Six Functors and Gestalten

Talk 10 — Gestalten Seminar

after [Sch26], Lecture IX (*up to the extension to stacks*)

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Goal of the talk

Starting from a (presentable) six-functor formalism $\mathrm{D}\colon \mathrm{Span}(C) \to 1\mathrm{Pr}$, we construct *canonically* a functor

$$[-]_{\mathrm{D}}\colon C \longrightarrow \mathrm{Gest} := \mathrm{StRing}^{\mathrm{op}}, \qquad X \longmapsto [X]_{\mathrm{D}} = \mathrm{Gest}(A_{\mathrm{D},X}),$$

the *transmutation* of D, through which D factors as "geometry + quasicoherent sheaves". The engine is the *category of kernels* $\mathbb{K}_{\mathrm{D}}$; iterating it produces the Stefanich ring $A_{\mathrm{D},X}$. We show $[-]_{\mathrm{D}}$ commutes with finite limits, that every $[f]_{\mathrm{D}}$ is 1-étale and 1-proper, and that $\mathcal{O}(-)_1 \circ [-]_{\mathrm{D}}$ recovers the $(f^*, \otimes)$-part of D (Theorem 3.4). We stop just before the extension to stacks.

Conventions

Categories. Throughout, "category" means ∞-category, and C is a small one with finite limits.
The formalism. $\mathrm{D}\colon \mathrm{Span}(C) \to 1\mathrm{Pr}$ is a *presentable six-functor formalism* — a lax symmetric monoidal functor on the span category — where 1Pr denotes presentable categories and $2\mathrm{Pr} = \mathrm{Mod}_{1\mathrm{Pr}}(1\mathrm{Pr})$. Following [Sch26] we fix the cardinal $\kappa = \aleph_1$ throughout, so $1\mathrm{Pr} = \mathrm{Pr}^{\mathrm{L}}_{\aleph_1}$ and $n\mathrm{Pr} = \mathrm{Mod}_{(n-1)\mathrm{Pr}}(1\mathrm{Pr})$; "presentable" means $\aleph_1$-presentable, and the "countable colimit" and "countably presented" hypotheses invoked below are taken with respect to this κ.
Kernels and Gestalten. $\mathbb{K}_{\mathrm{D}}$ is the *category of kernels* (Section A) and $\mathrm{Gest} = \mathrm{StRing}^{\mathrm{op}}$. We write $\mathbb{1}$ for the monoidal unit, and $\mathcal{O}(-)_1$ (also written D_{qc}) for "global sections / quasicoherent sheaves" — the bottom floor of a Stefanich ring.
D-properties. A morphism is D-*suave*, D-*prim*, D-*smooth*, D-*étale*, or D-*proper* when it has the corresponding property for D (Section A); the unprefixed adjectives (smooth, étale, proper, ...) denote the underlying geometric notions.

1 From motives to six-functor formalisms

In algebraic geometry there are many cohomology theories: the ℓ-adic ones, one for each prime ℓ; crystalline cohomology; de Rham and Betti cohomology. Grothendieck envisioned a *universal* system of coefficients that would dictate all the rest — an initial cohomology theory out of which every other one is read off. In its original, abelian-categorical form, the idea asks for a category Mot_k of *motives*, together with an assignment $X \mapsto \mathrm{Mot}(X)$ (the *motive* of X), such that every reasonable "cohomology functor" $h^\bullet\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathbb{A}$ valued in an abelian category $\mathbb{A}$ factors through it:

$$\begin{array}{ccc} \mathrm{Sch}_k^{\mathrm{op}} & \xrightarrow{\mathrm{Mot}} & \mathrm{Mot}_k \\ & \searrow^{h^\bullet} & \downarrow \\ & & \mathbb{A}. \end{array}$$

The dashed leg is a *realization*: each Weil cohomology — Betti, de Rham, ℓ-adic — is one realization of the universal motive. All the variation *among* cohomology theories is carried by this choice of realization, while the geometric input $X \mapsto \mathrm{Mot}(X)$ stays fixed. This shape of factorization — a fixed universal object, recovered into each concrete theory by a single leg — is the one we will keep meeting; the goal of the talk is its modern, categorified incarnation.

