical adjunction calculus; we sketch the suave case. The kernel 2-category has a built-in self-duality $\mathbb{K}_{\mathsf{D},Y} \simeq \mathbb{K}^{\mathrm{op}}_{\mathsf{D},Y}$ obtained by reversing 1-morphisms, under which a left adjoint becomes a right adjoint and vice versa; the bidual identity follows. The Hom formula is obtained by applying $\Psi_{\mathsf{D},Y}$ to the adjunction $A \dashv \mathsf{SD}_f(A)$ and unfolding via $f_! \dashv f^!$ together with the definition of internal hom. □

A more concrete characterization of suaveness and primness is available, useful in computations. It rests on the following standard 2-categorical lemma.

Lemma 2.3.25 (Adjoints in 2-categories, [HM24, Proposition D.2.8]). *Let $\mathbb{C}$ be a 2-category and $f : X \to Y$ a 1-morphism. The following are equivalent:*

(i) f is a left adjoint.

(ii) For each $Z \in \{X, Y\}$, the post-composition functor $f_*\colon \mathsf{Fun}_{\mathbb{C}}(Z, X) \to \mathsf{Fun}_{\mathbb{C}}(Z, Y)$, $h \mapsto f \circ h$, admits a right adjoint G_Z, and the canonical morphism $G_Y(\mathrm{id}_Y) \circ f \to G_X(f)$ becomes an isomorphism after applying $\mathrm{Hom}_{\mathsf{Fun}(X,X)}(\mathrm{id}_X, -)$.

(iii) For each $Z \in \mathbb{C}$, f_* admits a right adjoint G_Z, and for every $h\colon Z \to Z'$ in $\mathbb{C}$, the canonical map $h^* G_{Z'} \to G_Z h^*$ is an equivalence of functors.

The right adjoint of f is then $G_Y(\mathrm{id}_Y)\colon Y \to X$.

Proposition 2.3.26 (Concrete criterion for suave/prim objects). *Let D be a six-functor formalism on (C, E) and $f: X \to Y$ in E. Write $\pi_i\colon X \times_Y X \to X$ for the projections.*

(i) $A \in \mathsf{D}(X)$ is f-suave iff the natural map

$$\pi_1^* \underline{\mathrm{Map}}(A,\ f^! \mathbb{1}_Y) \otimes \pi_2^* A \longrightarrow \underline{\mathrm{Map}}(\pi_1^* A,\ \pi_2^! A)$$

becomes an isomorphism after applying $\mathrm{Hom}_{\mathsf{D}(X)}(\Delta_{f!}\mathbb{1}_X, -) \simeq \mathrm{Hom}_{\mathsf{D}(X)}(\mathbb{1}_X, \Delta_f^!(-))$. In this case, for every $Z \in (C_E)_{/Y}$ and $M \in \mathsf{D}(Z)$,

$$\mathrm{pr}_X^* \mathsf{SD}_f(A) \otimes \mathrm{pr}_Z^* M \xrightarrow{\sim} \underline{\mathrm{Map}}(\mathrm{pr}_X^* A,\ \mathrm{pr}_Z^! M) \quad \text{in } \mathsf{D}(X \times_Y Z).$$

(ii) $A \in \mathsf{D}(X)$ is f-prim iff the natural map

$$f_!\big(\pi_{2*} \underline{\mathrm{Map}}(\pi_1^* A,\ \Delta_{f!}\mathbb{1}_X) \otimes A\big) \longrightarrow f_* \underline{\mathrm{Map}}(A, A)$$

becomes an isomorphism after applying $\mathrm{Hom}_{\mathsf{D}(Y)}(\mathbb{1}_Y, -)$. In this case, $\mathsf{PD}_f(A) \simeq \pi_{2*}\underline{\mathrm{Map}}(\pi_1^* A, \Delta_{f!}\mathbb{1}_X)$, and for every Z, M,

$$\mathrm{pr}_{Z!}\big(\mathrm{pr}_X^* \mathsf{PD}_f(A) \otimes M\big) \xrightarrow{\sim} \mathrm{pr}_{Z*} \underline{\mathrm{Map}}\big(\mathrm{pr}_X^* A,\ M\big).$$

Proof. Apply Lemma 2.3.25 to the kernel 2-category $\mathbb{K}_{\mathsf{D},Y}$. □

Corollary 2.3.27 (Suave–prim duality pairing). *Let $f: X \to Y$ in E, with $A \in \mathsf{D}(X)$ f-suave and $B \in \mathsf{D}(X)$ f-prim. Then there is a natural isomorphism*

$$f_* \underline{\mathrm{Map}}(B,\ \mathsf{SD}_f(A)) \simeq f_* \underline{\mathrm{Map}}\big(\mathsf{PD}_f(B),\ A\big)^{\vee}.$$

Proof. Direct from Proposition 2.3.24 and Proposition 2.3.26. □

Proposition 2.3.28 (Closure properties). *Let $f: X \to Y$ be in E.*

- *$\mathsf{Suave}_f(X)$ and $\mathsf{Prim}_f(X)$ are closed under retracts in $\mathsf{D}(X)$.*
- *For any small $\mathcal{I}$ such that D is compatible with $\mathcal{I}$-indexed colimits and $\mathcal{I}^{\mathrm{op}}$-indexed limits, $\mathsf{Suave}_f(X)$ and $\mathsf{Prim}_f(X)$ are closed under such colimits and limits.*

In particular, when D is a stable three-functor formalism, $\mathsf{Suave}_f(X)$ and $\mathsf{Prim}_f(X)$ are closed under retracts, fibers, and cofibers, hence are thick subcategories of $\mathsf{D}(X)$.

Suave morphisms and prim morphisms

Specializing Definition 2.3.21 to $A = \mathbb{1}_X$ yields classes of *morphisms*.

Definition 2.3.29 (Suave and prim morphisms). Let $f: X \to Y$ be in E.

(i) f is D-**suave** if $\mathbb{1}_X \in \mathsf{Suave}_f(X)$. We then write $\omega_f := \mathsf{SD}_f(\mathbb{1}_X) \in \mathsf{D}(X)$ for the **dualizing complex** of f.

(ii) f is D-**prim** if $\mathbb{1}_X \in \mathrm{Prim}_f(X)$. We then write $\delta_f := \mathrm{PD}_f(\mathbb{1}_X) \in \mathrm{D}(X)$ for the **codualizing complex** of f.

***Remark* 2.3.30.** Combining Theorem 2.3.20 with Definition 2.3.29 gives

$$f \text{ is D-smooth} \iff f \text{ is D-suave and } \omega_f \text{ is invertible.}$$

Proposition 2.3.24 and Proposition 2.3.26 provide explicit formulas:

$$\omega_f \simeq f^!(\mathbb{1}_Y) \qquad \text{and} \qquad \delta_f \simeq \pi_{2*}\Delta_{f!}\mathbb{1}_X,$$

where $\pi_2 \colon X \times_Y X \to X$ is the projection onto the second factor.

For D-suave f, the formula $\omega_f \otimes f^*(-) \xrightarrow{\sim} f^!(-)$ is equivalent (by adjunction) to $f^*(-) \xrightarrow{\sim} \underline{\mathrm{Map}}(\omega_f, f^!(-))$. Plugging in $\mathbb{1}_Y$ yields $\mathbb{1}_X \xrightarrow{\sim} \underline{\mathrm{Map}}(\omega_f, \omega_f)$, and for dualizable ω_f this gives $\mathbb{1}_X \simeq \omega_f^\vee \otimes \omega_f$. We record this as an observation.

***Observation* 2.3.31.** For a D-suave morphism f, the dualizing complex ω_f is $\otimes$-invertible if and only if it is dualizable. Consequently the verification of D-smoothness reduces to checking suaveness and dualizability of ω_f.

We close the subsubsection with the base-change identities for suave and prim morphisms; they will guide the definitions of cohomologically étale and proper morphisms in the next subsubsection.

Lemma 2.3.32 (Base change for suave and prim morphisms, [HM24, Lemma 4.5.13])**.** *Let* D *be a six-functor formalism on* (C, E) *and consider a pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in C_E.

(i) *If f is* D-*suave, the natural transformations*

$$g'_* f'^* \simeq f^* g_*, \quad g'_! f'^! \simeq f^! g_!, \quad f'^* g^! \simeq g'^! f^*, \quad g'^* f^! \simeq f'^! g^*$$

are all isomorphisms.

(ii) *If f is* D-*prim, the natural transformations*

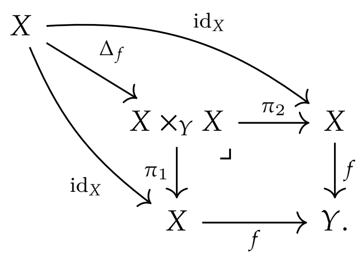
$$g^* f_* \simeq f'_* g'^*, \quad g^! f_! \simeq f'_! g'^!, \quad f_! g'_* \simeq g_* f'_!, \quad f_* g'_! \simeq g_! f'_*$$

are all isomorphisms.

Cohomological étaleness and properness

We now ask: under what conditions are there genuine isomorphisms $f^! \simeq f^*$ and $f_! \simeq f_*$, rather than mere twists by ω_f or δ_f?

A candidate map $f^! \to f^*$ is constructed using the diagonal. Consider



We have $f\pi_1 = f\pi_2$ and $\mathrm{id}_X = \pi_2\Delta_f$. Suppose inductively that Δ_f satisfies $\Delta_f^! \simeq \Delta_f^*$. Then

$$\begin{aligned} f^! \simeq \mathrm{id}_X^* f^! \simeq \Delta_f^* \pi_2^* f^! &\xrightarrow{\sim} \Delta_f^! \pi_2^* f^! \\ &\to \Delta_f^! \pi_1^! f^* \simeq (\pi_1 \Delta_f)^! f^* \simeq f^*, \end{aligned}$$

where $\pi_2^* f^! \to \pi_1^! f^*$ is induced (via right adjoints) from the base-change isomorphism $\pi_{1!}\pi_2^* \simeq f^* f_!$. By Lemma 2.3.32, when f is D-suave the latter is an isomorphism, and so $f^! \simeq f^*$.

A dual construction gives $f_! \to f_*$, an isomorphism when Δ_f satisfies $\Delta_{f!} \simeq \Delta_{f*}$ and f is D-prim. To make the recursion well-founded we impose a truncatedness hypothesis: the recursion terminates at -2-truncated morphisms (i.e., isomorphisms), where the conclusion is trivial.

Definition 2.3.33 (Cohomologically étale and proper morphisms). Let D be a six-functor formalism on (C, E), and let $f: X \to Y$ be a truncated morphism in E (i.e., n-truncated for some $n \in \mathbb{N}$).

- f is D-**étale** (or **cohomologically étale**) if f is D-suave and Δ_f is either D-étale or an isomorphism.
- f is D-**proper** (or **cohomologically proper**) if f is D-prim and Δ_f is either D-proper or an isomorphism.

These definitions are well-founded because n-truncated morphisms have $(n-1)$-truncated diagonals.

Proposition 2.3.34 (Characterization of étaleness and properness). *Let* D *be a six-functor formalism on* (C, E) *and* $f: X \to Y$ *in* E.

(i) Suppose Δ_f *is* D*-étale. There is a natural map* $f^! \to f^*$*, and the following are equivalent:*

(a) f *is* D*-étale;*

(b) $f^!\mathbb{1}_Y \simeq f^*\mathbb{1}_Y$ *in* $\mathrm{D}(X)$*;*

(c) $f^! \xrightarrow{\sim} f^*$ *as functors.*

(ii) Suppose Δ_f *is* D*-proper. There is a natural map* $f_! \to f_*$*, and the following are equivalent:*

(a′) f *is* D*-proper;*

(b′) $f_!\mathbb{1}_X \simeq f_*\mathbb{1}_X$ *in* $\mathrm{D}(Y)$*;*

(c′) $f_! \xrightarrow{\sim} f_*$ *as functors.*

Proof sketch. By duality it suffices to prove (b)⇒(a) for the étale case. We must show that $\mathbb{1}_X$ is f-suave, i.e., via Proposition 2.3.26, that $\pi_1^* f^! \mathbb{1}_Y \to \pi_2^! \mathbb{1}_X$ becomes an isomorphism after applying $\mathrm{Hom}_{\mathsf{D}(X)}(\mathbb{1}_X, \Delta_f^!(-))$. The right-hand side gives $\mathrm{Hom}_{\mathsf{D}(X)}(\mathbb{1}_X, \Delta_f^! \pi_2^! \mathbb{1}_X) = \mathrm{Hom}_{\mathsf{D}(X)}(\mathbb{1}_X, \mathbb{1}_X)$ since $\pi_2 \Delta_f = \mathrm{id}_X$. The left-hand side, since Δ_f is D-étale, gives

$$\Delta_f^! \pi_1^* f^! \mathbb{1}_Y \simeq \Delta_f^* \pi_1^* f^! \mathbb{1}_Y \simeq (\pi_1 \Delta_f)^* f^! \mathbb{1}_Y \simeq f^! \mathbb{1}_Y \simeq \mathbb{1}_X$$

by hypothesis (b), matching the right-hand side. □

2.4. The Drew–Gallauer universal property and realizations

With the six-functor formalism on SH now in hand, we return to the question advertised in the chapter introduction: in what precise sense is SH universal? The Drew–Gallauer theorem (Section 2.4.1) singles it out by three axioms; every cohomology theory of varieties satisfying those axioms then receives a unique *realization* of SH, and we survey the principal examples in Section 2.4.2.

2.4.1. *The Drew–Gallauer universal property*

Theorem 2.4.1 (Drew–Gallauer [DG22]; cf. [Sch26, Theorem 11.1]). *Let k be a Noetherian base ring with $\mathrm{Spec}(k)$ of finite Krull dimension, and let C be the category of separated finite-type k-schemes, equipped with I = open immersions and P = proper morphisms. Consider the category of presentable six-functor formalisms*

$$\mathsf{D}\colon \mathsf{Span}_2(C, E)^{\otimes}_{I,P} \longrightarrow \mathsf{Pr}^L_{\mathrm{st}}$$

(with E the class of morphisms locally of finite presentation, in line with Theorem 2.3.10) in which open immersions are cohomologically étale and proper maps are cohomologically proper (Definition 2.3.33), and which satisfy:

(i) ***Cohomological smoothness:*** *every smooth morphism is cohomologically smooth in* D *(Definition 2.3.14);*

(ii) ***Excision:*** *for any closed immersion $i\colon Z \hookrightarrow X$ with open complement $j\colon U \hookrightarrow X$, the sequence*

$$j_! \mathbb{1} \longrightarrow \mathbb{1} \longrightarrow i_* \mathbb{1}$$

is a fiber sequence in $\mathsf{D}(X)$;

(iii) $\mathbb{A}^1$***-homology trivial:*** *for $\pi\colon \mathbb{A}^1 \to \mathrm{Spec}(k)$, the counit map $\pi_! \pi^! \mathbb{1} \to \mathbb{1}$ is an equivalence.*

This category has an initial object, and that initial object is the motivic six-functor formalism SH *of Morel–Voevodsky.*

The proof of [DG22] is constructive, and recovers the construction of SH given in Section 2.1: enforcing the existence of left adjoints $f_\sharp$ to smooth pullbacks gives the stable presheaf category $\mathsf{PShv}(\mathsf{Sm}_X; \mathsf{Sp})$; excision implies Nisnevich descent and forces passage to $\mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_X; \mathsf{Sp})$; triviality of $\mathbb{A}^1$-homology forces $\mathbb{A}^1$-localization, yielding $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_X; \mathsf{Sp})$; finally, $\otimes$-inverting the Tate object yields SH. We refer the reader to [Sch26, §11] for a clean exposition of the proof sketch.

2.4.2. *Realizations of motives*

The power of Theorem 2.4.1 lies in its corollary: any other six-functor formalism on C satisfying axioms (1)–(3) automatically receives a unique symmetric monoidal functor from SH, called its **realization**. We survey the principal examples and identify the representing motivic spectrum for each.

Definition 2.4.2 (Realization)**.** A **realization** of SH valued in a six-functor formalism

$$\mathsf{D}\colon \mathsf{Span}_2(C, E)^{\otimes}_{I,P} \to \mathsf{Pr}^L_{\mathrm{st}}$$

satisfying the axioms of Theorem 2.4.1 is the unique symmetric monoidal natural transformation

$$\mathrm{real}_{\mathsf{D}}\colon \mathsf{SH} \longrightarrow \mathsf{D}$$

of functors $(\mathsf{Sch}^{\mathrm{qcqs}})^{\mathrm{op}} \to \mathsf{Pr}^L_{\mathrm{st}}$ supplied by the universal property; on each base X, $\mathrm{real}_{\mathsf{D},X}\colon \mathsf{SH}(X) \to \mathsf{D}(X)$ is colimit-preserving and symmetric monoidal. The image $\mathrm{real}_{\mathsf{D},X}(\mathbb{1}_X) \in \mathsf{D}(X)$ recovers the *cohomology theory of* X represented by D.

***Remark* 2.4.3** (Cohomology as representable in SH)**.** The right adjoint $\rho_{\mathsf{D},X}\colon \mathsf{D}(X) \to \mathsf{SH}(X)$ of $\mathrm{real}_{\mathsf{D},X}$ assigns to each cohomology theory in D a **realization spectrum** in $\mathsf{SH}(X)$ whose cohomology recovers the original theory. This is the modern explanation for why "every" reasonable cohomology theory admits motives: a six-functor formalism satisfying the DG axioms automatically produces a representing motivic spectrum.

ℓ-adic étale cohomology

For a base scheme S on which a prime ℓ is invertible, the ℓ-adic derived category

$$\mathsf{D}_{\text{ét}}(-, \mathbb{Z}_\ell)^\wedge_\ell\colon (\mathsf{Sch}^{\mathrm{qcqs}}_S)^{\mathrm{op}} \longrightarrow \mathsf{Pr}^L_{\mathrm{st}},$$

defined as the ℓ-completed derived category of ℓ-adic constructible sheaves (equivalently, lisse $\mathbb{Z}_\ell$-sheaves on the pro-étale site), is a presentable stable six-functor formalism. The categorical recasting of the classical Verdier–Deligne–SGA4 framework is given in [Sch26, Appendix to Lecture VII]: open immersions are cohomologically étale, smooth morphisms are cohomologically smooth in the ℓ-completion, and excision holds, so the DG axioms (Theorem 2.4.1) are met. Applying Theorem 2.4.1 to this formalism then yields the realization

$$\mathrm{real}_\ell\colon \mathsf{SH}(S) \longrightarrow \mathsf{D}_{\text{ét}}(S, \mathbb{Z}_\ell)^\wedge_\ell$$

sends a smooth S-scheme X to its ℓ-adic étale chains, and its right adjoint identifies, in $\mathsf{SH}(S)$, the **ℓ-adic realization spectrum** $\mathrm{H}\mathbb{Z}^{\text{ét}}_\ell$ representing ℓ-adic étale cohomology: $\mathrm{H}^{p,q}_{\text{ét}}(X, \mathbb{Z}_\ell) \simeq \pi_{2q-p}\mathrm{Map}_{\mathsf{SH}(S)}(M_S(X), \mathrm{H}\mathbb{Z}^{\text{ét}}_\ell(q))$ for X smooth over S. Variant coefficients — $\mathbb{Z}/\ell^n$, $\mathbb{Q}_\ell$, or $\overline{\mathbb{Q}}_\ell$ — give refinements of the same realization. The celebrated application is Voevodsky's solution to the Bloch–Kato conjecture.

Betti realization

For a $\mathbb{C}$-scheme X, the analytification $X(\mathbb{C})$ carries the structure of a topological space, and the assignment $X \mapsto \mathsf{D}(X(\mathbb{C}), \mathbb{Z})$ (the derived category of sheaves of abelian groups on $X(\mathbb{C})$ in the analytic topology) defines a six-functor formalism on $\mathsf{Sch}^{\mathrm{qcqs}}_{\mathbb{C}}$. Smooth morphisms become locally trivial fibrations on analytifications, hence cohomologically smooth; proper morphisms remain proper after analytification; the localization, excision, and $\mathbb{A}^1$-triviality axioms are inherited from classical sheaf theory on the contractible space $\mathbb{A}^1(\mathbb{C}) \simeq \mathbb{C}$. The induced realization

$$\mathrm{real}_{\mathrm{B}}\colon \mathsf{SH}(\mathbb{C}) \longrightarrow \mathsf{D}(-(\mathbb{C}), \mathbb{Z})$$

identifies the **Betti realization spectrum** $\mathrm{H}\mathbb{Z}^{\mathrm{B}} \in \mathsf{SH}(\mathbb{C})$, whose cohomology computes singular cohomology of complex points: $\mathrm{H}^{p,q}_{\mathrm{B}}(X, \mathbb{Z}) \simeq \mathrm{H}^p_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z})$ (with the weight q collapsed since the simplicial circle S^1 and Tate $\mathbb{G}_m$ have homotopy-equivalent topological realizations). Compatibility of the Betti realization with the slice filtration is one of the key ingredients in Voevodsky's formulation of motivic cohomology over $\mathbb{C}$.

de Rham realization

In characteristic zero, algebraic de Rham cohomology extends to a six-functor formalism via the theory of $\mathcal{D}$-modules. For a $\mathbb{Q}$-scheme S, the derived category of $\mathcal{D}$-modules

$$\mathsf{D}(\mathcal{D}_{-})\colon (\mathsf{Sch}_S^{\mathrm{qcqs}})^{\mathrm{op}} \longrightarrow \mathsf{Pr}^L_{\mathrm{st}}$$

is a six-functor formalism in the sense of Beilinson–Bernstein–Deligne. The induced realization $\mathrm{real}_{\mathrm{dR}}\colon \mathsf{SH}(S) \to \mathsf{D}(\mathcal{D}_{-})$ identifies the **de Rham realization spectrum** $\mathrm{H}\mathbb{Q}^{\mathrm{dR}} \in \mathsf{SH}(\mathbb{Q})$, whose cohomology recovers algebraic de Rham cohomology with the Hodge filtration encoded in the Tate weight: $\mathrm{H}^{p,q}_{\mathrm{dR}}(X,\mathbb{Q}) \simeq \mathrm{H}^p(X,\Omega^{\geq q}_{X/\mathbb{Q}})$ for X smooth. In characteristic $p > 0$, algebraic de Rham is no longer $\mathbb{A}^1$-invariant; the appropriate replacements (crystalline, rigid, prismatic) lie outside SH but are partially captured by the MotSp framework of Section 2.1.5.

Berkovich 2-motives

In the non-Archimedean setting, Aoki [Aok26] has recently constructed a six-functor formalism on rigid-analytic spaces over a non-Archimedean local field, called **Berkovich 2-motives** and built on a *normed ring stack* encoding the structure of analytic functions. The corresponding realization

$$\mathrm{real}_{\mathrm{Berk}}\colon \mathsf{SH}(K) \longrightarrow \mathsf{2Mot}_{\mathrm{Berk}}(K)$$

provides a motivic-theoretic basis for Scholze–Weinstein-style p-adic analytic geometry; see [Aok26, §10] for details (with Berkovich-geometric preliminaries in [Aok26, §9]) and [Man22] for the closely related p-adic six-functor formalism on rigid-analytic varieties.

Solid quasi-coherent realizations

Clausen–Scholze's theory of solid abelian groups and the associated solid derived categories produces realizations of SH that are particularly natural in arithmetic-geometric contexts (over $\mathbb{Z}$, in p-adic settings, and on analytic stacks). The framework is currently being developed within the language of *analytic stacks* of Clausen–Scholze [CS23], and the precise compatibility with the Drew–Gallauer axioms is the subject of ongoing work.

What lies beyond

The DG axioms (Theorem 2.4.1) impose $\mathbb{A}^1$-homotopy invariance, which excludes several cohomology theories of recent interest:

- **Algebraic K-theory** of singular schemes is not $\mathbb{A}^1$-invariant; its $\mathbb{A}^1$-localization KH (homotopy K-theory) is, and is represented in SH by KGL via the Bachmann–Elmanto–Morrow comparison (Proposition 3.1.7).
- **Prismatic cohomology** (Bhatt–Scholze) is not $\mathbb{A}^1$-invariant; capturing it within a six-functor formalism requires the non-$\mathbb{A}^1$-invariant setting of MotSp [AI25; AHI24b].
- **Topological Hochschild and cyclic homology** (THH, TC) are not $\mathbb{A}^1$-invariant; their motivic refinements likewise sit naturally in MotSp rather than SH.

The development of a Drew–Gallauer-style universal property for MotSp encompassing all of the above is currently an active area of research.