The first move is to categorify the coefficients. Studying a category C of geometric objects — say schemes of finite type over $\mathbb{Z}$ — on its own is hard; one probes it through linear invariants. The bare association $X \mapsto H^\bullet(X)$ is coarse, since many non-isomorphic spaces share the same cohomology. One remembers far more by attaching to each X a whole category of *coefficients*: *local systems* $\mathcal{L}$, which themselves carry cohomology $H^\bullet(X, \mathcal{L})$ and are transported along a map $f\colon Y \to X$ by fibrewise cohomology $f_*\mathcal{L}$. To allow arbitrary maps, not just smooth proper ones, one enlarges these into a category of *sheaves*. Cohomology then acquires a dual companion, *cohomology with compact support* $H_c^\bullet$, already visible in Poincaré duality.

The structure that packages all of this is a *six-functor formalism* (defined carefully in Section A). Informally, it assigns to each $X \in C$ a symmetric monoidal *category of coefficients* $\mathrm{D}(X)$ — in our setting $\mathrm{D}(X) \in 1\mathrm{Pr}$ — and to each map $f\colon Y \to X$ two adjoint pairs

$$f^* \dashv f_*, \qquad f_! \dashv f^!,$$

encoding pullback, pushforward (cohomology), exceptional pushforward (cohomology with compact support), and exceptional pullback, all tied to the tensor product $\otimes$ by base change and a projection formula.

We now look for the categorified analogue of Mot_k: a fixed target through which *every* six-functor formalism factors. A first candidate is the category AnStck of *analytic stacks*, through which the $(f^*, \otimes)$-part $\mathrm{D}^{*,\otimes}$ often factors. Grothendieck and Scholze are then after the same thing: a universal coefficient through which every theory factors. They differ only in *which leg of the triangle they hold fixed*:

	Grothendieck: motives	**Scholze: six functors**
factorization	$\mathrm{Sch}_k^{\mathrm{op}} \xrightarrow{\mathrm{Mot}} \mathrm{Mot}_k \overset{\text{real.}}{\dashrightarrow} \mathbb{A}$, $\quad h^\bullet \colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathbb{A}$	$C^{\mathrm{op}} \overset{X \mapsto X^?}{\dashrightarrow} \mathrm{AnStck}^{\mathrm{op}} \xrightarrow{\mathrm{D}_{qc}} \mathrm{CAlg}(1\mathrm{Pr})$, $\quad \mathrm{D}^{*,\otimes} \colon C^{\mathrm{op}} \to \mathrm{CAlg}(1\mathrm{Pr})$
universal coefficient	Mot_k	AnStck (later Gest)
front leg	Mot — *fixed*	transmutation $X \mapsto X^?$ — *varies with* D
recovering leg	realization $\mathrm{Mot}_k \to \mathbb{A}$ — *varies*	$\mathrm{D}_{qc} = \mathcal{O}(-)_1$ — *fixed*
variation lives in	the realization	the transmutation

The same triangle, with the fixed and varying legs interchanged. Scholze packages all the realizations into the single theory-independent leg $\mathcal{O}(-)_1 = \mathrm{D}_{qc}$ — the same for every D, the leg that reads the $(f^*, \otimes)$-part of D back off the transmutation (Theorem 3.4) — and studies the front leg instead: the *transmutation* $X \mapsto X^?$, after Bhatt [Bha22, Remark 2.3.8], which converts a geometric object into its universal coefficient and commutes with finite limits. So the classical Betti and de Rham realizations are just $\mathcal{O}(-)_1$ applied to the transmutations by the Betti and de Rham ring stacks. The one caveat is the dashed top leg: AnStck achieves this only for *some* D.

Before turning to the obstruction, we record when the dashed leg *does* exist. For schemes (separated of finite type over a field, say) such transmutations are produced from a *ring stack*, by the following criterion of Scholze.