Chapter 3.

Stable Motivic Homotopy Theory II — Comparisons

The previous chapter constructed $\mathsf{SH}(S)$ and characterized it abstractly via the Drew–Gallauer theorem. The present chapter turns from intrinsic structure to *comparisons*: how does $\mathsf{SH}(S)$ relate to other categories arising in algebraic geometry, and how do its internal structures echo the classical machinery of algebraic topology?

A useful compass is the analogy with sheaf theory on a topological space:

Sheaves on a space X		**Motivic world over** S
$\mathsf{Shv}(X, \mathsf{An})$	$\longleftrightarrow$	$\mathsf{H}(S)$
$\mathsf{Shv}(X, \mathsf{An}_*)$	$\longleftrightarrow$	$\mathsf{H}_*(S)$
$\mathsf{Shv}(X, \mathsf{Sp})$	$\longleftrightarrow$	$\mathsf{SH}(S)$

The bottom row is imperfect: the naive analogue of $\mathsf{Shv}(X, \mathsf{Sp})$ on the motivic side is $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$, and $\mathsf{SH}(S)$ differs from it precisely because we have inverted the Tate object rather than the simplicial circle alone. Pinning down the relationship between the two is the first comparison of the chapter.

The chapter is organized around three nested comparisons:

- **Categorical.** The Bachmann–Elmanto–Morrow equivalence (Proposition 3.1.7) identifies graded $\mathbb{A}^1$-invariant Nisnevich sheaves of spectra with $\mathbb{P}^1$-bundle formula and Bott-periodic structure with $\mathcal{P}'_S$-modules in $\mathsf{SH}(S)$. The resulting motivic Eilenberg–MacLane functor H lifts classical $\mathbb{A}^1$-invariant cohomology theories (de Rham, Betti, étale, ...) to motivic ring spectra; we apply it to construct the motivic K-theory spectrum KGL.
- **Filtration.** The homotopy t-structure on $\mathsf{SH}(S)$ and Voevodsky's slice filtration are the algebraic-geometric counterparts of the Postnikov t-structure and the chromatic filtrations of stable homotopy theory. The dictionary, mediated by $\sigma^\infty \dashv \omega^\infty$ and Morel's stable $\mathbb{A}^1$-connectivity theorem, culminates in Voevodsky's slice conjecture: the slice filtration of KGL is stable under arbitrary base change.
- **Orientation.** A motivic ring spectrum $E \in \mathsf{CAlg}(\mathrm{h}\mathsf{SH}(S))$ is *oriented* when the inclusion $\mathbb{T}_S \to \mathcal{P}ic$ admits a retraction after tensoring with E, mirroring the classical $P\mathbb{CP}^1 \hookrightarrow P\mathbb{CP}^\infty$. From this single retraction unfold Chern classes, the projective bundle formula, formal group laws, and Quillen universality, with $\mathrm{H}\mathbb{Z}$ (additive), KGL (multiplicative), and MGL (universal) populating the chromatic picture.

The three comparisons interlock: H builds KGL, whose slice filtration produces $\mathbb{A}^1$-motivic cohomology and ultimately the slice conjecture, and whose canonical orientation drives the projective bundle formula. The closing sections turn to low-weight $\mathbb{A}^1$-motivic cohomology and the universal motivic Thom spectrum MGL.

3.1. Relating $\mathsf{SH}(S)$ and $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$

A motivic spectrum $E \in \mathsf{SH}(S)$ is determined by its underlying graded-sheaf data $(E(j))_{j\in\mathbb{Z}}$ with bonding equivalences $E(j) \simeq \underline{\mathrm{Map}}(\mathbb{T}_S, E(j+1))$. On the graded-sheaf side, the $\mathbb{A}^1$-invariant cohomology theories of algebraic geometry — de Rham, Betti, étale, … — come equipped with additional structure: multiplicative products, first Chern classes, and the projective bundle formula. The main result of the section, due to Bachmann–Elmanto–Morrow (Proposition 3.1.7), is a symmetric monoidal equivalence

$$\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\mathrm{c,pbf}} \;\simeq\; \mathsf{Mod}_{\mathcal{P}'_S}(\mathsf{SH}(S))$$

identifying such graded data with modules over a commutative algebra $\mathcal{P}'_S \in \mathsf{CAlg}(\mathsf{SH}(S))$; composition with the forgetful functor yields the **motivic Eilenberg–MacLane functor** H, which lifts classical cohomology theories to motivic ring spectra.

Our path is indirect. The tautological adjunction $\sigma^{\infty,\mathrm{gr}} \dashv \omega^{\infty,\mathrm{gr}}$ fails to be $\mathbb{E}_2$-monoidal because the switch map on $\mathbb{T}_S \wedge \mathbb{T}_S$ is not homotopic to the identity in $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$; passing to Bott-periodic modules builds in the missing symmetry, and the equivalence drops out.

Notation 3.1.1 (Tensor symbols)**.** Throughout this section the symbol $\otimes$ denotes the ambient monoidal product, whose meaning (in $\mathsf{SH}(S)$, $\mathsf{GrSH}(S)$, $\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$, SSeq, etc.) is determined by typing information; $\wedge$ remains reserved for pure $\mathsf{H}_*(S)$-level expressions such as $\mathbb{T}_S \simeq S^1 \wedge \mathbb{G}_m$.

3.1.1. Graded algebras and the projective bundle formula

Construction 3.1.2. Equip $\mathbb{Z}^\delta$ with its discrete groupoid structure, so that $\mathsf{Fun}(\mathbb{Z}^\delta, \mathcal{C})$ acquires a symmetric monoidal structure via Day convolution with respect to the addition operation on $\mathbb{Z}$. Consider the square of commutative monoids in An:

$$\begin{array}{ccc} \mathbb{F} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow \\ \mathbb{S} & \longrightarrow & \mathbb{Z} \end{array}$$

where the vertical maps are group completions and the horizontal maps are induced by π_0. Restriction along $\mathbb{F} \to \mathbb{N} \to \mathbb{Z}$ yields a lax symmetric monoidal forgetful functor

$$\mathrm{res}\colon \mathsf{Gr}(\mathcal{C}) = \mathsf{Fun}(\mathbb{Z}^\delta, \mathcal{C}) \longrightarrow \mathsf{SSeq}(\mathcal{C}).$$

Let $E^\star \in \mathsf{CAlg}(\mathsf{Gr}(\mathcal{C}))$ be a $\mathbb{Z}$-graded commutative algebra in $\mathcal{C}$, and let $c\colon T \to E^1$ be a morphism in $\mathcal{C}$ (a **Bott element**). By the universal property of Sym, the element c extends uniquely to a morphism

$$\mathrm{Sym}(T_{\mathrm{sym}}) \longrightarrow \mathrm{res}(E)$$

of commutative algebras in $\mathsf{SSeq}(\mathcal{C})$, making $\mathrm{res}(E)$ canonically into a commutative algebra in $\mathsf{Sp}_T^{\mathrm{lax}}(\mathcal{C})$. Moreover, if the resulting bonding maps

$$E^n \otimes T \xrightarrow{1\otimes c} E^n \otimes E^1 \longrightarrow E^{n+1}$$

have adjoints that are equivalences, then (E, c) lifts to a commutative algebra in $\mathsf{Sp}_T(\mathcal{C})$ with $\omega^{\infty-n}(E, c) \simeq E^n$ for all $n \in \mathbb{Z}$.

The motivic case

We now specialize to $\mathcal{C} = \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ with $T = \Sigma^\infty \mathbb{T}_S$, where $\mathbb{T}_S$ is the Tate object of Definition 1.4.2 and Σ^∞ denotes the suspension functor $\Sigma^\infty \colon \mathsf{H}_*(S) \to \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$; a Bott element is then a morphism $\Sigma^\infty \mathbb{T}_S \to E^1$.

Construction 3.1.3 (Q_S-modules and the projective bundle formula). Let $\Sigma^\infty \mathbb{T}_S\langle 1\rangle \in \mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ denote the graded sheaf concentrated in degree 1 with value $\Sigma^\infty \mathbb{T}_S$, and set

$$Q_S \ :=\ \mathrm{Sym}\big(\Sigma^\infty \mathbb{T}_S\langle 1\rangle\big)\ \in\ \mathsf{CAlg}\big(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})\big).$$

One checks that $\sigma^\infty(Q_S) \simeq \mathrm{Sym}(\mathbb{T}_S\langle 1\rangle) \in \mathsf{Gr}(\mathsf{SH}(S))$, where on the right $\mathbb{T}_S$ stands for the image of $\Sigma^\infty \mathbb{T}_S$ under $\sigma^\infty \colon \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \to \mathsf{SH}(S)$, consistent with the notation of Section 2.1.

We therefore consider the category of Q_S-modules, for which we introduce the abbreviation

$$\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\mathrm{c}} \ :=\ \mathsf{Mod}_{Q_S}\big(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})\big).$$

Given $E \in \mathsf{CAlg}\big(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\mathrm{c}}\big)$, the bonding maps of Construction 3.1.2 take the form

$$b^n \colon E^n \otimes \Sigma^\infty \mathbb{T}_S \longrightarrow E^{n+1}, \qquad n \in \mathbb{Z}.$$

Passing to adjoints, these become

$$b^{n,\dagger} \colon E^n \longrightarrow \Omega_{\Sigma^\infty \mathbb{T}_S} E^{n+1} \ \simeq\ \mathrm{fib}\big(E^{n+1}(\mathbb{P}^1_-) \xrightarrow{\infty^*} E^{n+1}\big),$$

where ∞^* denotes restriction along the inclusion of the point at infinity. The requirement that every $b^{n,\dagger}$ is an equivalence is precisely the $\mathbb{P}^1$ **bundle formula** (pbf).

We write

$$\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\mathrm{c,pbf}}$$

for the full subcategory of Q_S-modules satisfying the $\mathbb{P}^1$ bundle formula.

The passage from such data to a genuine motivic spectrum is encoded in the functor H announced at the start of the section; we construct it in Section 3.1.4, after the Bott-periodic groundwork of Section 3.1.3 is in place.

3.1.2. The $\sigma^{\infty,\mathrm{gr}} \dashv \omega^{\infty,\mathrm{gr}}$ adjunction and periodization

To construct the Eilenberg–MacLane functor H, we approach the problem from the reverse direction: from a motivic spectrum $E \in \mathsf{SH}(S)$, extract a graded sheaf and ask when this graded sheaf lies in $\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\mathrm{c,pbf}}$.

By Corollary 2.1.13, for each $j \in \mathbb{Z}$, we can extract an $\mathbb{A}^1$-invariant Nisnevich sheaf from E via the formula

$$E(j) = \mathrm{map}_{\mathsf{SH}(S)}(M_S(-), E \otimes \mathbb{T}_S^{\otimes j}).$$

***Warning* 3.1.4.** Our notation is different from [BEM25, p. 3.2], where they use the "$\mathbb{G}_m$-twist" and then define $E(j)$ as $\mathrm{map}_{\mathsf{SH}(S)}(M_S(-), E \otimes \mathbb{G}_m^{\otimes j})$. Our notation is close to the T-spectra we construct in Section 2.1.3, and we believe that it is more natural.

Assembling these levels and packaging the bonding data, one obtains an adjunction

$$\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \underset{\omega^{\infty,\mathrm{gr}}}{\overset{\sigma^{\infty,\mathrm{gr}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{SH}(S)$$

with
$$\sigma^{\infty,\mathrm{gr}}(E^\star) = \bigoplus_{j\in\mathbb{Z}} \sigma^\infty(E^j)\otimes \mathbb{T}_S^{\otimes j}, \qquad \omega^{\infty,\mathrm{gr}}(E)^j \simeq E(j) \simeq \omega^\infty\big(E\otimes \mathbb{T}_S^{\otimes j}\big).$$

The right adjoint $\omega^{\infty,\mathrm{gr}}$ does not commute with pullback functors, and it is also poorly multiplicative. To diagnose the obstruction: under the identification $\mathsf{Gr}(C) \simeq \mathsf{PShv}(\mathbb{Z}^\delta)\otimes C$, an $\mathbb{E}_n$-monoidal structure on $\sigma^{\infty,\mathrm{gr}}$ corresponds to an $\mathbb{E}_n$-monoidal map $\mathbb{Z}\to \mathcal{P}ic(\mathsf{SH}(S))$ sending $1\mapsto \mathbb{T}_S$. For $n=1$, the free $\mathbb{E}_1$-monoid $\mathbb{Z}$ on an invertible generator delivers such a map canonically, and $\sigma^{\infty,\mathrm{gr}}$ becomes monoidal — so $\omega^{\infty,\mathrm{gr}}$ is lax monoidal and induces a functor
$$\mathsf{Alg}(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})) \longleftarrow \mathsf{Alg}(\mathsf{SH}(S)).$$

For $n=2$, however, the lift requires the swap on $\mathbb{T}_S\otimes\mathbb{T}_S$ to be homotopic to the identity in $\mathsf{SH}(S)$ — but Proposition 2.1.17 (and its proof) shows the swap is homotopic to $-\mathrm{id}$, so the desired $\mathbb{E}_2$-monoidal lift does not exist.

To bypass this obstruction we work with extra data on the $\mathsf{SH}(S)$ side.

Definition 3.1.5. For $E\in\mathsf{CAlg}(\mathsf{SH}(S))$, a **periodization** of E is a compatible commutative algebra structure on $E\otimes\mathbb{T}_S^{\otimes\star}\in\mathsf{GrSH}(S)$. We write $\mathsf{CAlg}(\mathsf{SH}(S))_{\mathrm{per}}$ for the category of pairs $(E,\text{periodization})$, with morphisms respecting the periodization.

By symmetric monoidality of ω^∞, the right adjoint upgrades on this enriched source to
$$\mathsf{CAlg}(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})) \longleftarrow \mathsf{CAlg}(\mathsf{SH}(S))_{\mathrm{per}}.$$

The category $\mathsf{CAlg}(\mathsf{SH}(S))_{\mathrm{per}}$ is still defined extrinsically (in terms of $\mathsf{GrSH}(S)$); the next subsection reinterprets it as the category of commutative algebras over a single commutative ring in $\mathsf{SH}(S)$ via the Bott-periodic-module construction.

3.1.3. Bott-periodic modules and the BEM equivalence

Definition 3.1.6 (Bott-periodic modules)**.** Let $E\in\mathsf{CAlg}(C)$ and let $\beta\colon T\to E$ be a morphism in C (the **Bott element**). The category of β**-periodic** E**-modules** is the monoidal Bousfield localization
$$P_\beta\mathsf{Mod}_E(C)\ \subset\ \mathsf{Mod}_E(C)$$
obtained by inverting the map $\beta\colon T\otimes E\to E$.

Mirroring Construction 3.1.3, consider $\mathcal{P}_S := \mathrm{Sym}(\mathbb{T}_S\langle 1\rangle)\in\mathsf{CAlg}(\mathsf{GrSH}(S))$ with Bott element $\beta\colon \mathcal{P}_S\otimes\mathbb{T}_S\langle 1\rangle\to\mathcal{P}_S$. By [BH21, Lemma 12.1], the Bott-periodization $P_\beta\mathcal{P}_S$ is computed by the telescope
$$\mathrm{colim}\Big(\mathcal{P}_S \xrightarrow{\beta} \mathcal{P}_S\otimes\mathbb{T}_S^{\otimes -1}\langle -1\rangle \xrightarrow{\beta} \mathcal{P}_S\otimes\mathbb{T}_S^{\otimes -2}\langle -2\rangle \xrightarrow{\beta}\cdots\Big).$$

Set $\mathcal{P}'_S := (P_\beta\mathcal{P}_S)^0$, the degree-zero component; in $\mathsf{GrSH}(S)$ one has $E\otimes\mathbb{T}_S^{\otimes\star}\simeq E\otimes_{\mathcal{P}'_S}P_\beta\mathcal{P}_S$.

This converts a periodization into a $\mathcal{P}'_S$-algebra structure and back. A map $\mathcal{P}'_S\to E$ in $\mathsf{CAlg}(\mathsf{SH}(S))$ produces $E\otimes_{\mathcal{P}'_S}P_\beta\mathcal{P}_S$, the periodization of E. Conversely, a periodization yields $\mathbb{T}_S\to E\otimes\mathbb{T}_S$ from the unit, which induces $\mathcal{P}_S\to E\otimes\mathbb{T}_S^{\otimes\star}$ and (since the target is Bott-periodic) factors as $P_\beta\mathcal{P}_S\to E\otimes\mathbb{T}_S^{\otimes\star}$; passing to the degree-zero component gives $\mathcal{P}'_S\to E$. We obtain an equivalence
$$\mathsf{CAlg}(\mathsf{SH}(S))_{\mathrm{per}}\ \simeq\ \mathsf{CAlg}(\mathsf{SH}(S))_{\mathcal{P}'_S/}\ =\ \mathsf{CAlg}\big(\mathsf{Mod}_{\mathcal{P}'_S}(\mathsf{SH}(S))\big),$$

identifying periodized motivic ring spectra with commutative algebras in $\mathsf{Mod}_{\mathcal{P}'_S}(\mathsf{SH}(S))$ — an intrinsic interpretation, as promised.

Proposition 3.1.7 ([BEM25, Proposition 3.15]). *There is a monoidal equivalence $\omega_{\otimes}^{\infty,\mathrm{gr}}$ filling the following commutative square of monoidal categories and lax monoidal functors:*

$$\begin{array}{ccc} \mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c,pbf}} & \xleftarrow{\omega_{\otimes}^{\infty,\mathrm{gr}}} & \mathsf{Mod}_{\mathcal{P}'_S}(\mathsf{SH}(S)) \\ \downarrow & & \downarrow \\ \mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp}) & \xleftarrow[\omega^{\infty,\mathrm{gr}}]{} & \mathsf{SH}(S). \end{array}$$

Moreover, $\omega_{\otimes}^{\infty,\mathrm{gr}}$ upgrades to a symmetric monoidal equivalence. Consequently, the motivic Eilenberg–MacLane functor H is the inverse of $\omega_{\otimes}^{\infty,\mathrm{gr}}$.

Proof. Set $v\colon Q_S \otimes \Sigma^\infty \mathbb{P}^1_S\langle 1\rangle \to Q_S$; by construction

$$\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c,pbf}} \;=\; P_v\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c}}.$$

Since Q_S is the unit, [Hoy20, Proposition 3.2] identifies this v-localization with the formal inversion (Proposition 2.1.3) of $Q_S \otimes \Sigma^\infty \mathbb{P}^1_S\langle 1\rangle$. Using $\sigma^\infty(Q_S) = \mathcal{P}_S$ and [BH21, Lemma 12.1],

$$\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c,pbf}} \;\simeq\; P_\beta\mathsf{Mod}_{\mathcal{P}_S}(\mathsf{GrSH}(S)) \;\simeq\; \mathsf{Mod}_{P_\beta\mathcal{P}_S}(\mathsf{GrSH}(S)).$$

The discussion preceding the proposition supplies the final equivalence $\mathsf{Mod}_{P_\beta\mathcal{P}_S}(\mathsf{GrSH}(S)) \simeq \mathsf{Mod}_{\mathcal{P}'_S}(\mathsf{SH}(S))$. □

3.1.4. The motivic Eilenberg–MacLane functor H

With Proposition 3.1.7 in hand, we can finally define H as the inverse of the equivalence $\omega_{\otimes}^{\infty,\mathrm{gr}}$.

Definition 3.1.8. The procedure above defines the **motivic Eilenberg–MacLane functor**

$$H\colon \mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c,pbf}} \to \mathsf{SH}(S).$$

Classical $\mathbb{A}^1$-invariant cohomology theories of algebraic geometry — algebraic de Rham, Betti, étale, … — furnish objects of $\mathsf{CAlg}(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c,pbf}})$, since they come equipped with:

- a family of $\mathbb{A}^1$-invariant Nisnevich $\mathrm{D}(\mathbb{Z})$-valued sheaves $F(j)$ together with multiplications $F(i)\otimes F(j) \to F(i+j)$;
- a first Chern class map $c_1\colon \Gamma_{\mathrm{Nis}}(-,\mathbb{G}_m)[1] \to F(1)$;
- the projective bundle formula: for every $X \in \mathsf{Sm}_S$, $j\in\mathbb{Z}$ and $n \geq 1$,

$$\sum_{i=0}^{n} c_1(\mathcal{O}(1))^i \colon \bigoplus_{i=0}^{n} F(j-i)(X) \xrightarrow{\sim} F(j)(\mathbb{P}^n_X),$$

  which for $n=1$ recovers the $\mathbb{P}^1$-bundle formula of Construction 3.1.3.

The multiplications assemble into a commutative algebra structure in $\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathrm{D}(\mathbb{Z}))$; forgetting the $\mathbb{Z}$-linear structure produces $F(\star) \in \mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})$. Restricting the first Chern class to $\mathcal{O}(1)\in \mathrm{Pic}(\mathbb{P}^1_S)$ yields a morphism

$$\Sigma^\infty\mathbb{P}^1_S \xrightarrow{[\mathcal{O}(1)]} \mathcal{P}ic \simeq \tau^{\leq 1}\Gamma_{\mathrm{Zar}}(-,\mathbb{G}_m)[1] \longrightarrow F(1),$$

upgrading $F(\star)$ to an object of $\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S,\mathsf{Sp})_{\mathrm{c,pbf}}$, which corresponds via H to a motivic spectrum.

3.1.5. *Promoting Bott-periodic algebras to T-spectra*

As a final application of the Bott-periodic framework, we record a general mechanism for promoting a Bott-periodic commutative algebra to a T-spectrum. This will be the key tool in the construction of motivic K-theory below.

Proposition 3.1.9 (Promotion to T-spectra)**.** *Let $E \in \mathsf{CAlg}(C)$ and $\beta\colon T \to E$ be a Bott element. Then the suspension functor induces an equivalence*

$$P_\beta \mathsf{Mod}_E(C) \;\simeq\; P_\beta \mathsf{Mod}_{\sigma^\infty(E)}\big(\mathsf{Sp}_T(C)\big).$$

Consequently, E admits an essentially unique promotion to a commutative algebra

$$\underline{E} \;\in\; \mathsf{CAlg}\big(\mathsf{Sp}_T(C)\big) \qquad \text{with } \omega^{\infty-j}\underline{E} \simeq E \text{ for all } j \in \mathbb{Z}.$$

3.1.6. *Example: the motivic K-theory spectrum* KGL

We now apply the Bott-periodic construction (Proposition 3.1.9) to build the motivic K-theory spectrum $\mathrm{KGL}_S \in \mathsf{CAlg}(\mathsf{SH}(S))$. Throughout, we follow the case convention of Convention 1.3.4: uppercase for non-connective, lowercase for connective; in particular, the effective cover of KGL constructed below (Definition 3.1.10) is identified with kgl, not KGL.

Define homotopy K-theory

$$\mathrm{KH}(X) \;\coloneqq\; \mathrm{L}_{\mathbb{A}^1}\mathrm{K}(X) \;=\; \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}}\, \mathrm{K}\big(X \times \Delta^n\big).$$

Restricted to Sm_S, KH defines an object of $\mathsf{CAlg}(\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}))$.

Functor-of-points recollection of $\mathcal{O}(1)$. The Serre twist $\mathcal{O}(1) \in \mathrm{Pic}(\mathbb{P}^1_S)$ is determined by its functor of points: for $x\colon \mathrm{Spec}(R) \to \mathbb{P}^1_S$ corresponding to a surjection $R^2 \twoheadrightarrow L$ of invertible R-modules, the pullback $x^*\mathcal{O}(1)$ is L. Equivalently, $\mathcal{O}(1)$ classifies a map $\mathbb{P}^1_S \to \mathcal{P}ic$ into the motivic Picard stack. Its dual $\mathcal{O}(-1) = \mathcal{O}(1)^\vee$ is the *tautological* line bundle, appearing as the kernel of $\mathcal{O}^{\oplus 2} \twoheadrightarrow \mathcal{O}(1)$; both are distinguished classes in $\mathrm{K}_0(\mathbb{P}^1)$.

The Bott element. Among the many non-trivial classes in $\mathrm{K}_0(\mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}\cdot\beta$, we single out

$$\beta \;\coloneqq\; 1 - [\mathcal{O}(-1)] \;\in\; \mathrm{fib}\big(\mathrm{K}(\mathbb{P}^1) \to \mathrm{K}(\infty)\big),$$

the **Bott element**. The sign and the choice of $\mathcal{O}(-1)$ are not cosmetic: this formula is fixed by compatibility with the canonical orientation map $1 - (-)^\vee\colon \Omega^\infty\mathcal{P}ic \to \Omega^\infty\mathrm{k}$ of Bachmann–Elmanto–Morrow [BEM25, Definition 4.4] (written there as K^{cn} in our notation k), whose composition with the classifying map $[\mathcal{O}(1)]\colon \mathbb{P}^1 \to \Omega^\infty\mathcal{P}ic$ assigns to the Serre twist its first Chern class $1 - [\mathcal{O}(1)^\vee] = 1 - [\mathcal{O}(-1)] = \beta$. We will see this explicitly in Example 3.2.29; for now it suffices that under the motivic identification $\mathbb{T}_S \simeq \mathbb{P}^1_S/\{\infty\}$, the class β lifts canonically to a morphism

$$\beta\colon \mathbb{T}_S \longrightarrow \mathrm{KH}$$

in $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$.

Why single out this element. Two facts make β the right choice. First, iterated multiplication by β realizes the Bott equivalence $\mathrm{KH} \simeq \Omega_{\mathbb{T}_S}\mathrm{KH}$, which is what promotes KH to a $\mathbb{T}_S$-spectrum. Second, β is the restriction to the $\mathbb{T}_S$-cell of the canonical KGL-orientation, hence the universal first Chern class of $\mathcal{O}(1)$ in KH-theory; this is developed in Example 3.2.29.

Definition 3.1.10 (KGL). The pair (KH, β) is Bott-periodic, and Proposition 3.1.9 promotes it to a commutative algebra

$$\mathrm{KGL}_S \;\in\; \mathsf{CAlg}\big(\mathsf{SH}(S)\big)$$

with $\omega^{\infty-j}\mathrm{KGL}_S \simeq \mathrm{KH}$ for all $j \in \mathbb{Z}$, reflecting the Bott periodicity $\mathbb{T}_S \otimes \mathrm{KGL}_S \simeq \mathrm{KGL}_S$.

Proposition 3.1.11 (Base change for KGL). *For any morphism of schemes $f : Y \to X$, the natural map*

$$f^*\mathrm{KGL}_X \xrightarrow{\sim} \mathrm{KGL}_Y$$

is an equivalence in $\mathsf{SH}(Y)$.

To show this, we need the following proposition.

Proposition 3.1.12 (Bhatt–Lurie [Elm+20, Appendix A]). *The connective* K*-theory functor*

$$\mathrm{k}\colon \mathrm{CRing}_R \longrightarrow \mathsf{Sp}$$

is left Kan extended from its restriction to the subcategory SmRing_R *of smooth R-algebras.*

Proof of Proposition 3.1.12. It suffices to show that the rank-n vector bundle stack $\mathsf{Vect}_n : \mathrm{CRing}_R \to$ An is left Kan extended from SmRing_R. A presheaf is left Kan extended from a subcategory iff it is a colimit of representables from that subcategory. Consider the chain of morphisms of stacks

$$\coprod_k U_{n,k} \longrightarrow \coprod_k \mathsf{Gr}_n(\mathbb{A}^k) \longrightarrow \mathsf{Vect}_n,$$

where $\mathsf{Gr}_n(\mathbb{A}^k)$ is the Grassmannian of rank-n subbundles of $\mathcal{O}^k$ (classifying rank-n vector bundles equipped with a surjection from $\mathcal{O}^k$), and $U_{n,k} \subset \mathsf{Gr}_n(\mathbb{A}^k)$ is the standard smooth affine open chart. On affine schemes the second map is surjective (any rank-n vector bundle on an affine scheme is a quotient of some trivial bundle), and the first map is an open cover by smooth affines; the composite is therefore an effective epimorphism, and Vect_n is the colimit of its Čech nerve, consisting of disjoint unions of smooth affines. □

Proof of Proposition 3.1.11. By Proposition 3.1.12, k is left Kan extended from smooth affine algebras; the same holds for KH after $\mathbb{A}^1$-localization. Left Kan extension is automatically compatible with base change, so $f^*\mathrm{KH}_X \simeq \mathrm{KH}_Y$. Since the Bott element is also natural, the Bott-periodic construction (Proposition 3.1.9) gives $f^*\mathrm{KGL}_X \simeq \mathrm{KGL}_Y$. □

Corollary 3.1.13 (Cisinski). *The presheaf* KH *is a cdh sheaf on* $\mathsf{Sch}^{\mathrm{qcqs}}$.

Proof. By Corollary 2.3.12, SH is a cdh sheaf on $\mathsf{Sch}^{\mathrm{qcqs}}$. Combined with Proposition 3.1.11, the assignment $X \mapsto \mathrm{KGL}_X \in \mathsf{SH}(X)$ is a cdh-local section of the cocartesian fibration $\int \mathsf{SH}$. Since the unit $\mathbb{1}$ is preserved by pullback, the functor $\mathrm{Map}_{\mathsf{SH}(-)}(\mathbb{1}, -)$ applied to this section preserves cdh descent. Combined with the identification $\omega^{\infty}\mathrm{KGL}_X|_{\mathsf{Sm}_X} \simeq \mathrm{KH}|_{\mathsf{Sm}_X}$ (Definition 3.1.10), we obtain

$$\mathrm{KH}(X) \;\simeq\; \mathrm{Map}_{\mathsf{SH}(X)}(\mathbb{1}, \mathrm{KGL}_X),$$

which inherits cdh descent. □

3.1.7. *The homotopy t-structure*

We end this section with the homotopy t-structure on $\mathsf{SH}(S)$ and compare it, via the $\sigma^\infty \dashv \omega^\infty$ adjunction, with the Postnikov t-structure on $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$. We follow [BH21, Appendix B].

Notation 3.1.14 (Bigrading in this section)**.** For the t-structure discussion it is convenient to use the **Bachmann–Hoyois bigrading** [BH21], which separates the simplicial degree from the Tate degree:

$$S^{a,b}_{\mathrm{BH}} \simeq (S^1)^{\otimes a} \otimes \mathbb{G}_m^{\otimes b}.$$

This differs from the Morel convention $S^{p,q}$ of Definition 1.4.4 by $(p,q) = (a+b, b)$. Throughout the present section we adopt the (a,b) labeling; outside this section the Morel convention is in force. Unless stated otherwise, $\otimes$ denotes the monoidal product of $\mathsf{SH}(S)$ and $\wedge$ that of $\mathsf{H}_*(S)$.

Notation 3.1.15. Let $I \subset \mathbb{Z} \times \mathbb{Z}$ be a subset, where $(i,j) \in I$ represents the bidegree $(a,b) =$ (simplicial, Tate). We denote by $\mathsf{SH}(S)_{\geq I}$ the full subcategory of $\mathsf{SH}(S)$ generated under colimits and extensions by

$$\{\, S^{i+j,j} \otimes M_S(X) \mid X \in \mathsf{Sm}_S,\ (i,j) \in I \,\}.$$

Example 3.1.16. When $I = \varnothing$, we have $\mathsf{SH}(S)_{\geq \varnothing} = \{0\}$.

Since $\mathsf{SH}(S)$ is presentable stable and the generating family above is small, [Lur17, Proposition 1.4.4.11] implies that $\mathsf{SH}(S)_{\geq I}$ is the connective part of an accessible t-structure on $\mathsf{SH}(S)$. We denote the corresponding coconnective part by $\mathsf{SH}(S)_{<I}$, consisting of all $E \in \mathsf{SH}(S)$ such that $\mathrm{Map}(F, E) \simeq *$ for every $F \in \mathsf{SH}(S)_{\geq I}$. We denote by $\tau_{\geq I}$ the right adjoint to $\mathsf{SH}(S)_{\geq I} \hookrightarrow \mathsf{SH}(S)$ and $\tau_{<I}$ the left adjoint to $\mathsf{SH}(S)_{<I} \hookrightarrow \mathsf{SH}(S)$.

Different choices of I produce different t-structures adapted to different purposes. Among them, one is distinguished by its role in Morel's work [Mor05] on the stable $\mathbb{A}^1$-connectivity theorems.

Definition 3.1.17 (Homotopy t-structure)**.** The **homotopy t-structure** on $\mathsf{SH}(S)$ is the t-structure determined by $I = \{0\} \times \mathbb{Z}$. We write $\mathsf{SH}(S)_{\geq 0} := \mathsf{SH}(S)_{\geq \{0\}\times\mathbb{Z}}$, and more generally $\mathsf{SH}(S)_{\geq d} := \mathsf{SH}(S)_{\geq \{d\}\times\mathbb{Z}}$ for $d \in \mathbb{Z}$. We denote by $\tau_{\geq d}$ the right adjoint to $\mathsf{SH}(S)_{\geq d} \hookrightarrow \mathsf{SH}(S)$ and by $\tau_{<d}$ the left adjoint to $\mathsf{SH}(S)_{<d} \hookrightarrow \mathsf{SH}(S)$.

Unpacking the notation, $\mathsf{SH}(S)_{\geq d}$ is the full subcategory generated under colimits and extensions by the objects $\Sigma^{d+j,j} M_S(X)$ with $X \in \mathsf{Sm}_S$ and $j \in \mathbb{Z}$.

Next, we record some functoriality properties of $\mathsf{SH}(S)_{\geq I}$. For every morphism $f \colon S' \to S$, the functor $f^* \colon \mathsf{SH}(S) \to \mathsf{SH}(S')$ sends the generators $\Sigma^{i+j,j} M_S(X)$ to $\Sigma^{i+j,j} M_{S'}(X \times_S S')$, which are again generators; hence f^* is right t-exact, and consequently its right adjoint f_* is left t-exact. The following lemmas improve on these observations.

Lemma 3.1.18 ([BH21, B.1])**.** *If $f \colon S' \to S$ is a pro-smooth morphism, then $f^* \colon \mathsf{SH}(S) \to \mathsf{SH}(S')$ is t-exact.*

Proof. Right t-exactness was noted above, so we must show that f^* preserves the coconnective part $\mathsf{SH}(S)_{<I}$.

We first treat the smooth case. If f is smooth, then f^* admits a left adjoint $f_\# \colon \mathsf{SH}(S') \to \mathsf{SH}(S)$ sending $\Sigma^{i+j,j} M_{S'}(X') \mapsto \Sigma^{i+j,j} M_S(X')$ (where X' is regarded as smooth over S via f). Thus $f_\#$ is right t-exact, and consequently f^* is left t-exact. Combined with right t-exactness, f^* is t-exact in the smooth case.

Now suppose $f: S' \to S$ is pro-smooth, with $S' = \lim_{\alpha \in A} S_\alpha$ a cofiltered limit of smooth S-schemes $f_\alpha: S_\alpha \to S$ with affine transition maps. Let $E \in \mathsf{SH}(S)_{<I}$; we show $f^*E \in \mathsf{SH}(S')_{<I}$, equivalently $\mathrm{Map}(F, f^*E) \simeq *$ for every $F = \Sigma^{i+j,j} M_{S'}(X')$ with $X' \in \mathsf{Sm}_{S'}$ and $(i,j) \in I$.

By [Gro67, 8.8.2 and 17.7.8], there exists an index $\alpha_0 \in A$ and a smooth S_{α_0}-scheme X_{α_0} with $X' \simeq S' \times_{S_{\alpha_0}} X_{\alpha_0}$. By continuity of $\mathsf{SH}(-)$ [CD19, Proposition 4.3.4],

$$\mathrm{Map}_{\mathsf{SH}(S')}(\Sigma^{i+j,j} M_{S'}(X'), f^*E) \simeq \operatorname*{colim}_{\alpha \geq \alpha_0} \mathrm{Map}_{\mathsf{SH}(S_\alpha)}(\Sigma^{i+j,j} M_{S_\alpha}(X_\alpha), f_\alpha^* E),$$

where $X_\alpha := S_\alpha \times_{S_{\alpha_0}} X_{\alpha_0}$. Since each f_α is smooth, the already established smooth case gives $f_\alpha^* E \in \mathsf{SH}(S_\alpha)_{<I}$, so every term in the colimit is contractible. □

Lemma 3.1.19 ([BH21, B.2]). *If $\iota: Z \to S$ is a closed immersion, then $\iota_*: \mathsf{SH}(Z) \to \mathsf{SH}(S)$ is t-exact.*

Proof. Left t-exactness follows from the general fact noted above. For right t-exactness, since ι_* preserves colimits, the full subcategory $\{E \in \mathsf{SH}(Z) \mid \iota_* E \in \mathsf{SH}(S)_{\geq I}\}$ is closed under colimits and extensions. It therefore suffices to show $\iota_*(\Sigma^{a+j,j} M_Z(Y)) \in \mathsf{SH}(S)_{\geq I}$ for all $Y \in \mathsf{Sm}_Z$ and $(a,j) \in I$. By [Gro67, p. 18.1.1], locally on Y we can write $Y \simeq X \times_S Z$ for some $X \in \mathsf{Sm}_S$, so it suffices to show that $\iota_* \iota^*$ preserves $\mathsf{SH}(S)_{\geq I}$.

Consider the localization exact sequence

$$j_! j^* E \longrightarrow E \longrightarrow \iota_* \iota^* E,$$

where $j: U \hookrightarrow S$ is the open complement of Z. Since j is an open immersion, $j_! \simeq j_\#$ is right t-exact (as in the smooth case of Lemma 3.1.18), and j^* is right t-exact generally, so $j_! j^*$ is right t-exact. If $E \in \mathsf{SH}(S)_{\geq I}$, then $j_! j^* E \in \mathsf{SH}(S)_{\geq I}$, and so by the exact sequence $\iota_* \iota^* E \in \mathsf{SH}(S)_{\geq I}$. □

By the lemmas above, one can derive the following property, which says that motivic spectra are detected by their pullbacks to points.

Proposition 3.1.20 ([BH21, B.3]). *Let S be a qcqs scheme locally of finite Krull dimension and let $E \in \mathsf{SH}(S)$. Then $E \in \mathsf{SH}(S)_{\geq I}$ if and only if $s^*E \in \mathsf{SH}(\kappa(s))_{\geq I}$ for every point $s \in S$, where $s: \operatorname{Spec} \kappa(s) \to S$ is the canonical map.*

Proof. Necessity is clear: $s^*: \mathsf{SH}(S) \to \mathsf{SH}(\kappa(s))$ is right t-exact.

For sufficiency, we proceed by induction on $d := \dim S$. Assume $s^*E \in \mathsf{SH}(\kappa(s))_{\geq I}$ for every $s \in S$; we show $\tau_{<I} E \simeq *$.

By Proposition 1.1.5 and Theorem 2.3.5, $\tau_{<I} E \simeq *$ can be checked on the henselizations $S_s^h := \operatorname{Spec} \mathcal{O}_{S,s}^h$ for $s \in S$. Since $S_s^h \to S$ is pro-smooth, Lemma 3.1.18 gives $(\tau_{<I} E)|_{S_s^h} \simeq \tau_{<I}(E|_{S_s^h})$, so we may replace S by S_s^h and assume S is local Henselian with closed point $\iota: \operatorname{Spec} \kappa(s) \hookrightarrow S$ and open complement $j: U \hookrightarrow S$, where $\dim U < \dim S$.

Consider the localization cofiber sequence

$$j_! j^* E \longrightarrow E \longrightarrow \iota_* \iota^* E.$$

For any $u \in U$, $(j^*E)_u \simeq E_{j(u)} \in \mathsf{SH}(\kappa(u))_{\geq I}$ by assumption, so by the induction hypothesis $j^*E \in \mathsf{SH}(U)_{\geq I}$, and hence $j_! j^* E \in \mathsf{SH}(S)_{\geq I}$ (since j is an open immersion, $j_! \simeq j_\#$ is right t-exact, as in the proof of Lemma 3.1.19). Also, $\iota^* E = s^* E \in \mathsf{SH}(\kappa(s))_{\geq I}$ by assumption, and Lemma 3.1.19 gives $\iota_* \iota^* E \in \mathsf{SH}(S)_{\geq I}$. Both outer terms of the cofiber sequence lie in $\mathsf{SH}(S)_{\geq I}$, hence so does E. □

Postnikov t-structure and Homotopy t-structure

Recall the adjunction

$$\sigma^\infty \colon \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \rightleftarrows \mathsf{SH}(S) \colon \omega^\infty$$

introduced in Section 2.1.4. To match the homotopy t-structure on $\mathsf{SH}(S)$ with something on the unstable side, we equip $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ with the following natural t-structure.

Definition 3.1.21 (Postnikov t-structure). The **Postnikov t-structure** on $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ has connective part

$$\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\geq 0} := \left\{ F \,\middle|\, \pi_i(F) = 0 \text{ as Nisnevich sheaves for all } i < 0 \right\},$$

where $\pi_i(F)$ denotes the Nisnevich sheafification of the presheaf $X \mapsto \pi_i F(X)$.

***Remark* 3.1.22.** It is not automatic from the definition that the connective subcategory above is the non-negative part of an accessible t-structure on $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$: one needs to verify that it is presentable and closed under colimits and extensions, and that the corresponding truncation functors exist. This requires Morel's stable $\mathbb{A}^1$-connectivity theorem (Theorem 3.1.29 below) to control how $\mathrm{L}_{\mathbb{A}^1}$ interacts with Nisnevich connectivity.

The Postnikov t-structure is the one we care about because it pairs well with σ^∞:

Proposition 3.1.23. *The functor $\sigma^\infty \colon \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \to \mathsf{SH}(S)$ is right t-exact from the Postnikov t-structure to the homotopy t-structure.*

Proof. Let $F \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\geq 0}$. We must show $\sigma^\infty F \in \mathsf{SH}(S)_{\geq 0}$, equivalently $\mathrm{Map}_{\mathsf{SH}(S)}(\sigma^\infty F, E) \simeq *$ for every $E \in \mathsf{SH}(S)_{<0}$. By adjunction this is $\mathrm{Map}(F, \omega^\infty E) \simeq *$.

It therefore suffices to show that $\omega^\infty E$ lies in $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{<0}$, i.e. has $\pi_i = 0$ as a Nisnevich sheaf for $i \geq 0$. By definition of $\mathsf{SH}(S)_{<0}$ (the right orthogonal of $\mathsf{SH}(S)_{\geq \{0\}\times\mathbb{Z}}$), we have

$$\mathrm{Map}_{\mathsf{SH}(S)}\big(\Sigma^{j,j} M_S(X),\, E\big) \simeq * \quad \text{for all } X \in \mathsf{Sm}_S,\ j \in \mathbb{Z}.$$

Taking $j = 0$, we get $\mathrm{Map}(M_S(X), E) \simeq *$, i.e. $\pi_i \omega^\infty E(X) = 0$ for all $i \geq 0$ and $X \in \mathsf{Sm}_S$. The presheaves $\pi_i \omega^\infty E$ thus vanish for $i \geq 0$, and so do their Nisnevich sheafifications. □

The homotopy t-structure on $\mathsf{SH}(S)$ also admits a description directly in terms of the bigraded homotopy sheaves of E.

Definition 3.1.24 (Bigraded homotopy sheaves). For $E \in \mathsf{SH}(S)$ and integers $a, b \in \mathbb{Z}$, the **(a, b)-th homotopy sheaf** of E is the Nisnevich sheaf $\pi_{a,b}(E)$ associated with the presheaf

$$X \longmapsto \pi_0 \mathrm{Map}_{\mathsf{SH}(S)}\big(\Sigma^{a,b}_{\mathrm{BH}} M_S(X),\, E\big),$$

where $\Sigma^{a,b}_{\mathrm{BH}} = (S^1)^{\otimes a} \otimes \mathbb{G}_m^{\otimes b}$ is the BH-bigraded sphere of Notation 3.1.14.

By [Mor03, Proposition 5.1.14], the family $\{\pi_{a,b}\}_{a,b\in\mathbb{Z}}$ is jointly conservative on $\mathsf{SH}(S)$: a morphism in $\mathsf{SH}(S)$ is an equivalence if and only if it induces isomorphisms on all bigraded homotopy sheaves.

To relate the bigraded picture to the Postnikov one, we use the T-spectrum structure on $\mathsf{SH}(S)$ with $T = \mathbb{T}_S = \Sigma^\infty \mathbb{P}^1_S$. Recall from Corollary 2.1.13 that a motivic spectrum E has a family of "levels"

$$E(j) := \omega^{\infty - j} E \simeq \omega^\infty\big(E \otimes \mathbb{T}_S^{\otimes j}\big) \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}),$$

one for each $j \in \mathbb{Z}$.

Definition 3.1.25 (Tate-graded homotopy sheaves). For $E \in \mathsf{SH}(S)$ and $i, j \in \mathbb{Z}$, set

$$\pi_i(E)_j \;\coloneqq\; \pi_i\, E(j) \;=\; \pi_i\, \omega^\infty\big(E \otimes \mathbb{T}_S^{\otimes j}\big),$$

the i-th (classical) Nisnevich homotopy sheaf of the j-th level $E(j)$ of E.

***Remark* 3.1.26.** Unwinding the definition,

$$\pi_i(E)_j \;=\; \pi_{i-j,-j}(E),$$

since $\mathbb{T}_S = S^1 \wedge \mathbb{G}_m \simeq S^{1,1}_{\mathrm{BH}}$. Some authors (e.g. [BEM25, §3.2]) instead use the "$\mathbb{G}_m$-twist" and define $E(j) = \omega^\infty(E \otimes \mathbb{G}_m^{\otimes j})$, in which case the analogous formula becomes $\pi_i(E)_j = \pi_{i,-j}(E)$. Our choice aligns with the general T-spectrum formalism of Section 2.1.3, where $\mathbb{T}_S$ is the invertible object. In either convention the simplicial degree a of $\pi_{a,b}(E)$ agrees with the Morel weight $p - q$ of $\pi_{p,q}(E)$ under the dictionary $(p, q) = (a + b, b)$.

The conservativity of $\{\pi_{a,b}\}$ on $\mathsf{SH}(S)$ (Proposition above) translates, via the substitution $(a, b) \leftrightarrow (i, j)$ of Definition 3.1.25, into the conservativity of $\{\pi_i(E)_j\}_{i,j}$: a morphism in $\mathsf{SH}(S)$ is an equivalence iff it induces isomorphisms on $\pi_i(-)_j$ for all i, j. This is the precise sense in which the Postnikov t-structure on each Tate level $E(j)$ controls E.

The two t-structures — the homotopy t-structure on $\mathsf{SH}(S)$ and the Postnikov t-structure on each level — agree, in a precise sense, as soon as the base S satisfies Morel's $\mathbb{A}^1$-connectivity property.

Morel originally isolated a fundamental property of the base scheme S which governs the interaction between $\mathbb{A}^1$-localization and Nisnevich connectivity.

Definition 3.1.27 ($\mathbb{A}^1$-connectivity property, [Mor05, Definition 1]). A scheme S is said to satisfy the **$\mathbb{A}^1$-connectivity property** if the $\mathbb{A}^1$-localization functor

$$\mathrm{L}_{\mathbb{A}^1} \colon \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S, \mathsf{Sp}) \longrightarrow \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$$

is t-exact for the Postnikov t-structures on both sides; equivalently, if $\mathrm{L}_{\mathbb{A}^1}$ preserves 0-connective objects.

Theorem 3.1.28 (Stable $\mathbb{A}^1$-connectivity, [Mor05, Theorem 6.1.8]). *Every field k satisfies the $\mathbb{A}^1$-connectivity property. That is, for $S = \operatorname{Spec} k$ and every $F \in \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S, \mathsf{Sp})_{\geq n}$, one has $\mathrm{L}_{\mathbb{A}^1} F \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\geq n}$.*

Morel conjectured that the $\mathbb{A}^1$-connectivity property holds more generally; it is known to fail for some schemes, but the following *shifted* version recovers a uniform statement for all qcqs schemes of finite valuative dimension.

Theorem 3.1.29 (Shifted stable $\mathbb{A}^1$-connectivity, [BK25]). *Let S be a qcqs scheme with $d \coloneqq \mathrm{v.dim}(S) < \infty$. For every integer n and every $F \in \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S, \mathsf{Sp})_{\geq n}$, the $\mathbb{A}^1$-localization satisfies*

$$\mathrm{L}_{\mathbb{A}^1} F \;\in\; \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\geq n-d}.$$

In particular, $\mathrm{L}_{\mathbb{A}^1}$ loses at most d degrees of connectivity, and recovers Theorem 3.1.28 when $d = 0$.

It follows from a theorem of Hoyois that, when S satisfies the $\mathbb{A}^1$-connectivity property, the homotopy t-structure on $\mathsf{SH}(S)$ is detected by the Tate-graded homotopy sheaves:

Proposition 3.1.30 ([Hoy15, Theorem 2.3]). *Let S be a scheme satisfying the $\mathbb{A}^1$-connectivity property, and let $E \in \mathsf{SH}(S)$. Then:*

(i) $E \in \mathsf{SH}(S)_{\geq d}$ *if and only if* $\pi_i(E)_j = 0$ *for all* $i - j < d$.