Theorem 1.1 ([Sch25, Theorem 10.6])**.** *Work in the category of analytic stacks over some base analytic stack S. Let k be a ring and let R be a k-algebra stack satisfying the following conditions.*

(1) *The map $f\colon R \to *$ is !-able and* D*-smooth, and the map $\mathbb{1} \to f_! f^! \mathbb{1}$ is an equivalence.*

(2) *The origin $i\colon * \xrightarrow{0} R$ is a closed immersion, the units $j\colon R^\times \hookrightarrow R$ yield an open immersion,*[1] *and*

$$j_! \mathbb{1} \to \mathbb{1} \to i_* \mathbb{1}$$

is a fibre sequence in $\mathrm{D}_{qc}(R)$.

This induces a transmutation functor

$$X \mapsto X_R$$

taking any separated scheme X of finite type over k to the 0*-truncated analytic stack over S given by the sheafification of the functor*

$$\mathrm{AnRing}_S \ni A \mapsto X(R(A)).$$

[1] Implicit here is that the composite is zero, i.e. $R^\times$ and the origin 0 have empty intersection.

The functor $X \mapsto X_R$ commutes with finite limits and Zariski gluing. Moreover, the functor $X \mapsto X_R$ sends étale maps to D*-étale maps, proper maps to* D*-proper maps, and smooth maps to* D*-smooth maps, and thus $X \mapsto \mathrm{D}_{qc}(X_R)$ yields a six-functor formalism with these properties.*

For instance, the Betti stack and the solid de Rham stack arise from such ring stacks.

The fixed second leg D_{qc} satisfies the *categorical Künneth formula*: by [Kes25], on analytic stacks over an affine base the external product is an equivalence.[2] Hence, *if* D factors as above, then D itself satisfies categorical Künneth:

$$\mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(X \times Y) \qquad \text{for all } X, Y.$$

For SH, for IndCoh, and for ℓ-adic sheaves, however, the categorical Künneth formula *fails* — so none of these factors through AnStck in this way.

Remark 1.2 (Why the Künneth formula fails for SH, IndCoh, and ℓ-adic sheaves)**.** In each case the external product $\mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y) \xrightarrow{\boxtimes} \mathrm{D}(X \times Y)$ fails to be an equivalence — it is not essentially surjective — so the kernel category $\mathbb{K}_{\mathrm{D}}$ is strictly larger than $\mathrm{Mod}_{\mathrm{D}(*)}(1\mathrm{Pr})$ (Theorem 2.3). (For ℓ-adic sheaves the external product is moreover fully faithful [Gai25]; for IndCoh and SH the cited sources assert only non-equivalence.)

- *ℓ-adic sheaves* [Gai25]. The functor $\mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y') \xrightarrow{\boxtimes} \mathrm{Shv}(Y \times Y')$ is *never* an equivalence once Y, Y' are positive-dimensional of finite type. For a witness take $Y' = Y$ *smooth* (so that $T^*(Y \times Y)$ and singular support are available) and the diagonal $\Delta\colon Y \hookrightarrow Y \times Y$: the constructible sheaf $\Delta_* \mathbb{1}_Y$ is compact, with singular support the conormal $N^*_\Delta \subset T^*(Y \times Y)$. Every external product satisfies $\mathrm{SS}(F \boxtimes G) \subseteq \mathrm{SS}(F) \times \mathrm{SS}(G)$ [KS90, §5.4]. But N^*_Δ is not contained in any finite union of products of conic Lagrangians: under $N^*_\Delta \cong T^*Y$, a point $(y, \xi, y, -\xi) \in \Lambda \times \Lambda'$ forces $(y, \xi) \in \Lambda \cap a(\Lambda')$ (a the fibrewise antipode), a locus of dimension at most $d = \dim Y$, while N^*_Δ has dimension $2d$. Since $\boxtimes$ is fully faithful and preserves colimits, any preimage of the compact object $\Delta_* \mathbb{1}_Y$ would itself be compact — hence a retract of a finite colimit of external products $F \boxtimes G$ — which the singular-support bound forbids. So $\Delta_* \mathbb{1}_Y$ escapes the image. (We argue with the topological singular support of [KS90, §5.4]; for ℓ-adic coefficients the corresponding theory is Beilinson's [Bei16].)
- *Ind-coherent sheaves*. The obstruction sits at the bottom (1Pr) level, and is exactly the failure of Künneth: over a base S the formalism $X \mapsto \mathrm{IndCoh}(X)$ takes values in $1\mathrm{Pr}_S = \mathrm{Mod}_{\mathrm{QCoh}(S)}(1\mathrm{Pr})$, the 2-category of $\mathrm{QCoh}(S)$-linear presentable categories (= 2-$\mathrm{QCoh}(S)$), and its right-lax monoidal structure is the external product