(ii) $E \in \mathsf{SH}(S)_{\leq d}$ *if and only if* $\pi_i(E)_j = 0$ *for all* $i - j > d$.

Equivalently, substituting $(a, b) = (i - j, -j)$*:* $E \in \mathsf{SH}(S)_{\geq d}$ *iff* $\pi_{a,b}(E) = 0$ *for all* $a < d$*, and similarly for the coconnective part.*

Combining Proposition 3.1.23 with Theorem 3.1.29, one sees that the composite

$$\sigma^\infty \circ \mathrm{L}_{\mathbb{A}^1} \colon \mathsf{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_S, \mathsf{Sp}) \longrightarrow \mathsf{SH}(S)$$

sends the n-connective part of the Postnikov t-structure into the $(n - d)$-connective part of the homotopy t-structure, and is t-exact over a base satisfying the $\mathbb{A}^1$-connectivity property.

3.2. Slice filtration and orientation

This section develops two interlocking pieces of the comparison program. Voevodsky's *slice filtration* on $\mathsf{SH}(S)$ (Construction 3.2.7) decomposes a motivic spectrum by its $\mathbb{T}$-cellular complexity, mirroring the way the Postnikov tower decomposes a classical spectrum by its homotopical complexity. Applied to KGL, the graded pieces define $\mathbb{A}^1$-motivic cohomology (Definition 3.2.32), and the comparison theorem of Bachmann–Elmanto–Morrow (Theorem 3.2.42) identifies it with the cdh-motivic cohomology built from K-theory directly, culminating in Voevodsky's slice conjecture (Theorem 3.2.44). *Orientation* (Definition 3.2.23) is the structural input that makes the slice machinery computable on projective bundles — it is the motivic avatar of complex orientation, and the source of the projective bundle formula, Chern classes, and formal group laws.

3.2.1. *Recollection of filtrations*

Definition 3.2.1. Let C be a category. The category of **(decreasing) filtrations** $\mathsf{Fil}(C)$ is defined to be

$$\mathsf{Fil}(C) := \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, C).$$

The category of **increasing filtrations** is defined similarly, with $\mathbb{Z}^{\mathrm{op}}$ replaced by $\mathbb{Z}$.

Intuitively, a filtration $F^\star$ looks like a sequence

$$\cdots \longrightarrow F^{s+1} \longrightarrow F^s \longrightarrow F^{s-1} \longrightarrow \cdots$$

The constant-filtration functor $C \to \mathsf{Fil}(C)$ is fully faithful and exhibits C as a full subcategory of $\mathsf{Fil}(C)$.

Definition 3.2.2. Let C be a category. A **filtration on an object** $X \in C$ is a filtration $F^\star$ together with a map $F^\star \to X$ of filtrations (regarding X as a constant filtration).

More generally, let $\mathbb{Z}^{\mathrm{op}} \cup \{-\infty\}$ denote the poset obtained from $\mathbb{Z}^{\mathrm{op}}$ by adjoining an object $-\infty$ together with a unique morphism $n \to -\infty$ for every $n \in \mathbb{Z}$. Then the **category of pairs** $(F^\star \to X)$ **of filtrations on objects of** C is the functor category $\mathsf{Fun}(\mathbb{Z}^{\mathrm{op}} \cup \{-\infty\}, C)$.
The picture to keep in mind is the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F^{s+1} & \longrightarrow & F^s & \longrightarrow & F^{s-1} & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & X. & & & & \end{array}$$

Definition 3.2.3. Let C be a category.

- A filtration $F^\star$ on X is called **exhaustive** if the canonical map $\mathrm{colim}_{s\in\mathbb{Z}^{\mathrm{op}}} F^s \to X$ is an equivalence.
- A filtration $F^\star$ is called **complete** if $F^\infty := \lim_s F^s \simeq 0$.

Definition 3.2.4. Let C be a category with a terminal object $*$ that admits cofibers, and let $F^\star \in \mathsf{Fil}(C)$ be a filtration. For $i \in \mathbb{Z}$, the ***i*-th associated graded piece** is

$$\mathrm{gr}^i_F = \mathrm{cofib}\left(F^{i+1} \to F^i\right).$$

The associated graded functor is denoted $\mathrm{gr}^\star \colon \mathsf{Fil}(C) \to \mathsf{Gr}(C)$.

3.2.2. *Effective motivic spectra and the slice filtration*

In motivic homotopy theory, a filtration of particular interest is the slice filtration.

Definition 3.2.5. The category of **effective motivic spectra**

$$\mathsf{SH}^{\mathrm{eff}}(S) \subseteq \mathsf{SH}(S)$$

is the smallest full subcategory of $\mathsf{SH}(S)$ closed under colimits and extensions containing $M_S(X)$ for all $X \in \mathsf{Sm}_S$. More generally, for any $j \in \mathbb{Z}$, the category of ***j*-effective motivic spectra** is

$$\mathsf{SH}^{\mathrm{eff}}(S)(j) := \mathbb{T}_S^{\otimes j} \otimes \mathsf{SH}^{\mathrm{eff}}(S).$$

By construction, $\mathsf{SH}^{\mathrm{eff}}(S)$ coincides with $\mathsf{SH}(S)_{\geq \mathbb{Z}\times\{0\}}$ in the sense of Notation 3.1.15, and more generally $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ coincides with $\mathsf{SH}(S)_{\geq \mathbb{Z}\times\{j\}}$. The subcategory $\mathsf{SH}^{\mathrm{eff}}(S)$ is closed under the tensor product of $\mathsf{SH}(S)$, and hence inherits a presentably symmetric monoidal structure.

Proposition 3.2.6. *The inclusion functor $\iota^j \colon \mathsf{SH}^{\mathrm{eff}}(S)(j) \hookrightarrow \mathsf{SH}(S)$ admits a right adjoint r^j.*

Proof. By construction, $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ is a full subcategory of $\mathsf{SH}(S)$ closed under colimits, hence ι^j preserves colimits.

Moreover, $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ is presentable: it is generated under colimits by the set $\{\mathbb{T}_S^{\otimes j} \otimes M_S(X)[i] \mid X \in \mathsf{Sm}_S,\ i \in \mathbb{Z}\}$, and each such generator is compact in $\mathsf{SH}(S)$ (since X_+ is compact in $\mathsf{PShv}(\mathsf{Sm}_S)_*$, and both $\mathrm{L}_{\mathrm{mot}} \colon \mathsf{PShv}(\mathsf{Sm}_S)_* \to \mathsf{H}_*(S)$ and $\sigma^\infty \circ \Sigma^\infty \colon \mathsf{H}_*(S) \to \mathsf{SH}(S)$ preserve compact objects by [Lur09, Lemma 5.5.1.4]). The adjoint functor theorem [Lur09, Corollary 5.5.2.9] then gives a right adjoint r^j. □

Construction 3.2.7 (Slice filtration)**.** By Proposition 3.2.6, there is a sequence of colimit-preserving inclusions

$$\cdots \subset \mathsf{SH}^{\mathrm{eff}}(S)(j+1) \subset \mathsf{SH}^{\mathrm{eff}}(S)(j) \subset \mathsf{SH}^{\mathrm{eff}}(S)(j-1) \subset \cdots \subset \mathsf{SH}(S),$$

giving rise, for every motivic spectrum $E \in \mathsf{SH}(S)$, to a functorial filtration

$$\cdots \to \iota^{j+1} \circ r^{j+1} E \to \iota^j \circ r^j E \to \iota^{j-1} \circ r^{j-1} E \to \cdots \to E.$$

We denote this filtration by $\mathrm{Fil}^\star_{\mathrm{slice}}(E)$ and refer to it as the **slice filtration** of E. We will call $\mathrm{Fil}^0_{\mathrm{slice}}(E)$ the **effective cover** of E.

The slice filtration satisfies the following properties:

- The formation $E \mapsto \mathrm{Fil}^{\star}_{\mathrm{slice}}(E)$ is functorial in E and hence provides a functor $\mathsf{SH}(S) \to \mathsf{FilSH}(S)$.
- $\mathrm{Fil}^{j}_{\mathrm{slice}}(E) = \iota^j \circ r^j(E) \to E$ is universal for maps out of a j-effective motivic spectrum into E: any map $F \to E$ with $F \in \mathsf{SH}^{\mathrm{eff}}(S)(j)$ factors uniquely through $\mathrm{Fil}^{j}_{\mathrm{slice}}(E) \to E$.
- The graded piece

$$s^j(E) \ := \ \mathrm{cofib}\big(\mathrm{Fil}^{j+1}_{\mathrm{slice}}(E) \to \mathrm{Fil}^{j}_{\mathrm{slice}}(E)\big),$$

  called the ***j*-th slice** of E, satisfies $\mathsf{SH}^{\mathrm{eff}}(S)(j+1) \perp s^j(E)$, i.e. $\mathrm{Map}_{\mathsf{SH}(S)}(F, s^j(E)) \simeq *$ for every $F \in \mathsf{SH}^{\mathrm{eff}}(S)(j+1)$.
- $\mathrm{Fil}^{\star}_{\mathrm{slice}}(E)$ is exhaustive (Definition 3.2.3): since every generator $M_S(X)$ of $\mathsf{SH}^{\mathrm{eff}}(S)$ lies in $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ for j sufficiently small (in fact, for all $j \leq 0$), the functors $\iota^j r^j$ are eventually the identity on any fixed compact generator, so $\mathrm{colim}_{j \to -\infty} \iota^j r^j E \simeq E$.

In general, the slice filtration is not complete. The question of convergence is subtle, and addressing it will motivate the introduction of the *effective homotopy t-structure* and, in particular, of *very effective* motivic spectra, to which we now turn.

Applying the functor

$$\mathsf{SH}(S) \xrightarrow{\omega^{\infty}} \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \xrightarrow{\Gamma} \mathsf{Sp}, \tag{3.2.1}$$

where Γ denotes the global sections, we equip $(\omega^{\infty}E)(S)$ with the filtration

$$\mathrm{Fil}^{\star}_{\mathrm{slice}}(\omega^{\infty}E)(S) := \big(\omega^{\infty}\mathrm{Fil}^{\star}_{\mathrm{slice}}(E)\big)(S),$$

which, by a slight abuse of terminology, we will also refer to as the slice filtration on $(\omega^{\infty}E)(S)$. Since ω^{∞} is exact, the graded pieces of this filtration are $\mathrm{gr}^{j}_{\mathrm{slice}}(\omega^{\infty}E)(S) \simeq \omega^{\infty}s^j(E)(S) \in \mathsf{Sp}$ for $j \in \mathbb{Z}$.

***Remark* 3.2.8** (Multiplicativity). For any scheme S, the formation of the slice filtration defines a lax symmetric monoidal functor

$$\mathrm{Fil}^{\star}_{\mathrm{slice}} \colon \mathsf{SH}(S) \to \mathsf{FilSH}(S),$$

where the target is endowed with the Day convolution symmetric monoidal structure. Similarly, passing to graded pieces defines a symmetric monoidal functor

$$\mathrm{gr}^{\star}_{\mathrm{slice}} \colon \mathsf{SH}(S) \to \mathsf{GrSH}(S).$$

Furthermore, each of the functors in (3.2.1) is lax symmetric monoidal.

***Remark* 3.2.9** (Base change for the slice filtration). Let $f : S' \to S$ be a morphism of qcqs schemes. Since f^* preserves colimits and $f^*M_S(X) \simeq M_{S'}(X \times_S S')$ by base change, f^* sends $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ into $\mathsf{SH}^{\mathrm{eff}}(S')(j)$ for every $j \in \mathbb{Z}$.

Consequently, the canonical map $f^*\mathrm{Fil}^{\star}_{\mathrm{slice}}(E) \to f^*E$ of filtered spectra factors uniquely, by the universal property of $\mathrm{Fil}^{\star}_{\mathrm{slice}}(f^*E)$, through a map

$$f^*\mathrm{Fil}^{\star}_{\mathrm{slice}}(E) \ \longrightarrow \ \mathrm{Fil}^{\star}_{\mathrm{slice}}(f^*E),$$

which in turn induces on graded pieces a map of graded spectra

$$f^*s^{\star}E \ \longrightarrow \ s^{\star}f^*E.$$

If f is smooth, then f^* admits a left adjoint $f_\sharp\colon \mathsf{SH}(S') \to \mathsf{SH}(S)$, which preserves j-effective spectra (it preserves colimits, and $f_\sharp M_{S'}(X') \simeq M_S(X')$ for $X' \in \mathsf{Sm}_{S'}$, regarded as smooth over S via f). By the adjunctions $f_\sharp \dashv f^*$ and $\iota^j \dashv r^j$, this gives a natural equivalence $f^* \circ r^j \simeq r^j \circ f^*$, so f^* commutes with the formation of the slice filtration and its slices.

By continuity of $\mathsf{SH}(-)$ ([CD19, Proposition 4.3.4]), the same conclusion holds more generally when f is a pro-smooth affine morphism.

We can also construct the unstable [1] analogue of the slice filtration. For any qcqs scheme S and $j \geq 0$, set

$$\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})(j) \subset \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$$

be the subcategory generated under colimits by $\Sigma^\infty \mathbb{T}_S^{\wedge j} \wedge E$, for $E \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$.

Definition 3.2.10. We say an $\mathbb{A}^1$-invariant Nisnevich sheaf $E \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ is **unstably j-effective**, if $E \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})(j)$.

By the same argument as in Proposition 3.2.6, the inclusion $\iota^j\colon \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})(j) \hookrightarrow \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ admits a right adjoint r^j. In the same way as for the slice filtration, this induces a functorial, exhaustive, multiplicative filtration for $E \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$, now $\mathbb{N}$-indexed. We denote this filtration by

$$\cdots \to \mathrm{Fil}^{j+1}_{\text{un-slice}}(E) \to \mathrm{Fil}^{j}_{\text{un-slice}}(E) \to \mathrm{Fil}^{j-1}_{\text{un-slice}}(E) \to \cdots \to \mathrm{Fil}^{0}_{\text{un-slice}}(E) \simeq E$$

and write $s^j_u(E)$ for the j-th graded piece, which we call the **unstable slices of** E.

We now compare the unstable slice filtration to the slice filtration on $\mathsf{SH}(S)$. Observing that

$$\sigma^\infty(\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})(j)) \subset \mathsf{SH}^{\mathrm{eff}}(S)(j),$$

we obtain Beck–Chevalley style transformations $\iota^j \circ \omega^\infty \to \omega^\infty \circ \iota^j$. They induce, for each $E \in \mathsf{SH}(S)$ and $j \geq 0$, natural maps

$$\mathrm{Fil}^j_{\text{un-slice}}\omega^\infty E \to \omega^\infty \mathrm{Fil}^j_{\text{slice}}(E) \quad \text{and} \quad s^j_u \omega^\infty E \to \omega^\infty s^j E.$$

Since all these functors and transformations are compatibly lax monoidal, the transformations above upgrade to lax monoidal transformations of $\mathbb{N}$-indexed filtered and graded objects respectively:

$$\mathrm{Fil}^j_{\text{un-slice}}\omega^\infty \to \omega^\infty \mathrm{Fil}^j_{\text{slice}} \quad \text{and} \quad s^j_u \omega^\infty \to \omega^\infty s^j.$$

Effective homotopy t-structure and very effective spectra

The category $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ inherits a natural t-structure from the homotopy t-structure on $\mathsf{SH}(S)$.

Definition 3.2.11 (Effective homotopy t-structure)**.** The **effective homotopy t-structure** on $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ is given by

$$\mathsf{SH}^{\mathrm{eff}}(S)(j)_{\geq d} := \mathsf{SH}^{\mathrm{eff}}(S)(j) \cap \mathsf{SH}(S)_{\geq d}, \quad \mathsf{SH}^{\mathrm{eff}}(S)(j)_{\leq d} := \mathsf{SH}^{\mathrm{eff}}(S)(j) \cap \mathsf{SH}(S)_{\leq d},$$

[1]"unstable" here means unstable under $\wedge \mathbb{T}_S$

for $d \in \mathbb{Z}$. Unwinding the definition via Notation 3.1.15, and using that $\iota^j : \mathsf{SH}^{\mathrm{eff}}(S)(j) \hookrightarrow \mathsf{SH}(S)$ preserves colimits so that the intersection of connective classes is generated by the common generator at (d, j), one computes

$$\mathsf{SH}^{\mathrm{eff}}(S)(j)_{\geq d} \;=\; \mathsf{SH}(S)_{\geq\{(d,j)\}}.$$

The connective part of the effective homotopy t-structure on $\mathsf{SH}^{\mathrm{eff}}(S)$ itself (i.e. $j = 0$, $d = 0$) is of independent interest.

Definition 3.2.12 (Very effective motivic spectra)**.** The category of **very effective motivic spectra**

$$\mathsf{SH}^{\mathrm{veff}}(S) \subseteq \mathsf{SH}^{\mathrm{eff}}(S)$$

is the smallest full subcategory of $\mathsf{SH}(S)$ closed under colimits and extensions containing $M_S(X)$ for all $X \in \mathsf{Sm}_S$. More generally, for $j \in \mathbb{Z}$, set

$$\mathsf{SH}^{\mathrm{veff}}(S)(j) := \mathbb{T}_S^{\otimes j} \otimes \mathsf{SH}^{\mathrm{veff}}(S).$$

By construction, $\mathsf{SH}^{\mathrm{veff}}(S) = \mathsf{SH}(S)_{\geq\{(0,0)\}} = \mathsf{SH}^{\mathrm{eff}}(S)_{\geq 0}$, so very effective spectra are precisely the connective objects in $\mathsf{SH}^{\mathrm{eff}}(S)$ for the effective homotopy t-structure.

Remark **3.2.13** (Shift of grading on twists)**.** The tensor product with $\mathbb{T}_S^{\otimes j}$ shifts the bigrading: since $\mathbb{T}_S \simeq S^{2,1}$, we have $\mathbb{T}_S^{\otimes j} \otimes M_S(X) = \Sigma^{2j,j} M_S(X)$, which lies in bidegree (j, j) in the (simplicial, Tate) coordinates of Notation 3.1.15. Consequently, under our inherited convention for the effective homotopy t-structure on $\mathsf{SH}^{\mathrm{eff}}(S)(j)$, the very effective part sits at $\geq j$ rather than ≥ 0:

$$\mathsf{SH}^{\mathrm{veff}}(S)(j) \;=\; \mathsf{SH}(S)_{\geq\{(j,j)\}} \;=\; \mathsf{SH}^{\mathrm{eff}}(S)(j)_{\geq j}.$$

Some authors instead define the t-structure on $\mathsf{SH}^{\mathrm{eff}}(S)(j)$ by transferring along $\mathbb{T}_S^{\otimes j}$, under which convention very effective objects would sit at ≥ 0. We do not adopt this convention, as the ambient homotopy t-structure on $\mathsf{SH}(S)$ is what we ultimately care about.

Proposition 3.2.14. *Let S be a Noetherian scheme of dimension $\leq d$.*

(i) Given $E \in \mathsf{SH}^{\mathrm{veff}}(S)$, then $\omega^\infty(E)$ is $(-d)$-connective (Definition 3.1.21); more generally, $\omega^{\infty,\mathrm{gr}}(E)^j = E(j)$ is $(j-d)$-connective for any $j \geq 0$.

(ii) Suppose $d = 1$, and let $E \in \mathsf{SH}^{\mathrm{eff}}(S)$. If $\omega^\infty(E)$ is connective, then $E \in \mathsf{SH}^{\mathrm{veff}}(S)$.

Proof. (i) For Noetherian S with $\mathrm{Kr.dim}(S) \leq d$ we have $\mathrm{v.dim}(S) = \mathrm{Kr.dim}(S) \leq d$ (see the discussion following **??**), so Theorem 3.1.29 applies with this d. The claim then follows by combining it with Proposition 3.1.23, tracking connectivity through the $\sigma^\infty \dashv \omega^\infty$ adjunction on each Tate level.

(ii) The functor $\omega^\infty : \mathsf{SH}^{\mathrm{eff}}(S) \to \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ is conservative and preserves colimits. Set $T := \sigma^\infty \circ \omega^\infty$; by the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5], ω^∞ is comonadic, so the augmented simplicial object

$$T^{\circ \bullet+1} E \longrightarrow E$$

is ω^∞-split, and hence

$$\operatorname*{colim}_{m \in \mathbf{\Delta}^{\mathrm{op}}} T^{\circ m+1} E \xrightarrow{\sim} E.$$

Now assume that $\omega^\infty(E)$ is connective, and set $E_\bullet := T^{\circ\bullet+1}E$. By [Lur11, Remark 4.3.4],

$$E \;\simeq\; |E_\bullet| \;\simeq\; \operatorname*{colim}_{m\to\infty}\Big(\operatorname*{colim}_{\Delta^{\mathrm{op}}_{\le m}} E_\bullet|_{\Delta^{\mathrm{op}}_{\le m}}\Big),$$

and filtered colimits preserve connectivity, so it suffices to show that each finite colimit $\operatorname{colim}_{\Delta^{\mathrm{op}}_{\le m}} E_\bullet|_{\Delta^{\mathrm{op}}_{\le m}}$ is very effective. By *loc. cit.*, there is a cofiber sequence

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}_{\le(m-1)}} E_\bullet|_{\Delta^{\mathrm{op}}_{\le(m-1)}} \;\longrightarrow\; \operatorname*{colim}_{\Delta^{\mathrm{op}}_{\le m}} E_\bullet|_{\Delta^{\mathrm{op}}_{\le m}} \;\longrightarrow\; M_m[m],$$

where $M_m[m]$ is a summand of E_m. Since $\mathsf{SH}^{\mathrm{veff}}(S)$ is closed under colimits and extensions, induction on m reduces the problem to showing that each $M_m[m]$ is very effective; since M_m is a summand of $E_m = T^{\circ m+1}E$, it further reduces to proving

$$T^{\circ m+1}E \;\in\; \mathsf{SH}(S)_{\ge -m} \qquad \text{for every } m \ge 0. \tag{3.2.2}$$

Indeed, granting (3.2.2), we have $M_m \in \mathsf{SH}(S)_{\ge -m}$, hence $M_m[m] \in \mathsf{SH}(S)_{\ge 0}$; since $M_m[m]$ is also effective (as a summand of a shift of an effective spectrum), it is very effective. We prove (3.2.2) by induction on m. For $m = 0$: by hypothesis $\omega^\infty E \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\ge 0}$, so by Proposition 3.1.23 we have $TE = \sigma^\infty\omega^\infty E \in \mathsf{SH}(S)_{\ge 0}$. For the inductive step, assume $T^{\circ m}E \in \mathsf{SH}(S)_{\ge -(m-1)}$, i.e. $T^{\circ m}E[m-1] \in \mathsf{SH}^{\mathrm{veff}}(S)$. Applying (i) with $d = 1$, we obtain $\omega^\infty(T^{\circ m}E[m-1]) \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\ge -1}$, i.e. $\omega^\infty T^{\circ m}E \in \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})_{\ge -m}$. By Proposition 3.1.23,

$$T^{\circ m+1}E = \sigma^\infty\omega^\infty T^{\circ m}E \;\in\; \mathsf{SH}(S)_{\ge -m},$$

which is (3.2.2).

□

***Remark* 3.2.15** (Motivic recognition principle). The category $\mathsf{SH}^{\mathrm{veff}}(S)$ admits a concrete unstable description, analogous to the classical identification $\mathsf{Sp}_{\ge 0} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{An})$ between connective spectra and grouplike $\mathbb{E}_\infty$-spaces. In the motivic setting, the role played classically by finite sets (via Σ^∞_+) is taken over by *framed correspondences* $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$, leading to an equivalence

$$\mathsf{SH}^{\mathrm{veff}}(S) \;\simeq\; \mathsf{Shv}^{\mathrm{fr}}_{\mathrm{Nis},\mathbb{A}^1}\big(\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)\big)^{\mathrm{gp}}$$

under suitable hypotheses on S. Stabilizing along the $\mathbb{T}_S$-direction then yields Theorem 4.2.1. We will return to this circle of ideas in due course; see [Elm+21] for details.

3.2.3. *Slice filtration for* KGL

Recall from Definition 3.1.10 that the motivic K-theory spectrum KGL is constructed as the β-periodization of KH at $\beta\colon \mathbb{T}_S \xrightarrow{1-[\mathcal{O}(-1)]} \mathrm{KH}$. We now study its slice filtration.