$$\mathrm{IndCoh}(X_1) \otimes_{\mathrm{QCoh}(S)} \mathrm{IndCoh}(X_2) \longrightarrow \mathrm{IndCoh}(X_1 \times_S X_2).$$

[2] Precisely, [Kes25, Cor. 1.5.1]: over a base with representable diagonal admitting an affine !-cover — in particular over an affine base — every !-able object is Künneth. Over a general stacky base the *relative* formula may fail, but that is not what is at stake here.

As Gaitsgory–Rozenblyum–Varshavsky note [GRV25], even when S, X_1, X_2 are smooth this is *not* an equivalence as soon as $X_1 \times_S X_2$ fails to be smooth, since IndCoh agrees with QCoh only on smooth targets. Hence Stefanich's 2- IndCoh(S) [Ste25] properly enlarges $1\mathrm{Pr}_S$. (Contrast QCoh, which *does* satisfy Künneth on perfect stacks [BFN10, Thm. 4.14].)

- *Motivic spectra* [Aok26]. Aoki shows the kernel 2-category $\mathbb{K}_{\mathrm{SH}}$ of SH (his $2\mathrm{SH}(\mathbb{Z})$) is, *as a stable presentably symmetric monoidal* 2*-category*, *freely* generated by a ring stack that is homologically trivial, smooth, and sutured — and is emphatically not the unit, giving a moduli description of the genuinely 2-categorical gestalt of SH. Were Künneth to hold, $\mathbb{K}_{\mathrm{SH}}$ would collapse to the single floor $\mathrm{Mod}_{\mathrm{SH}(*)}(1\mathrm{Pr})$.

These three failures are the gap we set out to fill. We enlarge the rigid target AnStck to the category of *Gestalten* Gest = $\mathrm{StRing}^{\mathrm{op}}$, and factor *every* presentable six-functor formalism D through a canonically associated transmutation:

$$\begin{array}{ccc} C^{\mathrm{op}} & \overset{[-]_{\mathrm{D}}}{\dashrightarrow} & \mathrm{Gest}^{\mathrm{op}} \\ & {\scriptstyle \mathrm{D}^{*,\otimes}} \searrow & \downarrow {\scriptstyle \mathcal{O}(-)_1} \\ & & \mathrm{CAlg}(1\mathrm{Pr}). \end{array}$$

In contrast with the analytic-stack picture, which applies only to *some* D, this construction — following Scholze [Sch26] and Stefanich [Ste25] — works for *every* presentable D. We write its front leg $[-]_{\mathrm{D}}$: this is the generic $X \mapsto X^{?}$ of the table, now produced canonically for every D rather than from an ad hoc ring stack R. Gestalten is in this sense the *universal coefficient category* for presentable six-functor formalisms, the categorified counterpart of Grothendieck's Mot_k. Quasicoherent sheaves on Gest — the fixed functor $\mathcal{O}(-)_1$ — is the *packaged realization* through which every 1Pr-valued D factors. The price is that $[X]_{\mathrm{D}}$ is now a genuinely higher-categorical object, obtained by iterating the category of kernels $\mathbb{K}_{\mathrm{D}}$ into a *Stefanich ring* $A_{\mathrm{D},X}$. The very failure of Künneth that obstructed the factorization through AnStck is exactly what the higher target absorbs (Theorem 2.3) — level by level, as the next section makes precise.