Fix a qcqs scheme S, and denote the effective cover of KGL_S by

$$\mathrm{kgl}_S := \mathrm{Fil}^0_{\mathrm{slice}}\mathrm{KGL}_S \;\in\; \mathsf{CAlg}(\mathsf{SH}(S)).$$

For any $E \in \mathsf{SH}(S)$, the slice filtration satisfies the functorial identifications

$$\mathrm{Fil}^j_{\mathrm{slice}}(\mathbb{T}_S \otimes E) \;\simeq\; \mathbb{T}_S \otimes \mathrm{Fil}^{j-1}_{\mathrm{slice}}(E), \qquad s^j(\mathbb{T}_S \otimes E) \;\simeq\; \mathbb{T}_S \otimes s^{j-1}(E).$$

Applied to $E = \mathrm{KGL}_S$ and combined with Bott periodicity $\beta\colon \mathbb{T}_S \otimes \mathrm{KGL}_S \xrightarrow{\sim} \mathrm{KGL}_S$, this yields

$$\mathbb{T}_S \otimes \mathrm{Fil}^{j-1}_{\mathrm{slice}}\mathrm{KGL}_S \xrightarrow{\sim} \mathrm{Fil}^{j}_{\mathrm{slice}}\mathrm{KGL}_S, \qquad \mathbb{T}_S \otimes s^{j-1}(\mathrm{KGL}_S) \xrightarrow{\sim} s^{j}(\mathrm{KGL}_S).$$

This observation leads to an alternative description of the slice filtration on KGL_S. The Bott element $\beta\colon \mathbb{T}_S \to \mathrm{kgl}_S$ together with the ring structure on kgl_S defines

$$\beta\colon \mathbb{T}_S \otimes \mathrm{kgl}_S \xrightarrow{\beta\otimes\mathrm{can}} \mathrm{kgl}_S \otimes \mathrm{kgl}_S \xrightarrow{\mathrm{mult}} \mathrm{kgl}_S,$$

and by construction the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T}_S \otimes \mathrm{kgl}_S & \xrightarrow{\sim} & \mathrm{Fil}^{1}_{\mathrm{slice}}\mathrm{KGL}_S \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\mathrm{can}} & \\ & \mathrm{kgl}_S. & \end{array}$$

More generally, for each $j \geq 0$ there is a compatible commutative diagram in which $\mathbb{T}_S$ is replaced by $\mathbb{T}_S^{\otimes j}$ and β by its j-fold iterate $\beta^j\colon \mathbb{T}_S^{\otimes j} \otimes \mathrm{kgl}_S \to \mathrm{kgl}_S$. We thus obtain an equivalence between the slice filtration on KGL_S

$$\mathrm{Fil}^{\star}_{\mathrm{slice}}\mathrm{KGL}_S = \cdots \longrightarrow \mathrm{Fil}^{j+1}_{\mathrm{slice}}\mathrm{KGL}_S \longrightarrow \mathrm{Fil}^{j}_{\mathrm{slice}}\mathrm{KGL}_S \longrightarrow \mathrm{Fil}^{j-1}_{\mathrm{slice}}\mathrm{KGL}_S \longrightarrow \cdots$$

and the **Bott filtration**

$$\mathbb{T}_S^{\otimes\star} \otimes \mathrm{kgl}_S = \cdots \longrightarrow \mathbb{T}_S^{\otimes j+1} \otimes \mathrm{kgl}_S \longrightarrow \mathbb{T}_S^{\otimes j} \otimes \mathrm{kgl}_S \longrightarrow \mathbb{T}_S^{\otimes j-1} \otimes \mathrm{kgl}_S \longrightarrow \cdots.$$

Passing to associated gradeds yields a description of the slices of KGL_S as Tate twists of the 0-th slice:

$$s^j\mathrm{KGL}_S \simeq \mathbb{T}_S^{\otimes j} \otimes s^0\mathrm{KGL}_S \quad \text{for } j \geq 0.$$

From this we deduce convergence of the slice filtration in low dimensions.

Lemma 3.2.16. *Let S be a regular Noetherian scheme of Krull dimension ≤ 1. Then $\mathrm{kgl}_S \in \mathsf{SH}^{\mathrm{veff}}(S)$, and $\omega^\infty\mathrm{Fil}^{j}_{\mathrm{slice}}\mathrm{KGL}_S$ is $(j - \mathrm{Kr.dim}(S))$-connective as a Nisnevich sheaf of spectra.*

Proof. For the first claim, by Proposition 3.2.14((ii)) it suffices to show that $\omega^\infty\mathrm{kgl}_S$ is connective. By construction $\mathrm{kgl}_S = \iota^0 r^0\mathrm{KGL}_S$, and for every $X \in \mathsf{Sm}_S$ the spectrum $M_S(X) \in \mathsf{SH}^{\mathrm{eff}}(S)$ lies in the essential image of ι^0. Combining the adjunction $\iota^0 \dashv r^0$ with full faithfulness of ι^0 gives

$$\omega^\infty\mathrm{kgl}_S(X) = \mathrm{Map}_{\mathsf{SH}(S)}\big(M_S(X), \iota^0 r^0\mathrm{KGL}_S\big) \simeq \mathrm{Map}_{\mathsf{SH}(S)}\big(M_S(X), \mathrm{KGL}_S\big) = \mathrm{KH}(X),$$

so $\omega^\infty\mathrm{kgl}_S \simeq \mathrm{KH}|_{\mathsf{Sm}_S}$. Since smooth schemes over a regular Noetherian base remain regular, $\mathrm{KH}(X) \simeq \mathrm{K}(X)$ on Sm_S, which is connective by Proposition 1.3.10.

For the second claim, we have $\mathrm{Fil}^{j}_{\mathrm{slice}}\mathrm{KGL}_S \simeq \mathbb{T}_S^{\otimes j} \otimes \mathrm{kgl}_S$ by the identification established above. Since $\mathrm{kgl}_S \in \mathsf{SH}^{\mathrm{veff}}(S)$, Proposition 3.2.14((i)) applied to the Tate level of index j gives that $\omega^\infty(\mathbb{T}_S^{\otimes j} \otimes \mathrm{kgl}_S)$ is $(j - \mathrm{Kr.dim}(S))$-connective. □

3.2.4. Orientation

A recurring theme in algebraic topology is that a cohomology theory becomes effectively computable on vector bundles — and on the projective bundles built from them — precisely when it is equipped with *orientation* data. At the heart of this theory lies a single, strikingly simple idea: orientation is a *retraction*. Concretely, the inclusion of the basepoint $\mathbb{P}^1 \hookrightarrow \mathcal{P}ic$ (or its topological shadow $P\mathbb{CP}^1 \hookrightarrow P\mathbb{CP}^\infty \simeq BU(1)$) becomes split after tensoring with a ring spectrum E exactly when E carries an orientation. From this retraction, the Thom isomorphism, Chern classes, the projective bundle formula, and formal group laws all unfold as formal consequences.

We first develop the picture in the topological setting (Section 3.2.4), referring to [MS74; BT82] for classical proofs and to [Lin25] for the modern functorial treatment. We then transcribe the whole story motivically (Section 3.2.4), following [BH21; AI25]. Throughout we work with homotopy commutative ring spectra $E \in \mathrm{CAlg}(\mathrm{hSp})$ (resp. $\mathrm{CAlg}(\mathrm{hSH}(S))$); every ring spectrum relevant to these notes carries a genuine $\mathbb{E}_\infty$-enhancement in $\mathrm{CAlg}(\mathrm{Sp})$ (resp. $\mathrm{CAlg}(\mathrm{SH}(S))$). For $E \in \mathrm{SH}(S)$ and $X \in \mathrm{SH}(S)$ we continue to write $E(j)(X) = \mathrm{map}_{\mathrm{SH}(S)}(X, E \otimes \mathbb{T}_S^{\otimes j})$ for the weight-j E-cohomology spectrum, in line with Section 3.1.2.

The topological case

Write $\mathcal{P}ic(\mathbb{S}) \subset \mathrm{Sp}^{\simeq}$ for the **Picard anima** of the sphere spectrum, the grouplike $\mathbb{E}_\infty$-monoid of invertible objects of Sp; its connected components are indexed by $\mathbb{Z}$ via $\Sigma^n\mathbb{S} \mapsto n$. The universal complex line bundle $\gamma_1 \to BU(1) \simeq P\mathbb{CP}^\infty$ fits into the diagram

$$P\mathbb{CP}^1 \hookrightarrow P\mathbb{CP}^\infty \xrightarrow{\gamma_1} \mathcal{P}ic(\mathbb{S}),$$

and its Thom spectrum identifies with $MU(1) \simeq \Sigma^\infty P\mathbb{CP}^\infty[-2]$ [Lin25, Corollary 4.3]; under this identification, the inclusion $P\mathbb{CP}^1 \hookrightarrow P\mathbb{CP}^\infty$ becomes a map of spectra $\mathbb{S} \to MU(1)$ realising the inclusion of a fiber [Lin25, Lemma 4.1].

Definition 3.2.17 (Complex orientation as retraction [Lin25, Definition 1.5])**.** A **complex orientation** of $E \in \mathrm{CAlg}(\mathrm{hSp})$ is a retraction

$$r_E \colon MU(1) \otimes E \longrightarrow \mathbb{S} \otimes E \simeq E$$

of the unit map $\mathbb{S} \otimes E \to MU(1) \otimes E$ induced by $P\mathbb{CP}^1 \hookrightarrow P\mathbb{CP}^\infty$. Equivalently, via the $(-)\otimes E$ adjunction, it is a map $x_E \colon MU(1) \to E$ whose restriction along $\mathbb{S} \to MU(1)$ is the unit of E; in cohomological language, $x_E \in E^2(P\mathbb{CP}^\infty)$ restricting to $1 \in \pi_0 E \simeq \widetilde{E}^2(P\mathbb{CP}^1)$. We say E is *complex-orientable* when such a retraction exists.

***Remark* 3.2.18** (Equivalent formulations of complex orientation). The retraction r_E of Definition 3.2.17 is one of several equivalent encodings of complex orientability. Specialising to the universal complex line bundle γ_1, r_E is exactly the data of a Thom class for γ_1 in the sense of [Lin25, Definition 3.1, Proposition 4.4]; the same notion of *E-orientation* extends to any complex vector bundle $\zeta \colon X \to BU$ via either of the following:

(i) (**Thom class.**) A class $\tau \in E^0(M\zeta)$ whose pullback along every fiber inclusion $\mathbb{S}^{2n} \hookrightarrow M\zeta$ (n the local rank) is, after the canonical $2n$-shift, a unit in $\pi_0 E$ [Lin25, Definition 3.1];

(ii) (**Trivialization.**) A natural equivalence $\zeta \otimes E \simeq \mathrm{const}_E$ of the X-parametrized family of E-module spectra [Lin25, Proposition 3.3].

The equivalence ((i))⇔((ii)) is [Lin25, Proposition 3.3 & Exercise 3.8]; specialising to $\zeta = \gamma_1$, both reduce to the retraction r_E via [Lin25, Proposition 4.4]. The retraction formulation is the most economical, the Thom-class formulation the most classical, and the trivialization formulation the most conceptual.

Theorem 3.2.19 (Thom isomorphism, topological [Lin25, Proposition 3.4]). *Let $\zeta\colon X \to \mathrm{B}U$ be E-orientable and $M\zeta$ its Thom spectrum. There is a natural equivalence*

$$M\zeta \otimes E \;\simeq\; \Sigma^\infty_+ X \otimes E.$$

For a rank-n complex vector bundle $V \to B$ this recovers the classical Thom isomorphism $E^q(B) \xrightarrow{\sim} \widetilde{E}^{q+2n}(\mathrm{Th}(V))$; pulling the Thom class along the zero section defines the ***Euler class*** *$e(V) \in E^{2n}(B)$, and the cofiber sequence of the unit sphere bundle $\mathbb{S}(V) \to B \to \mathrm{Th}(V)$ produces the* ***Gysin sequence***

$$\cdots \to E^q(B) \xrightarrow{p^*} E^q(\mathbb{S}(V)) \xrightarrow{\int} E^{q-2n+1}(B) \xrightarrow{\smile e(V)} E^{q+1}(B) \to \cdots.$$

Proof sketch. Taking colimits over X of the trivialization Remark 3.2.18((ii)) identifies $M\zeta\otimes E$ with $\mathrm{colim}_X \mathrm{const}_E = \Sigma^\infty_+ X \otimes E$. □

The entire theory of complex characteristic classes specializes from this functorial picture by iterating the retraction.

Theorem 3.2.20 (Chern classes and projective bundle formula, topological [Lin25, Theorem 1.9]). *Let (E, r_E) be complex-oriented.*

(i) *For a line bundle $\mathcal{L}$ on B classified by $[\mathcal{L}]\colon B \to P\mathbb{CP}^\infty$, the pullback $c_1(\mathcal{L}) \coloneqq [\mathcal{L}]^* x_E \in E^2(B)$ is the* ***first Chern class***.

(ii) *There is a unique family of classes $c_i(V) \in E^{2i}(B)$ for rank-n complex vector bundles V, extending c_1 and satisfying the* ***projective bundle formula***

$$E^*(\mathbb{P}(V)) \;\simeq\; E^*(B)[\![t]\!]\big/\big(t^n - c_1(V)t^{n-1} + \cdots + (-1)^n c_n(V)\big),$$

where $t = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$.

(iii) *The universal computation reads*

$$E^*(\mathrm{B}U(n)) \;\simeq\; E^*[\![c_1, \ldots, c_n]\!], \qquad |c_i| = 2i.$$

***Remark* 3.2.21** (Top Chern class and splitting principle, topological). The top Chern class coincides with the Euler class, $c_n(V) = e(V) \in E^{2n}(B)$. The **splitting principle** holds: the map $\mathrm{B}U(1)^{\times n} \to \mathrm{B}U(n)$ induces an injection on E-cohomology, and pulling a rank-n bundle V back along the flag bundle $\mathrm{Fl}(V) \to B$ realizes it as an iterated extension of line bundles, so any polynomial identity among Chern classes valid for sums of line bundles is universally valid.

***Remark* 3.2.22** (Formal group law and Quillen universality, topological). The tensor product of line bundles is classified by a map $\otimes\colon P\mathbb{CP}^\infty \times P\mathbb{CP}^\infty \to P\mathbb{CP}^\infty$; pulling x_E back along $\otimes$ yields a power series

$$F_E(x, y) \;\coloneqq\; \otimes^* x_E \;\in\; E^*[\![x, y]\!]$$

computing $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F_E(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2))$. The unital, commutative, associative structure of $\otimes$ makes F_E a graded **formal group law** over E^*. Singular cohomology $\mathrm{H}\mathbb{Z}$ yields the additive law $F = x + y$; complex K-theory $\mathrm{K}U$ yields the multiplicative law $F = x + y + xy$;

complex cobordism $\mathrm{M}U$ is itself complex-orientable and carries the *universal* formal group law. **Quillen's theorem** [Lin25, Theorem 1.14] asserts that the map

$$\mathrm{Hom}_{\mathsf{CAlg}(\mathrm{h}\mathsf{Sp})}(\mathrm{M}U, E) \longrightarrow \{\text{complex orientations of } E\}, \qquad f \mapsto f^*(x_{\mathrm{M}U}),$$

is a bijection, and the composite $L \to \pi_* MU$ from Lazard's ring is an isomorphism. This universality is the organizing principle of chromatic stable homotopy theory [Rav86; Lur10; Lin25].

The motivic case

In the motivic setting, the sphere spectrum $\mathbb{S}$ is replaced by the motivic sphere $\mathbb{1}_S$ and Sp by $\mathsf{SH}(S)$; cohomology classes live in the bigraded theory E^{**} of a motivic ring spectrum, and the Tate weight records $\mathbb{G}_m$-twists invisible under Betti realization. The classical Picard anima $\mathcal{P}ic(\mathbb{S})$ is replaced by the motivic **Picard spectrum** $\mathcal{P}ic$, the connective spectrum object in $\mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp})$ whose underlying $\mathbb{E}_\infty$-space $\Omega^\infty\mathcal{P}ic$ is the motivic Picard stack $\mathrm{B}\mathbb{G}_m$, classifying line bundles up to motivic equivalence. By a slight abuse of notation we also write $\mathcal{P}ic$ for its image in $\mathsf{SH}(S)$ under σ^∞, so that expressions like $\mathcal{P}ic \otimes E$ for $E \in \mathsf{SH}(S)$ make sense; whether $\mathcal{P}ic$ refers to the underlying motivic space, the connective spectrum, or its image in $\mathsf{SH}(S)$ will always be clear from context. The role of $P\mathbb{CP}^1 \hookrightarrow P\mathbb{CP}^\infty$ is played by the map $\mathbb{T}_S \to \mathcal{P}ic$ in $\mathsf{SH}(S)$ classified by $\mathcal{O}(1)$ on $\mathbb{P}^1_S$ (under $\mathbb{T}_S \simeq \mathbb{P}^1_S$). The correspondence between the topological and motivic Picard objects is structural — both are the $\mathbb{E}_\infty$-monoids of invertible objects in the relevant stable monoidal world — rather than a Betti realization. We adopt throughout the abbreviation

$$\widetilde{M}_S \;\coloneqq\; \sigma^\infty_{\mathbb{T}_S} \circ \mathrm{L}_{\mathrm{mot}} \colon \mathsf{PShv}(\mathsf{Sm}_S)_* \longrightarrow \mathsf{SH}(S)$$

for the **reduced motivic spectrum functor**, related to the unreduced functor M_S of Section 2.1 by $M_S(X) \simeq \widetilde{M}_S(X_+)$ for $X \in \mathsf{Sm}_S$. With these conventions the whole architecture of Section 3.2.4 transcribes verbatim, with the retraction of Definition 3.2.17 as its core.

Definition 3.2.23 (Motivic orientation as retraction [AI25]). An **orientation** of $E \in \mathsf{CAlg}(\mathrm{h}\mathsf{SH}(S))$ is a retraction

$$r_E \colon \mathcal{P}ic \otimes E \;\longrightarrow\; \mathbb{T}_S \otimes E$$

of the map $[\mathcal{O}(1)] \otimes \mathrm{id}_E \colon \mathbb{T}_S \otimes E \to \mathcal{P}ic \otimes E$ induced by the Serre twist $\mathcal{O}(1)$ on $\mathbb{P}^1_S$. We say E is *orientable* if such a retraction exists, and call (E, r_E) an *oriented* motivic ring spectrum.

***Remark* 3.2.24** (Equivalent formulations of orientation). By [AI25, Lemma 3.1.4], Definition 3.2.23 admits several equivalent formulations. The data of an orientation on E is equivalent to any of the following:

(i) A retraction r_E of $[\mathcal{O}(1)] \otimes \mathrm{id}_E \colon \mathbb{T}_S \otimes E \to \mathcal{P}ic \otimes E$ in $\mathsf{SH}(S)$;

(ii) A **universal first Chern class** morphism

$$c_E \colon \mathcal{P}ic \longrightarrow E \otimes \mathbb{T}_S$$

in $\mathsf{SH}(S)$ fitting into the commutative triangle

$$\begin{array}{ccc} \mathcal{P}ic & \xrightarrow{\;c_E\;} & E \otimes \mathbb{T}_S \\ {\scriptstyle [\mathcal{O}(1)]}\uparrow & \nearrow_{\eta_E \otimes \mathbb{T}_S} & \\ \mathbb{T}_S & & \end{array}$$

where $\eta_E\colon \mathbb{1}_S \to E$ is the unit. Concretely, c_E is the composite $\mathcal{P}ic \xrightarrow{\eta_E\otimes\mathcal{P}ic} \mathcal{P}ic \otimes E \xrightarrow{r_E} \mathbb{T}_S\otimes E$, so the equivalence ((i))⇔((ii)) is the unit-counit adjunction; the Tate twist appearing in the target reflects that r_E lands in $\mathbb{T}_S \otimes E$ rather than E;

(iii) A Thom class for the tautological line bundle on $\mathcal{P}ic$ whose restriction along $\mathbb{T}_S \to \mathcal{P}ic$ is the unit;

(iv) A section of $E^{\mathcal{P}ic} \to E^{\mathbb{T}_S}$ in $\mathsf{SH}(S)$.

The formulation ((ii)) exhibits an orientation as a universal first Chern class; the retraction formulation ((i)) is more economical and is the one we take as a definition.

***Remark* 3.2.25** (Three commentaries on the definition). First, the target $\mathbb{T}_S\otimes E$ rather than E encodes the *motivic weight shift*: since $\mathbb{T}_S \simeq S^{2,1}$, an orientation produces classes of bidegree $(2,1)$, refining the classical fact that c_1 lives in H^2. Second, the normalization built into the retraction — namely, that r_E splits the inclusion of the basepoint $\mathbb{T}_S \to \mathcal{P}ic$ — is the motivic avatar of the classical requirement that $c_1(\mathcal{O}_{P\mathbb{CP}^1}(1))$ generate $H^2(P\mathbb{CP}^1;\mathbb{Z})$ (compare the topological case of Definition 3.2.17). Third, although $\mathcal{P}ic$ looks much larger than $\mathbb{T}_S$, the $\mathbb{T}_S$*-cellular splitting principle* for $\mathcal{P}ic$ as a filtered colimit

$$\operatorname*{colim}_m \widetilde{M}_S(\mathbb{P}^m_S) \xrightarrow{\sim} \mathcal{P}ic$$

— with transition maps $\mathbb{P}^m_S \hookrightarrow \mathbb{P}^{m+1}_S$ the linear inclusions and each map $\mathbb{P}^m_S \to \Omega^\infty\mathcal{P}ic$ classifying $\mathcal{O}(1)$ — forces the single retraction on the $\mathbb{T}_S$ cell to propagate to a Chern class on every $\mathbb{P}^m_S$. Each $\widetilde{M}_S(\mathbb{P}^m_S)$ is 1-effective (since the cofiber sequences $\widetilde{M}_S(\mathbb{P}^{m-1}_S) \to \widetilde{M}_S(\mathbb{P}^m_S) \to \mathbb{T}_S^{\otimes m}$ build it as an iterated extension of Tate twists $\mathbb{T}_S^{\otimes i}$ for $1 \le i \le m$), and the Chern class c_E of Remark 3.2.24((ii)) therefore refines through the 1-effective cover $\mathrm{Fil}^1_{\mathrm{slice}}(E\otimes\mathbb{T}_S) \to E\otimes\mathbb{T}_S$ of the slice filtration (Construction 3.2.7). This $\mathbb{T}_S$-cellular splitting — reducing $\mathcal{P}ic$ to its $\mathbb{T}_S$-cell — is conceptually parallel to, but distinct from, the higher-rank *splitting principle* for vector bundles via flag bundles discussed in Remark 3.2.30.

For a line bundle $\mathcal{L}$ on a smooth S-scheme Y, the classifying map $[\mathcal{L}]\colon Y \to \Omega^\infty\mathcal{P}ic$ composes with the Chern class map c_E of Remark 3.2.24((ii)) to give the **first Chern class**

$$c_1(\mathcal{L}) \in \pi_0\, E(1)(M_S(Y)).$$

Multiplication in E combined with the diagonal $\Delta\colon Y \to Y\times_S Y$ defines a cup-product map

$$c_1(\mathcal{L})\cdot - \colon \underline{\mathrm{Map}}\big(M_S(Y),E\big) \longrightarrow \underline{\mathrm{Map}}\big(M_S(Y),E\otimes\mathbb{T}_S\big)$$

in $\mathsf{SH}(S)$ (note that source and target differ by a Tate twist, so this is not literally an endomorphism — each cup product with c_1 raises weight by one), iterating to powers $c_1(\mathcal{L})^i \in \pi_0\, E(i)(M_S(Y))$ living in weight-i cohomology. Applied to $Y = \mathbb{P}^d_S$ with $\mathcal{L} = \mathcal{O}(1)$, these powers are the building blocks of the projective bundle formula.

Definition 3.2.26 (Projective bundle formula). An oriented motivic commutative ring spectrum $(E, r_E) \in \mathsf{CAlg}(\mathsf{hSH}(S))$ **satisfies the projective bundle formula** if, for every $d \ge 1$, the map

$$\sum_{i=0}^{d} c_1(\mathcal{O}(1))^i\cdot\pi^*\colon \bigoplus_{i=0}^{d} \mathbb{T}_S^{\otimes(-i)}\otimes E \longrightarrow \underline{\mathrm{Map}}\big(M_S(\mathbb{P}^d_S),\, E\big)$$

is an equivalence in $\mathsf{SH}(S)$, where $\pi\colon \mathbb{P}^d_S \to S$ is the structure map.