The whole picture is one *ladder of categorical levels*, the skeleton of the rest of the talk: each invariant of X records data one categorical degree higher than the last — and so supports correspondingly richer algebraic operations — while the transmutation $[-]_{\mathrm{D}}$ records every level at once.

Level	Invariant $X \mapsto$	n-categorical algebra	Available operations
0	$H^\bullet(X)$	cohomology ring (numbers)	$+$, $\times$
1	$\mathrm{D}(X)$	symmetric monoidal category	limits, colimits, $\otimes$
2	$\mathbb{K}_{\mathrm{D},X}$	category of kernels	limits, colimits, $\otimes$, adjoints, monads
∞	$[X]_{\mathrm{D}}$	Stefanich ring (a Gestalt)	all of the above, at every level

2 The category of kernels $\mathbb{K}_{\mathrm{D}}$

We now construct the finite-limit-preserving transmutation $[-]_{\mathrm{D}}\colon C \to \mathrm{Gest}$ at the heart of the universal-coefficient picture above. The most naive candidate uses D directly — send each X to the Gestalt of its coefficient category, i.e. compose

$$C \hookrightarrow \mathrm{Span}(C) \xrightarrow{\mathrm{D}} 1\mathrm{Pr} \longrightarrow \mathrm{Gest}, \qquad X \longmapsto \mathrm{Gest}(\mathrm{D}(X))$$

(precisely, the $(f^*, \otimes)$-part $\mathrm{D}^{*,\otimes}\colon C^{\mathrm{op}} \to \mathrm{CAlg}(1\mathrm{Pr})$ postcomposed with the bottom-floor inclusion $\mathrm{CAlg}(1\mathrm{Pr}) \hookrightarrow \mathrm{StRing} = \mathrm{Gest}^{\mathrm{op}}$). But this need not preserve pullbacks: the product $X \times Y$ is sent to $\mathrm{Gest}(\mathrm{D}(X \times Y))$, whereas the product of $\mathrm{Gest}(\mathrm{D}(X))$ and $\mathrm{Gest}(\mathrm{D}(Y))$ in Gest is $\mathrm{Gest}\big(\mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y)\big)$ — products in $\mathrm{Gest} = \mathrm{StRing}^{\mathrm{op}}$ are relative tensor products of coefficients. The two coincide exactly when D satisfies the *categorical Künneth formula* $\mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(X \times Y)$ — which, as Theorem 1.2 showed, generally fails (Theorem 2.3). The *category of kernels* $\mathbb{K}_{\mathrm{D}}$ is the universal fix, but it acts one categorical degree at a time. Passing from D to $\mathbb{K}_{\mathrm{D}}$ need not make Künneth hold outright; it *lifts* the obstruction one categorification up, so that the kernel theory $X \mapsto \mathbb{K}_{\mathrm{D},X}$ may itself fail Künneth at the next level. Iterating the construction into the Stefanich ring (Section 3) then clears the obstruction level by level, yielding a transmutation that genuinely preserves finite limits.

The kernel 2-category is constructed in Section A (Theorem A.9); here $\mathbb{K}_{\mathrm{D}}$ is its absolute case, over the terminal object of C, taken in the presentable form of the following definition (the precise comparison with the small enriched 2-category of Theorem A.9 is the subject of Aoki's theorem below).

Definition 2.1. Let C be a category with finite limits. Regarding the six-functor formalism D as a commutative algebra object in $\mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr})$, equipped with the Day convolution symmetric monoidal structure, the 2-*category of kernels of* D is the module category

$$\mathbb{K}_{\mathrm{D}} \coloneqq \mathrm{Mod}_{\mathrm{D}}(\mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr})).$$