Theorem 3.2.27 (Projective bundle formula for oriented spectra). *Every oriented* $(E, r_E) \in \mathrm{CAlg}(\mathrm{hSH}(S))$ *satisfies the projective bundle formula.*

Proof sketch. The argument is a cell induction on d. By Corollary 1.4.8, $\mathbb{P}^d_S/\mathbb{P}^{d-1}_S \simeq S^{2d,d}$, which after stabilization is $\mathbb{T}_S^{\otimes d}$; the cofiber sequence $\mathbb{P}^{d-1}_S \hookrightarrow \mathbb{P}^d_S \to \mathbb{P}^d_S/\mathbb{P}^{d-1}_S$ yields

$$M_S(\mathbb{P}^{d-1}_S) \longrightarrow M_S(\mathbb{P}^d_S) \longrightarrow \mathbb{T}_S^{\otimes d}.$$

Applying $\underline{\mathrm{Map}}(-, E)$ gives a fiber sequence

$$\mathbb{T}_S^{\otimes(-d)} \otimes E \longrightarrow \underline{\mathrm{Map}}\big(M_S(\mathbb{P}^d_S), E\big) \longrightarrow \underline{\mathrm{Map}}\big(M_S(\mathbb{P}^{d-1}_S), E\big).$$

The key claim is that this fiber sequence *splits* canonically once E is oriented. Set $t := c_1(\mathcal{O}(1)) \in \pi_0\, E(1)(M_S(\mathbb{P}^d_S))$ and consider its d-th power

$$t^d : \mathbb{T}_S^{\otimes(-d)} \otimes E \longrightarrow \underline{\mathrm{Map}}\big(M_S(\mathbb{P}^d_S), E\big).$$

Splitting on the top cell. Post-composing t^d with the quotient map $M_S(\mathbb{P}^d_S) \to M_S(\mathbb{P}^d_S)/M_S(\mathbb{P}^{d-1}_S) \simeq \mathbb{T}_S^{\otimes d}$ amounts, via the triangle of Remark 3.2.24((ii)), to evaluating the universal first Chern class $c_E : \mathcal{P}ic \to E \otimes \mathbb{T}_S$ on the basepoint inclusion $\mathbb{T}_S \to \mathcal{P}ic$. This evaluation is the unit $\eta_E \otimes \mathbb{T}_S$ *by the very definition of the retraction* (the triangle commutes: $r_E \circ ([\mathcal{O}(1)] \otimes \mathrm{id}_E) = \mathrm{id}_{\mathbb{T}_S \otimes E}$), so t acts as the identity on the $\mathbb{T}_S$ cell tensored with η_E. Iterating, t^d is an equivalence on the top cell, providing the desired splitting onto the $\mathbb{T}_S^{\otimes(-d)} \otimes E$ summand.

Splitting on the bottom cell. Restriction along $M_S(\mathbb{P}^{d-1}_S) \hookrightarrow M_S(\mathbb{P}^d_S)$ sends $t = c_1(\mathcal{O}_{\mathbb{P}^d_S}(1))$ to $c_1(\mathcal{O}_{\mathbb{P}^{d-1}_S}(1))$ by naturality of the universal Chern class.

Inducting on d starting from $d = 1$ (which is the defining retraction) assembles the displayed direct-sum decomposition. □

***Remark* 3.2.28** (Relation to the pbf condition of Proposition 3.1.7). The $\mathbb{P}^1$-bundle formula attached to a Q_S-module in Construction 3.1.3 is precisely the $d = 1$ case of Definition 3.2.26; Theorem 3.2.27 then says that the higher-degree formulas are automatic consequences of orientation.

Example 3.2.29 (KGL is oriented; the Bott element as restricted first Chern class). The motivic K-theory spectrum KGL_S carries a canonical orientation

$$r^{\mathrm{KGL}} : \mathcal{P}ic \otimes \mathrm{KGL}_S \longrightarrow \mathbb{T}_S \otimes \mathrm{KGL}_S$$

coming from the Bachmann–Elmanto–Morrow orientation map $1 - (-)^\vee : \Omega^\infty \mathcal{P}ic \to \Omega^\infty \mathrm{k}$ of [BEM25, Definition 4.4] (written there as K^{cn} in our notation k), which (upon passage to $\mathrm{SH}(S)$ and Bott periodization) classifies, for each smooth S-scheme Y and line bundle $\mathcal{L}$ on Y, the first Chern class

$$c_1^{\mathrm{KGL}}(\mathcal{L}) \;=\; 1 - [\mathcal{L}^\vee] \;\in\; \mathrm{KH}_0(Y).$$

Specializing to $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^1_S$ gives

$$c_1^{\mathrm{KGL}}(\mathcal{O}(1)) \;=\; 1 - [\mathcal{O}(1)^\vee] \;=\; 1 - [\mathcal{O}(-1)] \;=\; \beta,$$

recovering the Bott element of Section 3.1.6. Conversely, r^{KGL} is the universal extension of β to all line bundles: the existence of a retraction defining r^{KGL} comes from the K-theoretic projective bundle formula

$$\mathrm{KH}(\mathbb{P}^\infty_S) \;\simeq\; \lim_m \mathrm{KH}(\mathbb{P}^m_S) \;\simeq\; \mathrm{KH}(S)[\![t]\!]$$

with $t = 1-[\mathcal{O}(-1)]$, which propagates the single retraction on the $\mathbb{T}_S$ cell through the filtered colimit $\Omega^\infty \mathcal{P}ic \simeq \operatorname{colim}_m \mathbb{P}^m_S$ in motivic spaces (equivalently, $\mathcal{P}ic \simeq \operatorname{colim}_m \widetilde{M}_S(\mathbb{P}^m_S)$ in $\mathsf{SH}(S)$; cf. Remark 3.2.25). Combined with Theorem 3.2.27, the orientation yields the projective bundle formula for KGL_S in all degrees.

***Remark* 3.2.30** (Higher Chern classes, Euler class, and formal group law). Once Theorem 3.2.27 and Example 3.2.29 are in hand, the classical template promotes the first Chern class to higher characteristic classes. Given a rank-r vector bundle $\mathcal{E}$ on a smooth S-scheme Y, the projective bundle $\mathbb{P}(\mathcal{E}) \to Y$ admits the universal quotient (Serre) line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and Theorem 3.2.27 applied relatively (a formal consequence of the absolute statement, cf. [AI25, Lemma 3.2.3]) produces a unique identity

$$c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^r \;=\; \sum_{i=1}^{r} (-1)^{i-1} c_i(\mathcal{E}) \cdot c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^{r-i}$$

in $\pi_0 E(r)(M_S(Y))$, defining the **higher Chern classes** $c_i(\mathcal{E}) \in \pi_0 E(i)(M_S(Y))$ for $1 \le i \le r$. The top Chern class $c_r(\mathcal{E})$ is the **Euler class** $e(\mathcal{E}) \in \pi_0 E(r)(M_S(Y))$, paralleling the topological identity $c_n(V) = e(V)$ of Remark 3.2.21. The **splitting principle** for vector bundles holds: pulling back $\mathcal{E}$ to the flag bundle $\mathrm{Fl}(\mathcal{E}) \to Y$ reduces computations to the case where $\mathcal{E}$ is a direct sum of line bundles, exactly as in topology. This is the rank-r analogue of the $\mathbb{T}_S$-cellular splitting for $\mathcal{P}ic$ in Remark 3.2.25: there the orientation propagates from $\mathbb{T}_S$ to all of $\mathcal{P}ic$; here it propagates from line bundles to all vector bundles.

The tensor product of line bundles, classified by $\mu\colon \mathcal{P}ic \times \mathcal{P}ic \to \mathcal{P}ic$, pulled back along c_1 produces a power series

$$F_E(x,y) \;=\; \mu^* c_1 \;\in\; E^*(S)[\![x,y]\!]$$

— the **formal group law** of E — computing $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F_E(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2))$. Distinguished examples:

- $\mathrm{H}\mathbb{Z}$, with the additive law $F(x,y) = x + y$;
- KGL, with the multiplicative law $F(x,y) = x + y - xy$, computed from Example 3.2.29 via

  $$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = 1-[\mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee] = 1-(1-c_1(\mathcal{L}_1))(1-c_1(\mathcal{L}_2)) = c_1(\mathcal{L}_1)+c_1(\mathcal{L}_2)-c_1(\mathcal{L}_1)c_1(\mathcal{L}_2);$$

  the Bott equivalence $\mathrm{KGL} \otimes \mathbb{T}_S \simeq \mathrm{KGL}$ is implicit, collapsing the bigrading so that both sides live in the same graded component. This matches the formula $\otimes^*(c) = x_1 + x_2 - \beta\, x_1 x_2$ of [AHI24a, Lemma 8.3] (with $\beta = 1$ after inverting). The sign opposite to the topological KU formula $F = x + y + xy$ in Remark 3.2.22 is purely a convention: choosing the Bott element $1 - [\mathcal{O}(-1)] = 1 - [\mathcal{O}(1)^\vee]$ rather than $1 - [\mathcal{O}(1)]$ gives c_1 in terms of duals, and the dual flips the sign in the multiplicative law. The two formulas are isomorphic over $\mathbb{Z}$ via $x \mapsto -x$, hence define the same isomorphism class of formal group law.
- MGL, whose formal group law will be shown in Section 3.4 to be universal in the Lazard sense.

MGL provides a rich intermediate layer between $\mathrm{H}\mathbb{Z}$ and KGL, the basis of the motivic chromatic picture.

***Remark* 3.2.31** (Motivic Thom isomorphism, Euler class, and Gysin sequence). The motivic Thom isomorphism, Euler class, and Gysin sequence transcribe *verbatim* from their topological avatars in Theorem 3.2.19, with each rank-n complex vector bundle replaced by a motivic vector bundle of rank n and each topological shift by $2n$ replaced by a Tate twist $\mathbb{T}_S^{\otimes n}$:

- For an oriented E and an E-orientable motivic vector bundle $\zeta\colon X \to \mathrm{BGL}$ with Thom space $\mathrm{Th}(\zeta)$, there is a natural equivalence $\widetilde{M}_S(\mathrm{Th}(\zeta)) \otimes E \simeq M_S(X) \otimes E$, recovering for a rank-$n$ bundle $V \to B$ the Thom isomorphism $E^{p,q}(B) \xrightarrow{\sim} \widetilde{E}^{p+2n,q+n}(\mathrm{Th}(V))$;
- Pulling back the Thom class along the zero section gives the **motivic Euler class** $e(V) \in E^{2n,n}(B)$;
- The cofiber sequence of the unit sphere bundle (in the motivic sense) produces a **motivic Gysin sequence** relating $E^{*,*}(B)$, $E^{*,*}(B \setminus 0)$, and the Euler class.

See [BH21, §16.5] for the motivic Thom isomorphism in the framework of motivic Thom spectra, and [AHI24b, §2] for the modern Gysin-map theory underlying the motivic Gysin sequence.

The universal example is the motivic Thom spectrum MGL of Section 3.4: every orientation of $E \in \mathsf{CAlg}(\mathrm{h}\mathsf{SH}(S))$ factors uniquely through a morphism $\mathrm{MGL} \to E$, the motivic analogue of Quillen's theorem of Remark 3.2.22.

3.2.5. $\mathbb{A}^1$-motivic cohomology

Let S be a qcqs scheme. Applying the slice filtration of Construction 3.2.7 to the motivic K-theory spectrum KGL_S simultaneously assembles a motivic cohomology theory as its graded pieces and endows $\mathrm{KH}(S)$ with a natural multiplicative filtration.

Definition 3.2.32 ($\mathbb{A}^1$-motivic cohomology). For each qcqs scheme S and $j \in \mathbb{Z}$, the **$\mathbb{A}^1$-motivic cohomology of weight** j is

$$\mathbb{Z}(j)^{\mathbb{A}}(S) \;:=\; \mathrm{map}_{\mathsf{SH}(S)}\big(\mathbb{1}_S,\, s^j\mathrm{KGL}_S\big),$$

and the **$\mathbb{A}^1$-motivic filtration on** KH is

$$\mathrm{Fil}^{\star}_{\mathbb{A}}\mathrm{KH}(S) \;:=\; \mathrm{map}_{\mathsf{SH}(S)}\big(\mathbb{1}_S,\, \mathrm{Fil}^{\star}_{\mathrm{slice}}\mathrm{KGL}_S\big).$$

These assemble into presheaves of (respectively, filtered) spectra on $\mathsf{Sch}^{\mathrm{qcqs}}$.

***Remark* 3.2.33** (Levels of the motivic Eilenberg–MacLane spectrum). Write $\mathrm{H}\mathbb{Z}_S := s^0\mathrm{KGL}_S \in \mathsf{CAlg}(\mathsf{SH}(S))$. The Bott identification $s^j\mathrm{KGL}_S \simeq \mathbb{T}_S^{\otimes j} \otimes s^0\mathrm{KGL}_S$ established in the slice filtration for KGL above presents $\mathbb{Z}(j)^{\mathbb{A}}$ as the j-th level of $\mathrm{H}\mathbb{Z}_S$ in the $\mathbb{T}$-convention of Section 3.1.2:

$$\mathbb{Z}(j)^{\mathbb{A}}(S) \;\simeq\; \mathrm{map}_{\mathsf{SH}(S)}\big(\mathbb{1}_S,\, \mathrm{H}\mathbb{Z}_S \otimes \mathbb{T}_S^{\otimes j}\big) \;=\; \mathrm{H}\mathbb{Z}_S(j)(S).$$

Fixing S and letting $Y \in \mathsf{Sm}_S$ vary, the restriction of $\mathbb{Z}(j)^{\mathbb{A}}$ to Sm_S is the $\mathbb{A}^1$-invariant Nisnevich sheaf of spectra $\omega^{\infty}s^j\mathrm{KGL}_S$; in particular, using $\mathrm{KGL}_Y \simeq f_Y^*\mathrm{KGL}_S$ (Proposition 3.1.11) together with the fact that the slice filtration commutes with smooth pullback (Remark 3.2.9),

$$\mathbb{Z}(j)^{\mathbb{A}}(Y) \;\simeq\; \mathrm{map}_{\mathsf{SH}(S)}\big(M_S(Y),\, s^j\mathrm{KGL}_S\big), \qquad \mathrm{Fil}^{j}_{\mathbb{A}}\mathrm{KH}(Y) \;\simeq\; \mathrm{map}_{\mathsf{SH}(S)}\big(M_S(Y),\, \mathrm{Fil}^{j}_{\mathrm{slice}}\mathrm{KGL}_S\big)$$

for every $Y \in \mathsf{Sm}_S$.

***Remark* 3.2.34** (Multiplicativity and exhaustiveness). Since $\mathrm{Fil}^{\star}_{\mathrm{slice}}$ is lax symmetric monoidal (Construction 3.2.7) and $\mathrm{map}_{\mathsf{SH}(S)}(\mathbb{1}_S, -)$ is lax symmetric monoidal, $\mathrm{Fil}^{\star}_{\mathbb{A}}\mathrm{KH}$ is an $\mathbb{E}_\infty$-algebra in filtered presheaves of spectra, and $\mathbb{Z}(\star)^{\mathbb{A}}$ inherits the structure of a graded $\mathbb{E}_\infty$-algebra. Exhaustiveness of the slice filtration implies $\mathrm{colim}_{j\to-\infty}\mathrm{Fil}^{j}_{\mathbb{A}}\mathrm{KH} \simeq \mathrm{KH}$.

For applications over non-regular schemes and for cdh descent, we also need a variant of $\mathbb{Z}(\star)^{\mathbb{A}}$ built by pulling back the slice filtration from the absolute base. The need for such a variant comes from the fact that the slice filtration does not commute with arbitrary pullbacks (Remark 3.2.9); the KGL-spectra themselves, however, are stable under arbitrary pullback (Proposition 3.1.11).

Definition 3.2.35 ($\mathbb{A}^1$-cdh-motivic cohomology)**.** For any qcqs scheme S with structure map $f_S\colon S \to \mathrm{Spec}(\mathbb{Z})$ and $j \in \mathbb{Z}$, set

$$\mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}}(S) \;\coloneqq\; \mathrm{map}_{\mathsf{SH}(S)}\big(\mathbb{1}_S,\, f_S^* s^j \mathrm{KGL}_{\mathbb{Z}}\big),$$

and

$$\mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}(S) \;\coloneqq\; \mathrm{map}_{\mathsf{SH}(S)}\big(\mathbb{1}_S,\, f_S^* \mathrm{Fil}^{\star}_{\mathrm{slice}}\mathrm{KGL}_{\mathbb{Z}}\big).$$

These are presheaves of (respectively, filtered) spectra on $\mathsf{Sch}^{\mathrm{qcqs}}$.

***Remark* 3.2.36** (Comparison map and cdh descent)**.** By Proposition 3.1.11, $f_S^*\mathrm{KGL}_{\mathbb{Z}} \simeq \mathrm{KGL}_S$, so the base change transformation of Remark 3.2.9 yields a canonical multiplicative map

$$f_S^*\mathrm{Fil}^{\star}_{\mathrm{slice}}\mathrm{KGL}_{\mathbb{Z}} \;\longrightarrow\; \mathrm{Fil}^{\star}_{\mathrm{slice}}(f_S^*\mathrm{KGL}_{\mathbb{Z}}) \;\simeq\; \mathrm{Fil}^{\star}_{\mathrm{slice}}\mathrm{KGL}_S,$$

which is not an equivalence in general. Applying $\mathrm{map}_{\mathsf{SH}(S)}(\mathbb{1}_S, -)$ and passing to graded pieces, it induces natural comparison maps

$$\mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH} \longrightarrow \mathrm{Fil}^{\star}_{\mathbb{A}}\mathrm{KH}, \qquad \mathbb{Z}(\star)^{\mathbb{A},\mathrm{cdh}} \longrightarrow \mathbb{Z}(\star)^{\mathbb{A}}.$$

Since SH is a cdh sheaf on $\mathsf{Sch}^{\mathrm{qcqs}}$ (Corollary 2.3.12) and the targets $s^j\mathrm{KGL}_{\mathbb{Z}}$ and $\mathrm{Fil}^{\star}_{\mathrm{slice}}\mathrm{KGL}_{\mathbb{Z}}$ are fixed objects of $\mathsf{SH}(\mathrm{Spec}(\mathbb{Z}))$, the presheaves $\mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}}$ and $\mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}$ on $\mathsf{Sch}^{\mathrm{qcqs}}$ are cdh sheaves, justifying the “cdh” in the notation.

The following theorem summarizes the basic structural properties of both theories; we defer its proof to [BEM25]. The refinements concerning $\mathbb{Z}(0)^{\mathbb{A}}$ and $\mathbb{Z}(1)^{\mathbb{A}}$ — namely the $\mathbb{Z}$-linear structure and the motivic first Chern class of [BEM25, Theorem 4.39(3)&(4)] — are the subject of Section 3.3.

Theorem 3.2.37 ([BEM25, Theorem 4.39])**.** *Let S be a qcqs scheme. Then:*

(i) $\mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}(S)$ and $\mathrm{Fil}^{\star}_{\mathbb{A}}\mathrm{KH}(S)$ are functorial, multiplicative, exhaustive $\mathbb{Z}$-indexed filtrations on $\mathrm{KH}(S)$, with graded pieces naturally and multiplicatively given by

$$\mathrm{gr}^j_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}(S) \;\simeq\; \mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}}(S), \qquad \mathrm{gr}^j_{\mathbb{A}}\mathrm{KH}(S) \;\simeq\; \mathbb{Z}(j)^{\mathbb{A}}(S) \quad \text{for } j \in \mathbb{Z}.$$

The filtrations are related by a natural multiplicative map $\mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}(S) \to \mathrm{Fil}^{\star}_{\mathbb{A}}\mathrm{KH}(S)$, which on graded pieces recovers the comparison map of Remark 3.2.36.

(ii) For each $j \in \mathbb{Z}$, the presheaves $\mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}}$ and $\mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}$ are finitary, $\mathbb{A}^1$-invariant cdh sheaves on $\mathsf{Sch}^{\mathrm{qcqs}}$; meanwhile $\mathbb{Z}(j)^{\mathbb{A}}$ and $\mathrm{Fil}^{\star}_{\mathbb{A}}\mathrm{KH}$ are $\mathbb{A}^1$-invariant Nisnevich sheaves which commute with filtered colimits along smooth affine transition maps.

Corollary 3.2.38 (Projective bundle formula)**.** *For each $j \in \mathbb{Z}$ and $d \geq 0$, the map*

$$\sum_{i=0}^{d} c_1(\mathcal{O}(1))^i \cdot \pi^* \colon \bigoplus_{i=0}^{d} \mathbb{Z}(j-i)^{\mathbb{A},\mathrm{cdh}}(S) \;\xrightarrow{\sim}\; \mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}}(\mathbb{P}^d_S)$$

is an equivalence, and the analogous statement holds for $\mathbb{Z}(\star)^{\mathbb{A}}$.

Proof sketch. By Example 3.2.29, KGL_S carries a canonical orientation, and $f_S^*\mathrm{KGL}_{\mathbb{Z}} \simeq \mathrm{KGL}_S$ (Proposition 3.1.11). The slice filtration $\mathrm{Fil}^{\star}_{\mathrm{slice}}$ is lax symmetric monoidal (Construction 3.2.7), so the 0-th slice $\mathrm{H}\mathbb{Z}_S := s^0\mathrm{KGL}_S$ inherits an orientation from KGL_S, as does $f_S^* s^0\mathrm{KGL}_{\mathbb{Z}}$. Applying Theorem 3.2.27 to $\mathrm{H}\mathbb{Z}_S$ (resp. $f_S^* s^0\mathrm{KGL}_{\mathbb{Z}}$) and taking the j-th Tate level via Remark 3.2.33 yields the claim. ☐

Lisse and cdh motivic cohomology

Following [BEM25, §7], we record a second construction of motivic cohomology on qcqs schemes, obtained by left Kan extension from smooth schemes over a base and cdh sheafification. This variant will be the target of the main comparison theorem stated below.

Definition 3.2.39 (Lisse motivic cohomology [BEM25, Definition 7.1]). Let B be either a field or a mixed-characteristic Dedekind domain, and $j \in \mathbb{Z}$. The **weight-j lisse motivic cohomology** of qcqs B-schemes is the left Kan extension to $\mathsf{Sch}_B^{\mathrm{qcqs}}$ of the restriction of $\mathbb{A}^1$-motivic cohomology to smooth B-schemes:

$$\mathbb{Z}(j)_B^{\mathrm{lse}} := \mathrm{L}_B^{\mathrm{sm}}\big(\mathbb{Z}(j)^{\mathbb{A}}|_{\mathsf{Sm}_B}\big) : \big(\mathsf{Sch}_B^{\mathrm{qcqs}}\big)^{\mathrm{op}} \longrightarrow \mathsf{D}(\mathbb{Z}).$$

When $B = \mathrm{Spec}(\mathbb{Z})$ we drop the subscript and write $\mathbb{Z}(j)^{\mathrm{lse}}$.

Definition 3.2.40 (cdh-motivic cohomology [BEM25, Definition 7.8]). For $j \in \mathbb{Z}$, the **weight-j cdh-motivic cohomology** is the cdh sheafification of $\mathbb{Z}(j)^{\mathrm{lse}}$:

$$\mathbb{Z}(j)^{\mathrm{cdh}} := \mathrm{L}_{\mathrm{cdh}}\mathbb{Z}(j)^{\mathrm{lse}} : \big(\mathsf{Sch}^{\mathrm{qcqs}}\big)^{\mathrm{op}} \longrightarrow \mathsf{D}(\mathbb{Z}).$$

By [BEM25, Remark 4.20], $\mathbb{Z}(j)^{\mathbb{A}} = 0$ for $j < 0$; consequently $\mathbb{Z}(j)^{\mathrm{lse}} = 0$ and $\mathbb{Z}(j)^{\mathrm{cdh}} = 0$ in negative weights.

***Remark* 3.2.41** (Comparison map). By the universal property of $\mathbb{Z}(\star)^{\mathrm{cdh}}$ as the cdh sheafification of a left Kan extension, and since $\mathbb{Z}(\star)^{\mathbb{A},\mathrm{cdh}}$ is a cdh sheaf (Theorem 3.2.37((ii))), to produce a canonical multiplicative map of graded $\mathbb{E}_\infty$-algebras

$$\mathbb{Z}(\star)^{\mathrm{cdh}} \longrightarrow \mathbb{Z}(\star)^{\mathbb{A},\mathrm{cdh}}$$

it suffices to produce one on $\mathsf{Sm}_{\mathbb{Z}}$. On smooth $\mathbb{Z}$-schemes the slice-base-change comparison map $\mathbb{Z}(\star)^{\mathbb{A},\mathrm{cdh}} \to \mathbb{Z}(\star)^{\mathbb{A}}$ of Remark 3.2.36 is an equivalence (cf. Remark 3.2.9 for smooth f), and composing with the identity of $\mathbb{Z}(\star)^{\mathbb{A}}|_{\mathsf{Sm}_{\mathbb{Z}}}$ and taking the inverse yields the desired map.