This comes with a symmetric monoidal functor from $\mathrm{Span}(C)^{\mathrm{op}}$, given as the composite

$$\mathrm{Span}(C)^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr}) \to \mathrm{Mod}_{\mathrm{D}}(\mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr})) = \mathbb{K}_{\mathrm{D}},$$

namely the Yoneda embedding followed by the free D-module functor $\mathrm{D} \otimes (-)$. We denote it by $X \mapsto [X]_{\mathbb{K}_{\mathrm{D}}}$. Since $[X]_{\mathbb{K}_{\mathrm{D}}}$ is the free D-module on the representable presheaf at X, it is easy to map out of: by the free–forgetful adjunction composed with the Yoneda lemma, for any $X \in \mathrm{Span}(C)^{\mathrm{op}}$ and $M \in \mathbb{K}_{\mathrm{D}}$ one has

$$\mathrm{Hom}_{\mathbb{K}_{\mathrm{D}}}([X]_{\mathbb{K}_{\mathrm{D}}}, M) = M(X).$$

In particular, as the unit object of $\mathbb{K}_{\mathrm{D}}$ is D itself, taking $M = \mathbb{1}$ gives

$$\mathrm{Hom}_{\mathbb{K}_{\mathrm{D}}}([X]_{\mathbb{K}_{\mathrm{D}}}, \mathbb{1}) = \mathrm{D}(X).$$

Now every object of $\mathrm{Span}(C)$ is canonically self-dual ([HM24, p. 2.3.9]), and the embedding is symmetric monoidal, so $[Y]_{\mathbb{K}_D} \simeq [Y]^\vee_{\mathbb{K}_D}$ and $[X]_{\mathbb{K}_D} \otimes [Y]_{\mathbb{K}_D} \simeq [X \times Y]_{\mathbb{K}_D}$. Hence in general

$$\mathrm{Hom}_{\mathbb{K}_D}([X]_{\mathbb{K}_D}, [Y]_{\mathbb{K}_D}) \simeq \mathrm{Hom}_{\mathbb{K}_D}\big([X]_{\mathbb{K}_D} \otimes [Y]^\vee_{\mathbb{K}_D}, \mathbb{1}\big) \simeq \mathrm{Hom}_{\mathbb{K}_D}\big([X \times Y]_{\mathbb{K}_D}, \mathbb{1}\big) = \mathrm{D}(X \times Y),$$

which recovers Theorem A.9 (there the mapping object is written $\mathrm{Fun}_Y(X_1, X_2) = \mathrm{D}(X_1 \times_Y X_2)$; here Y is the terminal object, so $\mathrm{Hom}_{\mathbb{K}_D}([X]_{\mathbb{K}_D}, [Y]_{\mathbb{K}_D})$ is the absolute case $\mathrm{Fun}_*(X, Y)$). The direct computation above matches the two mapping objects on representables; the small enriched kernel 2-category of Theorem A.9 (Aoki [Aok26, Def. 2.4]) embeds fully faithfully into $\mathbb{K}_D$ as the representable objects $[X]_{\mathbb{K}_D}$, and the following theorem of Aoki identifies $\mathbb{K}_D$ itself with the 1Pr-enriched presheaves on it.

Theorem 2.2 (Aoki, [Aok26, Theorem 2.6])**.** *Let S be a small closed symmetric monoidal category and $V \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Let $\mathrm{D}\colon S \to V$ be a lax symmetric monoidal functor. Then there is a canonical equivalence*

$$\mathrm{PSh}^V(\tilde{S}^{\mathrm{op}}) \simeq \mathrm{Mod}_{\mathrm{D}}(\mathrm{Fun}(S, V)),$$

where $\tilde{S}$ is the V-enriched category obtained by base changing the self-enrichment of S along D.

Proof. Aoki credits this argument to H. Heine; we expand it, isolating the monadicity step and computing the resulting monad by hand.