The central comparison theorem of [BEM25] identifies, after $\mathbb{A}^1$-localization, the cdh-motivic cohomology of Definition 3.2.40 with the $\mathbb{A}^1$-cdh-motivic cohomology of Definition 3.2.35.

Theorem 3.2.42 ([BEM25, Theorem 9.1]). *The canonical comparison map*

$$\mathrm{L}_{\mathbb{A}^1}\mathbb{Z}(\star)^{\mathrm{cdh}} \xrightarrow{\sim} \mathbb{Z}(\star)^{\mathbb{A},\mathrm{cdh}}$$

is an equivalence of $\mathbb{E}_\infty$-algebras of graded cdh sheaves of complexes on qcqs schemes.

3.2.6. The slice conjecture and consequences

The comparison theorem Theorem 3.2.42, together with the vanishing of $\mathbb{Z}(\star)^{\mathrm{cdh}}$ in negative weights, has a striking consequence first conjectured by Voevodsky: the slice filtration on KGL is stable under arbitrary base change. The key intermediate input is a vanishing for the slice filtration on the 0-th slice $\mathrm{H}\mathbb{Z}_{\mathbb{Z}}$.

Corollary 3.2.43. *For every qcqs scheme S with structure map $f_S \colon S \to \mathrm{Spec}(\mathbb{Z})$,*

$$\mathrm{Fil}^1_{\mathrm{slice}}\big(f_S^* s^0 \mathrm{KGL}_{\mathbb{Z}}\big) \;\simeq\; 0.$$

Proof. Bott periodicity $\mathbb{T}_{\mathbb{Z}} \otimes \mathrm{KGL}_{\mathbb{Z}} \simeq \mathrm{KGL}_{\mathbb{Z}}$ together with the slice covariance $s^j(\mathbb{T}\otimes -) \simeq \mathbb{T}\otimes s^{j-1}(-)$ established in Section 3.2.3 gives $s^0\mathrm{KGL}_{\mathbb{Z}} \otimes \mathbb{T}_{\mathbb{Z}}^{-1} \simeq s^{-1}\mathrm{KGL}_{\mathbb{Z}}$. By definition of $\mathrm{Fil}^1_{\mathrm{slice}} = \iota^1 r^1$ and the generators $\mathbb{T}_S \otimes M_S(Y)[i]$ of $\mathsf{SH}^{\mathrm{eff}}(S)(1)$, the vanishing $\mathrm{Fil}^1_{\mathrm{slice}}(f_S^* s^0\mathrm{KGL}_{\mathbb{Z}}) \simeq 0$ amounts to

$$\mathrm{Map}_{\mathsf{SH}(S)}\big(\mathbb{T}_S \otimes M_S(Y)[i],\; f_S^* s^0 \mathrm{KGL}_{\mathbb{Z}}\big) \simeq 0 \qquad \text{for all } Y \in \mathsf{Sm}_S,\; i \in \mathbb{Z}.$$

Using the adjunction $\mathbb{T}_S \otimes - \dashv (-)\otimes \mathbb{T}_S^{-1}$, base change of $\otimes$, and the smooth-pullback identification $\mathrm{Map}_{\mathsf{SH}(S)}(M_S(Y), -) \simeq \mathrm{Map}_{\mathsf{SH}(Y)}(\mathbb{1}_Y, Y^*(-))$, the displayed mapping spectrum rewrites as

$$\mathrm{Map}_{\mathsf{SH}(Y)}\big(\mathbb{1}_Y,\; f_Y^* s^{-1}\mathrm{KGL}_{\mathbb{Z}}\big)[-i] \;=\; \mathbb{Z}(-1)^{\mathbb{A},\mathrm{cdh}}(Y)[-i].$$

By Theorem 3.2.42, $\mathbb{Z}(-1)^{\mathbb{A},\mathrm{cdh}} \simeq \mathrm{L}_{\mathbb{A}^1}\mathbb{Z}(-1)^{\mathrm{cdh}}$, and the latter is zero since cdh-motivic cohomology vanishes in negative weights (see the discussion preceding Remark 3.2.41). The claim follows. □

Theorem 3.2.44 (Slice conjecture). *For every morphism $f \colon S' \to S$ of qcqs schemes and every $j \in \mathbb{Z}$, the comparison map of Remark 3.2.9 is an equivalence:*

$$f^*\mathrm{Fil}^j_{\mathrm{slice}}\mathrm{KGL}_S \xrightarrow{\sim} \mathrm{Fil}^j_{\mathrm{slice}}\mathrm{KGL}_{S'}, \qquad f^* s^j \mathrm{KGL}_S \xrightarrow{\sim} s^j\mathrm{KGL}_{S'}.$$

Consequently, $\mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}} \simeq \mathbb{Z}(j)^{\mathbb{A}}$ on $\mathsf{Sch}^{\mathrm{qcqs}}$ for every $j \in \mathbb{Z}$.

Proof. By Proposition 3.1.11 we may reduce to $S = \mathrm{Spec}(\mathbb{Z})$ and write $E = \mathrm{KGL}_{\mathbb{Z}}$, $f = f_{S'}$. The natural map $f^*\mathrm{Fil}^0_{\mathrm{slice}}E \to \mathrm{Fil}^0_{\mathrm{slice}} f^*E$ fits in a cofiber sequence whose cofiber is $\mathrm{Fil}^0_{\mathrm{slice}}(f^*\mathrm{Fil}^{<0}_{\mathrm{slice}}E)$, where $\mathrm{Fil}^{<0}_{\mathrm{slice}}E := \mathrm{cofib}(\mathrm{Fil}^0_{\mathrm{slice}}E \to E)$. Since $\mathrm{Fil}^0_{\mathrm{slice}} = \iota^0 r^0$ is exact (r^0 is a right adjoint, hence preserves fiber sequences, and ι^0 preserves them by closure of $\mathsf{SH}^{\mathrm{eff}}$ under extensions), induction along the cofiber sequences $\mathrm{Fil}^{j-1}_{\mathrm{slice}}E \to \mathrm{Fil}^j_{\mathrm{slice}}E \to s^jE$ for $j < 0$ reduces the vanishing to

$$\mathrm{Fil}^0_{\mathrm{slice}}\big(f^* s^j \mathrm{KGL}_{\mathbb{Z}}\big) \;\simeq\; 0 \qquad \text{for every } j < 0.$$

By Bott periodicity and the formula $\mathrm{Fil}^0_{\mathrm{slice}}(\mathbb{T}_{S'}^{\otimes j} \otimes -) \simeq \mathbb{T}_{S'}^{\otimes j} \otimes \mathrm{Fil}^{-j}_{\mathrm{slice}}(-)$ from Remark 3.2.9,

$$\mathrm{Fil}^0_{\mathrm{slice}}\big(f^* s^j\mathrm{KGL}_{\mathbb{Z}}\big) \;\simeq\; \mathbb{T}_{S'}^{\otimes j} \otimes \mathrm{Fil}^{-j}_{\mathrm{slice}}\big(f^* s^0 \mathrm{KGL}_{\mathbb{Z}}\big).$$

For $j < 0$ we have $-j \geq 1$, so $\mathrm{Fil}^{-j}_{\mathrm{slice}}(f^*s^0\mathrm{KGL}_{\mathbb{Z}})$ is a subobject of $\mathrm{Fil}^1_{\mathrm{slice}}(f^*s^0\mathrm{KGL}_{\mathbb{Z}}) \simeq 0$ by Corollary 3.2.43, and therefore vanishes.

The same Bott-shift argument transports the equivalence at $j = 0$ to every $j \in \mathbb{Z}$, both for $\mathrm{Fil}^j_{\mathrm{slice}}$ and (passing to graded pieces) for s^j. Applying $\mathrm{map}_{\mathsf{SH}(-)}(\mathbb{1}, -)$ yields $\mathbb{Z}(j)^{\mathbb{A},\mathrm{cdh}} \simeq \mathbb{Z}(j)^{\mathbb{A}}$ on $\mathsf{Sch}^{\mathrm{qcqs}}$. □

Definition 3.2.45 (cdh-filtered K-theory). The **cdh-filtered K-theory** on qcqs $\mathbb{Z}$-schemes is

$$\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH} \;:=\; \mathrm{L}_{\mathrm{cdh}}\mathrm{L}^{\mathrm{sm}}\big(\mathrm{Fil}^\star_{\mathbb{A}}\mathrm{KH}|_{\mathsf{Sm}_{\mathbb{Z}}}\big),$$

where L^{sm} denotes left Kan extension along $\mathsf{Sm}_{\mathbb{Z}}^{\mathrm{op}} \hookrightarrow \mathsf{Sch}^{\mathrm{qcqs,op}}$.

Corollary 3.2.46. *On qcqs schemes,*

$$\mathrm{L}_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH} \xrightarrow{\sim} \mathrm{Fil}^{\star}_{\mathbb{A},\mathrm{cdh}}\mathrm{KH}.$$

Proof. Since the associated graded pieces of both filtrations are identified by Theorem 3.2.42 (in the $\mathbb{A}^1$-localized form supplied by Theorem 3.2.44), it suffices to check the comparison in cohomological degree 0. There it specializes to $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{cdh}}\mathrm{L}^{\mathrm{sm}}\mathrm{KH} \xrightarrow{\sim} \mathrm{KH}$, which holds because $\mathrm{KH} = \mathrm{L}_{\mathrm{cdh}}\mathrm{k}$ is $\mathbb{A}^1$-invariant and k is left Kan extended from smooth $\mathbb{Z}$-algebras (Proposition 3.1.12); both sides therefore satisfy the same universal property as the $\mathbb{A}^1$-cdh-localization of the connective K-theory presheaf. □

Corollary 3.2.47. *Let X be a qcqs scheme of valuative dimension $\leq d$. Then for every $i \in \mathbb{Z}$,*

$$\mathrm{Fil}^i_{\mathbb{A}}\mathrm{KH}(X) \in \mathsf{Sp}_{\geq i-d}.$$

Proof. By Theorem 3.2.44, $\mathrm{Fil}^i_{\mathrm{slice}}\mathrm{KGL}_X \simeq f_X^*\mathrm{Fil}^i_{\mathrm{slice}}\mathrm{KGL}_{\mathbb{Z}}$, and Bott periodicity together with Lemma 3.2.16 identifies the latter with $\mathbb{T}_X^{\otimes i} \otimes f_X^* s^0\mathrm{KGL}_{\mathbb{Z}} \in \mathsf{SH}^{\mathrm{veff}}(X)(i)$. The proof of Proposition 3.2.14((i)) only uses Theorem 3.1.29 together with Proposition 3.1.23, both of which apply verbatim to any qcqs scheme of valuative dimension $\leq d$; hence $\omega^{\infty}(\mathbb{T}_X^{\otimes i} \otimes f_X^* s^0\mathrm{KGL}_{\mathbb{Z}})$ is $(i-d)$-connective as a Nisnevich sheaf of spectra. Taking global sections yields $\mathrm{Fil}^i_{\mathbb{A}}\mathrm{KH}(X) \in \mathsf{Sp}_{\geq i-d}$. □

Definition 3.2.48 (Derived category of motives)**.** The **derived category of motives** over a qcqs scheme X is

$$\mathsf{DM}(X) := \mathsf{Mod}_{\mathrm{H}\mathbb{Z}_X}(\mathsf{SH}(X)) = \mathsf{Mod}_{s^0\mathrm{KGL}_X}(\mathsf{SH}(X)),$$

where $\mathrm{H}\mathbb{Z}_X = s^0\mathrm{KGL}_X$ is the motivic Eilenberg–MacLane spectrum of Remark 3.2.33.

Proposition 3.2.49 ([BEM25, Corollary 9.10])**.** *The assignment $X \mapsto \mathsf{DM}(X)$ upgrades to a six-functor formalism on qcqs schemes, compatible with the six-functor formalism on $\mathsf{SH}(-)$ of Theorem 2.3.10.*

Proof sketch. The traditional reference, via $\mathrm{H}\mathbb{Z}^{\mathbb{A}}$ being an absolute motivic spectrum, is [CD19, Proposition 11.4.7]; we record here the modern argument.

By Theorem 2.3.10 the motivic stable homotopy category SH is itself a Pr^L-valued six-functor formalism on qcqs schemes, with !-able morphisms those locally of finite type. Base-changing along the symmetric monoidal functor

$$-\otimes \mathrm{H}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})} : \mathsf{SH}(\mathbb{Z}) \longrightarrow \mathsf{DM}(\mathbb{Z})$$

defines a candidate six-functor formalism $X \mapsto \mathsf{SH}(X) \otimes_{\mathsf{SH}(\mathbb{Z})} \mathsf{DM}(\mathbb{Z})$. By Theorem 3.2.44, the natural map $f^*\mathrm{H}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})} \xrightarrow{\sim} \mathrm{H}\mathbb{Z}_X$ is an equivalence, so the canonical comparison

$$\mathsf{SH}(X) \otimes_{\mathsf{SH}(\mathbb{Z})} \mathsf{DM}(\mathbb{Z}) \xrightarrow{\sim} \mathsf{DM}(X)$$

is an equivalence in Pr^L [Lur17, Theorem 4.8.4.6], transporting the six-functor formalism from the left-hand side to DM. □

***Remark* 3.2.50** (Explicit description of DM [BEM25, Remark 9.11])**.** The category $\mathsf{DM}(X)$ is more accessible than it might initially appear. Under our convention (Definition 3.2.32, no degree shift), the motivic Eilenberg–MacLane spectrum is $\mathrm{H}\mathbb{Z}_X = H(\mathbb{Z}(\star)^{\mathbb{A}})$ via the functor of Definition 3.1.8, and Proposition 3.1.7 applied with Sp replaced by $\mathsf{D}(\mathbb{Z})$ yields

$$\mathsf{DM}(X) \;\simeq\; \mathsf{Mod}_{\mathbb{Z}(\star)^{\mathbb{A}}}\Big(\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_X, \mathsf{D}(\mathbb{Z}))_{\mathrm{c,pbf}}\Big).$$

Concretely, $\mathsf{DM}(X)$ is the ∞-category of $\mathbb{Z}(\star)^{\mathbb{A}}$-modules $M^\star$ in $\mathsf{GrShv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_X, \mathsf{D}(\mathbb{Z}))$ satisfying the $\mathbb{P}^1$-bundle formula: multiplication by the first Chern class $c_1^{\mathbb{A}}(\mathcal{O}(1)) \in \pi_0\mathbb{Z}(1)^{\mathbb{A}}(\mathbb{P}^1_X)$ induces an equivalence

$$M^j \;\xrightarrow{\sim}\; \mathrm{fib}\big(M^{j+1}(\mathbb{P}^1_-) \xrightarrow{\infty^*} M^{j+1}\big)$$

for every $j \in \mathbb{Z}$, in line with the bonding-map formulation of the pbf in Construction 3.1.3. In particular, if $\mathbb{A}^1$-motivic cohomology is accepted as a black box, $\mathsf{DM}(X)$ can be defined without ever mentioning $\mathsf{SH}(X)$.

3.3. Low weights

This section is the script of my talk in the Oberseminar. Content to appear.

3.4. The motivic Thom spectra MGL

Content to appear.

Chapter 4.

Framed Correspondences and Motivic Recognition Principle

In ordinary stable homotopy theory, the classical *recognition principle* identifies the category of connective spectra with the category of grouplike $\mathbb{E}_\infty$-spaces:

$$\mathsf{Sp}_{\geq 0} \;\simeq\; \mathsf{CMon}^{\mathrm{gp}}(\mathsf{An}),$$

the equivalence being implemented by Ω^∞ together with the bar construction B^∞ as inverse, see [Lur17, Theorem 5.2.6.15 & Remarks 5.2.6.12]. From the geometric side one recovers the same statement as $\Sigma^\infty_+ \dashv \Omega^\infty$ exhibiting $\mathsf{Sp}_{\geq 0}$ as the stabilization of the (Cartesian) symmetric monoidal category An_*.

This chapter takes up the motivic analogue. The naive translation — replacing An_* by the motivic homotopy category $\mathsf{H}_*(S)$ — fails: $\mathsf{H}_*(S)$ is not a topos in any reasonable sense, and grouplike $\mathbb{E}_\infty$-objects in it do not capture motivic spectra. The right replacement, isolated by Voevodsky and brought to a definitive form by Elmanto–Hoyois–Khan–Sosnilo–Yakerson [Elm+21], is to enlarge the category Sm_S of smooth S-schemes by adjoining *framed correspondences* as additional morphisms. The resulting category $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ then plays the role of Fin_* (or rather of pointed smooth manifolds with framed cobordisms): it is the universal site on which to formulate the motivic infinite loop space machine.

The main theorem of the chapter (Theorem 4.2.1) states, over a perfect field k, an equivalence of symmetric monoidal categories

$$\mathsf{SH}^{\mathrm{eff}}(k) \;\simeq\; \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}\big(\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_k),\, \mathsf{Sp}\big),$$

together with its very-effective refinement identifying $\mathsf{SH}^{\mathrm{veff}}(k)$ with grouplike framed motivic spaces. We sketch the structure of the proof, whose central ingredient is a motivic cancellation theorem of Voevodsky–Garkusha–Panin upgraded to the framed setting.

Section 4.1 introduces framed correspondences in the normally framed presentation, the category $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ and the framed presheaves $h^{\mathrm{fr}}_S(Y)$. Section 4.2 develops the recognition principle, first in the classical setting, then motivically.

4.1. Framed correspondences

A *framed correspondence* from a smooth S-scheme X to a smooth S-scheme Y is, informally, a span

$$X \xleftarrow{\;f\;} Z \xrightarrow{\;g\;} Y$$

in which f is finite syntomic and Z is equipped with a trivialization of its conormal data inside some affine space over X. Several inequivalent presentations of this data appear in the literature (equationally framed, tangentially framed, normally framed); after motivic localization they all coincide. We adopt the *normally framed* formulation of [Elm+21], which is technically cleanest.

Definition 4.1.1 (Normally framed correspondence [Elm+21, Definition 2.3.4]). Let $X, Y \in \mathsf{Sch}_S$ and $n \geq 0$. A **normally framed correspondence of level** n from X to Y over S consists of:

(i) a span

$$\begin{array}{ccc} & Z & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ X & & Y \end{array}$$

over S in which $f\colon Z \to X$ is finite syntomic;

(ii) a closed immersion $i\colon Z \hookrightarrow \mathbb{A}^n_X$ over X;

(iii) a trivialization $\tau\colon \mathcal{O}_Z^{\oplus n} \xrightarrow{\sim} \mathcal{N}_i$ of the conormal sheaf of i.

We denote the discrete groupoid of normally framed correspondences of level n by $\mathsf{Corr}_S^{\mathrm{nfr},n}(X, Y)$.

Remark **4.1.2** (Geometric content). The conormal trivialization τ is the categorical avatar of a *stable framing* of Z relative to X: since f is finite syntomic, its cotangent complex $\mathbb{L}_{Z/X}$ is concentrated in degree zero and locally free; the closed immersion $Z \hookrightarrow \mathbb{A}^n_X$ over X contributes the Euler sequence

$$0 \longrightarrow \mathbb{L}_{Z/X} \longrightarrow \mathcal{O}_Z^{\oplus n} \longrightarrow \mathcal{N}_i \longrightarrow 0,$$

so a trivialization of $\mathcal{N}_i$ together with the embedding presents $\mathbb{L}_{Z/X}$ as a direct summand of a trivial bundle, i.e. exhibits a stable framing of $\mathbb{L}_{Z/X}$. This recovers in algebraic geometry the classical picture of framed bordism in topology, where finite covers of smooth manifolds with stable framings correspond to maps to $\Sigma^\infty \mathbb{S}$.

Suspension by $\mathbb{A}^1$ defines a transition map

$$\sigma_{X,Y}\colon \mathsf{Corr}_S^{\mathrm{nfr},n}(X, Y) \hookrightarrow \mathsf{Corr}_S^{\mathrm{nfr},n+1}(X, Y), \qquad (Z, i, \tau) \longmapsto (Z, i \times \{0\}, \tau \oplus \mathrm{id}),$$

encoding the convention that the framed bordism category is stable under suspension; we set

$$\mathsf{Corr}_S^{\mathrm{fr}}(X, Y) \ := \ \operatorname*{colim}_n \mathsf{Corr}_S^{\mathrm{nfr},n}(X, Y).$$

Definition 4.1.3 (Category of framed correspondences). The (symmetric monoidal) category $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ has:

- objects: smooth S-schemes;
- morphism anima: $\mathsf{Map}_{\mathsf{Corr}^{\mathrm{fr}}}(X, Y) = \mathsf{Corr}_S^{\mathrm{fr}}(X, Y)$;
- composition: by pulling back framings along the structure span;
- symmetric monoidal structure: induced by the Cartesian product $\times_S$ on Sm_S and direct sum of framings.

The unit is S itself. There is a canonical symmetric monoidal functor

$$\gamma\colon \mathsf{Sm}_S \longrightarrow \mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$$

sending a morphism $f\colon X \to Y$ to the trivially framed span $X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y$ at level $n = 0$.

Notation 4.1.4 (Framed presheaves). For each $Y \in \mathsf{Sm}_S$ we write

$$h_S^{\mathrm{fr}}(Y) \;:=\; \mathsf{Corr}_S^{\mathrm{fr}}(-, Y) \;\in\; \mathsf{PShv}(\mathsf{Sm}_S),$$

the presheaf of framed correspondences into Y. By construction $h_S^{\mathrm{fr}}(Y)$ takes values in $\mathbb{E}_\infty$-anima: composition with the empty correspondence and disjoint union of framings provide the unit and product structure.

Proposition 4.1.5 (Sheaf properties, [Elm+21, Proposition 2.3.7]). *For every $Y \in \mathsf{Sm}_S$:*

(i) $h_S^{\mathrm{fr}}(Y)$ is an fpqc sheaf on Sm_S with values in $\mathbb{E}_\infty$-anima.

(ii) $h_S^{\mathrm{fr}}(Y)$ satisfies closed gluing.

(iii) For every Nisnevich covering sieve $R \hookrightarrow X$ generated by a single map, $h_S^{\mathrm{fr}}(R) \to h_S^{\mathrm{fr}}(X)$ is a Nisnevich equivalence; in particular, the Nisnevich sheafification $\mathrm{L}_{\mathrm{Nis}} h_S^{\mathrm{fr}}(Y)$ is computed levelwise on the normally framed level.

***Remark* 4.1.6** (Equivalence with other presentations). Two related presentations of framed correspondences appear in the literature:

- *Equationally framed* correspondences (Garkusha–Panin) record an open neighborhood $U \supseteq Z$ inside $\mathbb{A}^n_X$ together with explicit equations $\phi\colon U \to \mathbb{A}^n$ cutting out Z;
- *Tangentially framed* correspondences (Hoyois) record a trivialization of the cotangent complex $\mathbb{L}_{Z/S}$ rather than the conormal bundle.

[Elm+21, Corollary 2.3.25] shows that all three presentations agree after motivic localization, so we may use them interchangeably; the normally framed version is what we use.

The promotion of h_S^{fr} to a presheaf valued in spectra requires some care because $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ is naturally only *semiadditive*; spectrum-valued sheaves are recovered by group-completing and stabilizing. The technical setup is the following.

Definition 4.1.7 (Categories of framed motivic objects). Let $\mathsf{H}^{\mathrm{fr}}(S) \subseteq \mathsf{PShv}_\Sigma(\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S))$ denote the full subcategory of presheaves of anima on $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ that take finite coproducts in Sm_S to products of anima, are Nisnevich-local, and are $\mathbb{A}^1$-invariant. Its grouplike subcategory is denoted $\mathsf{H}^{\mathrm{fr}}(S)^{\mathrm{gp}}$. The corresponding category of **framed motivic S^1-spectra** is

$$\mathsf{SH}^{S^1,\mathrm{fr}}(S) \;:=\; \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}\big(\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S), \mathsf{Sp}\big),$$

the full subcategory of Sp-valued presheaves on $\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ which are Nisnevich-local and $\mathbb{A}^1$-invariant.

The functor $\gamma\colon \mathsf{Sm}_S \to \mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_S)$ induces an adjunction

$$\gamma^*\colon \mathsf{H}_*(S) \rightleftarrows \mathsf{H}^{\mathrm{fr}}(S) \colon \gamma_*,$$

and a corresponding S^1-stabilized adjunction

$$\gamma^*\colon \mathsf{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathsf{Sm}_S, \mathsf{Sp}) \rightleftarrows \mathsf{SH}^{S^1,\mathrm{fr}}(S) \colon \gamma_*.$$

The right adjoint γ_* sends a framed sheaf to its underlying Nisnevich sheaf with $\mathbb{E}_\infty$-structure; the left adjoint γ^* freely adjoins framed transfers.

4.2. Motivic infinite loop space machine

4.2.1. *The classical recognition principle*

Let C be a presentable Cartesian symmetric monoidal category. The bar construction supplies an adjunction

$$\mathrm{B}^\infty\colon \mathsf{CMon}(C) \rightleftarrows \mathsf{Stab}(C)\colon \Omega^\infty,$$

where $\mathsf{Stab}(C) = C \otimes \mathsf{Sp}$ is the stabilization. When C is a topos (in our default $(\infty,1)$-categorical sense), this adjunction restricts to an equivalence between grouplike commutative monoids and connective spectra:

$$\mathsf{CMon}^{\mathrm{gp}}(C) \;\simeq\; \mathsf{Stab}(C)_{\geq 0}.$$

Specializing to $C = \mathsf{An}$ recovers the classical formulation $\mathsf{Sp}_{\geq 0} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{An})$ [Lur17, Theorem 5.2.6.15].

The recognition principle in this generality applies, for instance, to $C = \mathsf{PShv}_{\mathrm{Nis}}(\mathsf{Sm}_S)$, identifying connective Nisnevich-sheaf spectra with grouplike sheaves of $\mathbb{E}_\infty$-spaces. It does *not* apply, however, to $C = \mathsf{H}(S)$: $\mathbb{A}^1$-localization fails to preserve the topos structure, and grouplike $\mathbb{E}_\infty$-objects in $\mathsf{H}(S)$ do not capture connective motivic S^1-spectra. This is the technical obstruction the framed transfer formalism repairs.

4.2.2. *The motivic recognition principle*

Theorem 4.2.1 (Motivic recognition principle, [Elm+21, Theorem 3.5.14]). *Let k be a perfect field. The pullback functor γ^* induces equivalences of symmetric monoidal categories:*

(i) (Unstable form.) $\mathsf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \xrightarrow{\sim} \mathsf{SH}^{\mathrm{veff}}(k)$;

(ii) (Stable form.) $\mathsf{SH}^{S^1,\mathrm{fr}}(k) \xrightarrow{\sim} \mathsf{SH}^{\mathrm{eff}}(k)$.

In both cases, the inverse equivalence is implemented by $\gamma_*\Sigma^\infty_{\mathbb{T}}$ (resp. $\gamma_*\Sigma^\infty_{\mathbb{G}_m,\mathrm{fr}}$); after $\mathbb{T}$-stabilization, the functor $\gamma^*\colon \mathsf{SH}(k) \to \mathsf{SH}^{\mathrm{fr}}(k)$ is itself an equivalence.

***Remark* 4.2.2** (Base extension to qcqs schemes). Hoyois [Hoy21] extends Theorem 4.2.1((i)) to arbitrary qcqs base schemes S via a noetherian-approximation argument. The stable form ((ii)) likewise extends, with the same proof outline below; we work over a perfect field for definiteness.

***Remark* 4.2.3** (The dictionary). Theorem 4.2.1 is the precise motivic counterpart of $\mathsf{Sp}_{\geq 0} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{An})$:

Classical		**Motivic**
$\mathsf{Sp}_{\geq 0}$	$\longleftrightarrow$	$\mathsf{SH}^{\mathrm{veff}}(k)$
$\mathsf{CMon}^{\mathrm{gp}}(\mathsf{An})$	$\longleftrightarrow$	$\mathsf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}$
finite sets Fin_*	$\longleftrightarrow$	$\mathsf{Corr}^{\mathrm{fr}}(\mathsf{Sm}_k)$
Σ^∞_+	$\longleftrightarrow$	$\Sigma^\infty_{\mathbb{T},\mathrm{fr}}$

The role of the disjoint-union biproduct in Fin_* is played by the framed transfers in $\mathsf{Corr}^{\mathrm{fr}}$: both promote a multiplicative structure on a category to a stable infinite-loop-space structure on its presheaves.

4.2.3. Outline of the proof

The proof of Theorem 4.2.1 proceeds in three steps, each of which is itself substantial.

Step 1: Cancellation. The first input is a motivic cancellation theorem: the functor $\Sigma^\infty_{\mathbb{G}_m} : \mathsf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathsf{SH}^{\mathrm{fr}}(k)$ is fully faithful. This is the framed analogue of the Voevodsky–Garkusha–Panin cancellation theorem for motives with transfers, established in this generality by Ananyevskiy–Garkusha–Panin and reproved categorically in [Elm+21, Theorem 3.5.8]. Geometrically, the input is that $\mathbb{G}_m$-suspension behaves well with respect to framed correspondences — the obstruction in the unframed setting is precisely the failure of cancellation of the $\mathbb{G}_m$-factor, which framed transfers repair.

Step 2: Reconstruction. The second step ([Elm+21, Theorem 3.5.12], the *reconstruction theorem*) establishes that $\gamma^* : \mathsf{SH}(k) \to \mathsf{SH}^{\mathrm{fr}}(k)$ is itself a symmetric monoidal equivalence. The proof reduces, by conservativity of γ_* and colimit preservation on both sides, to checking that the unit transformation $\mathrm{id} \to \gamma_*\gamma^*$ is an equivalence on $\Sigma^\infty_{\mathbb{T}} X_+$ for $X \in \mathsf{Sm}_k$. After motivic localization, the right-hand side is computed by a $\mathbb{G}_m$-prespectrum whose levels are the framed presheaves $h^{\mathrm{fr}}(\mathbb{G}_m^{\wedge n} \wedge X_+)$; the comparison map between this prespectrum and the standard $\mathbb{T}$-suspension prespectrum is the one identified by the geometric content of *loc. cit.* §3.5, and is shown to be a motivic equivalence using the cancellation theorem of Step 1 and a careful analysis of the multiplicative structure of h^{fr}.

Step 3: Image identification. Once γ^* is known to be an equivalence, the two restricted statements of Theorem 4.2.1 follow by identifying the images of the various stabilization functors:

- For ((i)), the image of $\gamma_*\Sigma^\infty_{\mathbb{T},\mathrm{fr}} : \mathsf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathsf{SH}(k)$ consists of the colimit closure of $\Sigma^\infty_{\mathbb{T}} X_+$ for $X \in \mathsf{Sm}_k$ in $\mathsf{SH}(k)$, which by [BE25, Remark after Proposition 4] is precisely $\mathsf{SH}^{\mathrm{veff}}(k)$.
- For ((ii)), the image is the colimit closure of $\Sigma^{-n}_{\mathbb{T}}\Sigma^\infty_{\mathbb{T}} X_+$ for $X \in \mathsf{Sm}_k$ and $n \geq 0$, which is by definition $\mathsf{SH}^{\mathrm{eff}}(k)$.

This completes the proof.

4.2.4. Consequences and applications

Corollary 4.2.4 (Sifted cocompletion description). *The category $\mathsf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ is the sifted cocompletion of the category of framed correspondences between affine smooth k-schemes (modulo $\mathbb{A}^1$-homotopy and Nisnevich descent). In particular, motivic spectra in $\mathsf{SH}^{\mathrm{veff}}(k)$ are determined by their values on smooth affine k-schemes.*

***Remark* 4.2.5** (The motivic Hurewicz map). The unit of the adjunction $\Sigma^\infty_+ \dashv \Omega^\infty$ in classical homotopy theory has a motivic counterpart: for any pointed motivic space $\mathcal{X} \in \mathsf{H}_*(k)$, the unit $\mathcal{X} \to \Omega^\infty_{\mathbb{T}}\Sigma^\infty_{\mathbb{T},\mathrm{fr}}\mathcal{X}$ in $\mathsf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ exhibits the right-hand side as the free framed motivic infinite loop space generated by $\mathcal{X}$, under Theorem 4.2.1((i)) corresponding to the very effective cover of the motivic suspension spectrum of $\mathcal{X}$. This is the source of explicit chain-level models for motivic stable invariants.

Example 4.2.6 (Explicit description of $\mathbb{1}_k$)**.** Under Theorem 4.2.1((i)), the motivic sphere spectrum $\mathbb{1}_k \in \mathsf{SH}^{\mathrm{veff}}(k)$ is identified with the group completion of $h^{\mathrm{fr}}(\mathrm{pt}) = \mathsf{Corr}^{\mathrm{fr}}_k(-, \mathrm{Spec}\, k)$, i.e. the Nisnevich-localized $\mathbb{A}^1$-invariant presheaf of finite syntomic schemes with stable framings. Computing $\pi_0(\mathbb{1}_k)$ via this presentation recovers the Grothendieck–Witt ring $\mathrm{GW}(k)$ [Elm+21, Corollary 3.5.16], in line with Morel's classical π_0-computation.

***Remark* 4.2.7** (Relation to motivic cohomology). The Eilenberg–MacLane spectrum $\mathrm{H}\mathbb{Z}$ of Remark 3.2.33 admits an explicit framed-transfer description: the underlying framed sheaf of $\mathrm{H}\mathbb{Z}$ is the linearization of h^{fr}, recovering Voevodsky's category of finite Nisnevich correspondences after collapsing framings. This is the natural starting point for the motivic versions of the Dold–Kan correspondence and the Eilenberg–Zilber theorem; we refer to [Elm+21, §5] for details.

Appendix A.

Norms on Motivic (Pre)spectra

This appendix sketches the construction of **norm functors** due to Bachmann–Hoyois [BH21], which were previewed in Remark 1.0.2 and underlie the multiplicative structure used at several points in Chapters 2 and 3. The construction proceeds in three steps—sifted cocompletion, motivic localization, and $\mathbb{T}$-stabilization—each a universal cocompletion. We therefore begin by recalling the cocompletion machinery in Section A.1, then turn to the norm functors themselves in Section A.2.

A.1. Sifted cocompletion and animation

A.1.1. *Algebraic structures as product-preserving presheaves*

A classical observation is that the category Grp of groups can be reconstructed from its full subcategory FinFree $\subseteq$ Grp of finitely generated free groups. Write $F_n \in$ FinFree for the free group on n generators; note that FinFree has finite coproducts, with $F_m \sqcup F_n \simeq F_{m+n}$.

Given a finite-product-preserving presheaf $\underline{G}\colon \mathsf{FinFree}^{\mathrm{op}} \to \mathsf{Set}$, set $G := \underline{G}(F_1)$. Since $F_n \simeq F_1^{\sqcup n}$ in FinFree, one has $\underline{G}(F_n) \simeq G^n$. The codiagonal-and-multiplication morphism $F_1 \to F_2$ in FinFree (sending the generator to ab) induces, after $\underline{G}$, the map

$$G \times G \;\simeq\; \underline{G}(F_2) \;\longrightarrow\; \underline{G}(F_1) \;\simeq\; G,$$

which is the group multiplication. The unit and inverse morphisms of FinFree likewise produce the unit and inverse of G, and one obtains an equivalence of 1-categories

$$\mathsf{Fun}^{\Pi}(\mathsf{FinFree}^{\mathrm{op}}, \mathsf{Set}) \;\simeq\; \mathsf{Grp},$$

where Fun^{Π} denotes the full subcategory of finite-product-preserving functors. The same pattern recovers CRing, Mod_R, and similar algebraic categories from their respective subcategories of free objects of finite rank. Sifted cocompletion is the universal-property reformulation of this observation.

A.1.2. *Sifted colimits*

Definition A.1.1 (Sifted category). A nonempty category $\mathcal{I}$ is **sifted** if the diagonal $\Delta\colon \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ is cofinal. A colimit indexed by a sifted $\mathcal{I}$ is called a **sifted colimit**.

The defining feature of sifted colimits is that they commute with finite products in An: for $X, Y\colon \mathcal{I} \to \mathsf{An}$, cofinality of Δ gives

$$\operatorname{colim}_{i\in\mathcal{I}}(X_i \times Y_i) \;\simeq\; \operatorname{colim}_{(i,j)\in\mathcal{I}\times\mathcal{I}}(X_i \times Y_j) \;\simeq\; \big(\operatorname{colim}_i X_i\big) \times \big(\operatorname{colim}_j Y_j\big).$$

Two paradigmatic examples are filtered categories and $\mathbb{\Delta}^{\mathrm{op}}$, and these in fact span the full notion:

Proposition A.1.2 ([Lur09, Corollary 5.5.8.17]). *A functor between categories with sifted colimits preserves all sifted colimits if and only if it preserves filtered colimits and geometric realizations (i.e., Δ^{op}-indexed colimits).*

In slogan form: *sifted colimits = filtered colimits + reflexive coequalizers*, the latter being the truncated form of geometric realizations.

A.1.3. Sifted cocompletion

Definition A.1.3 (Sifted cocompletion). Let C be a small category with finite coproducts. The **sifted cocompletion** of C is the smallest full subcategory

$$\mathcal{P}_\Sigma(C) \subseteq \mathsf{PShv}(C) = \mathsf{Fun}(C^{\mathrm{op}}, \mathsf{An})$$

containing the representables and closed under sifted colimits.

The same sifted-vs.-products commutation argument from Section A.1.1 applies: every $F \in \mathcal{P}_\Sigma(C)$, written as a sifted colimit $F \simeq \mathrm{colim}_{i \in I}\, y(X_i)$ of representables, satisfies $F(A \sqcup B) \simeq F(A) \times F(B)$. Conversely, every product-preserving presheaf arises this way:

Proposition A.1.4 ([Lur09, Proposition 5.5.8.10]). *Let C be a small category with finite coproducts. The Yoneda embedding identifies*

$$\mathcal{P}_\Sigma(C) \simeq \mathsf{Fun}^{\Pi}(C^{\mathrm{op}}, \mathsf{An}).$$

The defining universal property is then:

Proposition A.1.5 ([Lur09, Proposition 5.5.8.15]). *Let C be a small category with finite coproducts and $\mathcal{D}$ a category with all sifted colimits. Restriction along the Yoneda embedding induces an equivalence*

$$\mathsf{Fun}^{\mathrm{sift}}(\mathcal{P}_\Sigma(C), \mathcal{D}) \xrightarrow{\sim} \mathsf{Fun}(C, \mathcal{D}).$$

In words: $\mathcal{P}_\Sigma(C)$ is freely generated under sifted colimits by C. By the dual identification of Proposition A.1.4, it is equivalently *the universal way to add sifted colimits while preserving the existing finite coproducts*—which is the perspective taken in Remark 1.0.2.

***Remark* A.1.6** (Symmetric monoidal refinement). The sifted cocompletion is presentable, compactly generated by the image of the Yoneda embedding, and inherits a symmetric monoidal structure from any symmetric monoidal structure on C (via Day convolution). With this enhancement, $\mathcal{P}_\Sigma(C)$ becomes the free presentably symmetric monoidal category with sifted colimits generated by C [Lur17, §4.8.1]; this is the form in which we shall apply it.

A.1.4. Animation

The classical equivalence $\mathsf{Grp} \simeq \mathsf{Fun}^{\Pi}(\mathsf{FinFree}^{\mathrm{op}}, \mathsf{Set})$ lifts by replacing Set with An. Recall that an object P of a 1-category C with reflexive coequalizers is *compact projective* if $\mathrm{Hom}_C(P, -)$ preserves filtered colimits and reflexive coequalizers; equivalently (by Proposition A.1.2), if it preserves sifted colimits. Write $C^{\mathrm{cp}} \subseteq C$ for the full subcategory of compact projective objects. A 1-category C is *compactly projectively generated* when the canonical map $\mathcal{P}_\Sigma(C^{\mathrm{cp}}) \to C$ (using the 1-categorical sifted cocompletion) is an equivalence.

Definition A.1.7 (Animation)**.** Let C be a compactly projectively generated 1-category. The **animation** of C is

$$\mathsf{Ani}(C) \;:=\; \mathcal{P}_\Sigma(C^{\mathrm{cp}}) \;\simeq\; \mathsf{Fun}^{\Pi}\big((C^{\mathrm{cp}})^{\mathrm{op}}, \mathsf{An}\big),$$

where $\mathcal{P}_\Sigma$ is the (default) sifted cocompletion of Definition A.1.3, applied to C^{cp} viewed as a category.

Examples: $\mathsf{Ani}(\mathsf{Set}) = \mathsf{An}$; $\mathsf{Ani}(\mathsf{CRing})$ is the category of *animated commutative rings*, the home of derived/animated algebraic geometry; similarly for $\mathsf{Ani}(\mathsf{Mod}_R)$ and $\mathsf{Ani}(\mathsf{Grp})$.

***Remark* A.1.8** (Animation vs. derived category)**.** Lurie [Lur09, §5.5.8] calls the sifted cocompletion the *nonabelian derived category*. For an abelian category $\mathcal{A}$ with enough projectives, the connective derived category $\mathsf{D}_{\geq 0}(\mathcal{A}) \simeq \mathcal{P}^{\{\Delta^{\mathrm{op}}\}}(\mathcal{A}_{\mathrm{proj}})$ is obtained by freely adjoining geometric realizations to projective objects [Lur17, Lemma 1.3.3.17]. The animation $\mathsf{Ani}(\mathcal{A}^{\wedge}) \simeq \mathcal{P}_\Sigma(\mathcal{A}_{\mathrm{proj}})$ of the compactly projectively generated closure $\mathcal{A}^{\wedge}$ also adjoins filtered colimits, and so differs from $\mathsf{D}_{\geq 0}(\mathcal{A})$ by an Ind-completion step.

A.1.5. The general cocompletion framework

Sifted cocompletion is one instance of a general procedure. Let $\mathcal{K} \subseteq \mathcal{K}'$ be two classes of small categories, and write $\mathsf{Cat}(\mathcal{K})$ for the (very large) category of categories admitting all $\mathcal{K}$-indexed colimits, with morphisms preserving them. The forgetful functor $\mathsf{Cat}(\mathcal{K}') \to \mathsf{Cat}(\mathcal{K})$ admits a left adjoint:

Definition A.1.9 ($\mathcal{K}'$-cocompletion)**.** For $C \in \mathsf{Cat}(\mathcal{K})$, the **$\mathcal{K}'$-cocompletion** $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(C) \in \mathsf{Cat}(\mathcal{K}')$ is characterized by the universal property

$$\mathsf{Fun}^{\mathcal{K}'}\big(\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(C), \mathcal{D}\big) \;\simeq\; \mathsf{Fun}^{\mathcal{K}}(C, \mathcal{D})$$

for every $\mathcal{D} \in \mathsf{Cat}(\mathcal{K}')$, where $\mathsf{Fun}^{\mathcal{K}}$ denotes functors preserving all $\mathcal{K}$-colimits.

Familiar instances populate the framework:

Example A.1.10.

(i) $\mathcal{K} = \emptyset$, $\mathcal{K}' =$ all: $\mathcal{P}^{\mathrm{all}}(C) = \mathsf{PShv}(C)$.

(ii) $\mathcal{K} = \emptyset$, $\mathcal{K}' = \{\text{filtered}\}$: $\mathsf{Ind}(C)$.

(iii) $\mathcal{K} = \emptyset$, $\mathcal{K}' = \{\text{sifted}\}$: $\mathcal{P}_\Sigma(C)$.

(iv) $\mathcal{K} = \{\text{finite coproducts}\}$, $\mathcal{K}' =$ all: again $\mathcal{P}_\Sigma(C)$, by the dual perspective of Proposition A.1.4.

(v) $\mathcal{K} = \emptyset$, $\mathcal{K}' = \{\mathsf{Idem}\}$: the idempotent completion C^{idem}.

The fact that sifted cocompletion appears under two complementary descriptions (items 3 and 4 of Example A.1.10) reflects a general phenomenon: {sifted} and {finite coproducts} are *complementary* classes of colimits, in the sense that adjoining one to a category preserving the other gives the same answer. This is why $\mathcal{P}_\Sigma(C)$ may equally be thought of as "free sifted cocompletion of C" or as "free cocompletion preserving finite coproducts of C".

The norm-functor construction will use Definition A.1.9 in succession: each stage in the chain

$$\mathsf{Sm}_{T,+} \xrightarrow{(1)} \mathcal{P}_\Sigma(\mathsf{Sm}_T)_* \xrightarrow{(2)} \mathsf{H}_*(T) \xrightarrow{(3)} \mathsf{SH}(T)$$

is the $\mathcal{K}_{i+1}$-cocompletion of the previous one, for an enlarging tower of colimit classes (sifted; sifted plus motivic equivalences; the same plus $\mathbb{T}$-suspension). A symmetric monoidal functor on the source then extends uniquely whenever its image satisfies the next preservation condition.

A.2. Norm functors

A.2.1. Weil restriction

For a finite locally free morphism $p\colon T \to S$ in classical algebraic geometry, the **Weil restriction** $\mathrm{R}_p\colon \mathsf{Sm}_T \to \mathsf{Sm}_S$ is right adjoint to pullback $p^*\colon \mathsf{Sm}_S \to \mathsf{Sm}_T$, characterized by

$$\mathrm{Hom}_S(Y, \mathrm{R}_p X) \;\simeq\; \mathrm{Hom}_T(Y \times_S T, X).$$

In the pointed setting, Weil restriction extends to a symmetric monoidal functor $p_\otimes\colon \mathsf{Sm}_{T,+} \to \mathsf{Sm}_{S,+}$ with respect to the smash product. The core question of [BH21] is: can $p_\otimes$ be extended to motivic homotopy theory?

A.2.2. The norm functor theorem

Theorem A.2.1 (Bachmann–Hoyois [BH21, Theorems 3.3, 4.1])**.** *Let $p\colon T \to S$ be a finite locally free, universally open morphism. Then there exists a symmetric monoidal functor*

$$p_\otimes\colon \mathsf{H}_*(T) \longrightarrow \mathsf{H}_*(S)$$

satisfying:

(i) $p_\otimes$ preserves sifted colimits;

(ii) $p_\otimes(X_+) \simeq (\mathrm{R}_p X)_+$ for $X \in \mathsf{Sm}_T$;

(iii) if $T = S^{\sqcup n}$ and p is the fold map, then $p_\otimes$ is the n-fold smash product;

(iv) if p is moreover finite étale, $p_\otimes$ extends further to a symmetric monoidal functor $p_\otimes\colon \mathsf{SH}(T) \to \mathsf{SH}(S)$.

A.2.3. Construction via universal cocompletions

By Proposition A.1.5 and the symmetric monoidal refinement of Remark A.1.6, the symmetric monoidal Weil restriction $p_\otimes\colon \mathsf{Sm}_{T,+} \to \mathsf{Sm}_{S,+}$ extends along the universal sequence

$$\mathsf{Sm}_{T,+} \xrightarrow{(1)} \mathcal{P}_\Sigma(\mathsf{Sm}_T)_* \xrightarrow{(2)} \mathsf{H}_*(T) \xrightarrow{(3)} \mathsf{SH}(T)$$

in stages, provided the relevant preservation property holds at each stage:

(i) **Sifted cocompletion (unconditional).** Since Sm_T has finite coproducts, $p_\otimes$ extends along (1) uniquely to a symmetric monoidal sifted-cocontinuous functor on $\mathcal{P}_\Sigma(\mathsf{Sm}_T)_*$ by Proposition A.1.5.

(ii) **Motivic localization.** Since $\mathsf{H}_*(T) = \mathcal{P}_\Sigma(\mathsf{Sm}_T)_*[(\mathrm{Nis} \cup \mathbb{A}^1)^{-1}]$ is a localization, the extension of step (1) descends to $\mathsf{H}_*(T)$ if and only if $p_\otimes$ sends Nisnevich and $\mathbb{A}^1$-equivalences to motivic equivalences. This holds for finite locally free, universally open p [BH21, §6].

(iii) **$\mathbb{T}$-stabilization.** Since $\mathsf{SH}(T) = \mathsf{H}_*(T)[\mathbb{T}_T^{-1}]$ is the $\otimes$-inversion of $\mathbb{T}_T$ (Definition 2.1.5), the extension of step (2) descends to $\mathsf{SH}(T)$ if and only if $p_\otimes(\mathbb{T}_T)$ is $\otimes$-invertible in $\mathsf{H}_*(S)$. This holds for finite étale p [BH21, §9].

This is a paradigmatic example of how the universal-property approach pays off: a hard "can we extend?" question reduces to checking elementary preservation properties at each stage of the cocompletion tower.

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