Day convolution and the trivial enrichment. Equip $\mathrm{Fun}(S, V)$ with the Day convolution symmetric monoidal structure, for which a lax symmetric monoidal functor $S \to V$ is the same datum as a commutative algebra; thus $\mathrm{D} \in \mathrm{CAlg}(\mathrm{Fun}(S, V))$. Writing $U\colon \mathrm{An} \to V$ for the unit functor — the symmetric monoidal left adjoint with $U(*) = \mathbb{1}_V$ — the pointwise category $\mathrm{Fun}(S, V)$ is the V-presheaf category of S for its *trivial* enrichment S^{triv},

$$\mathrm{Fun}(S, V) = \mathrm{PSh}^V\left((S^{\mathrm{triv}})^{\mathrm{op}}\right), \qquad \underline{\mathrm{Hom}}_{S^{\mathrm{triv}}}(a, b) = U\big(\mathrm{Hom}_S(a, b)\big).$$

Change of enrichment. As S is closed it is enriched over itself with mapping objects $\underline{\mathrm{Hom}}_S(a, b)$, and a lax monoidal $G\colon S \to V$ base-changes this self-enrichment to the V-enrichment $a, b \mapsto G(\underline{\mathrm{Hom}}_S(a, b))$. Taking $G = U(\mathrm{Hom}_S(\mathbb{1}, -))$ recovers S^{triv} (since $\mathrm{Hom}_S(\mathbb{1}, \underline{\mathrm{Hom}}_S(a, b)) \simeq \mathrm{Hom}_S(a, b)$), while $G = \mathrm{D}$ yields $\tilde{S}$, with $\underline{\mathrm{Hom}}_{\tilde{S}}(a, b) = \mathrm{D}(\underline{\mathrm{Hom}}_S(a, b))$. Now $U(\mathrm{Hom}_S(\mathbb{1}, -))$ is the unit of $\mathrm{CAlg}(\mathrm{Fun}(S, V))$, so it admits a unique algebra map to D; base changing along it gives a V-enriched, identity-on-objects functor $S^{\mathrm{triv}} \to \tilde{S}$, whose opposite we call f. Being a base change of enrichment along a map of commutative algebras, f is symmetric monoidal, and passing to enriched presheaves yields an adjunction

$$f_! \dashv f^* \qquad \text{in } \mathrm{Mod}_V(\mathrm{Pr}^{\mathrm{L}}),$$

with $f^*\colon \mathrm{PSh}^V(\tilde{S}^{\mathrm{op}}) \to \mathrm{Fun}(S, V)$ the restriction of enrichment (lax monoidal, V-linear) and $f_!$ its V-linear, strong monoidal left adjoint, the enriched left Kan extension. Since $\underline{\mathrm{Hom}}_S(\mathbb{1}, a) \simeq a$, restriction of the monoidal unit recovers D:

$$(f^*\mathbb{1})(a) = \underline{\mathrm{Hom}}_{\tilde{S}}(\mathbb{1}, a) = \mathrm{D}\left(\underline{\mathrm{Hom}}_S(\mathbb{1}, a)\right) \simeq \mathrm{D}(a), \qquad \text{i.e. } f^*\mathbb{1} \simeq \mathrm{D} \text{ in } \mathrm{CAlg}(\mathrm{Fun}(S, V)).$$

Monadicity. Restriction along an identity-on-objects functor is computed objectwise, so f^* preserves all colimits and is conservative — a map of $\tilde{S}$-presheaves whose restriction to the (unchanged) collection of objects is an equivalence was already an objectwise equivalence. It is moreover V-linear and admits the left adjoint $f_!$, so the monadicity theorem for presentable module categories of Reutter–Zetto applies: the Eilenberg–Moore comparison is fully faithful [RZ25, Thm. 4.7] and, under these hypotheses, an equivalence [RZ25, Thm. 4.12]. Hence f^* is monadic, and with $T := f^* f_!$ the resulting V-linear monad on $\mathrm{Fun}(S, V)$,

$$\mathrm{PSh}^V(\tilde{S}^{\mathrm{op}}) \;\simeq\; \mathrm{Mod}_T(\mathrm{Fun}(S, V)).$$

It remains to identify T — first as an endofunctor, then as a monad.

The underlying endofunctor. Because f is the identity on objects, f^* does not move objects and $T(P)(s') = (f_!P)(s')$; the coend formula for the enriched left Kan extension $f_!$ then gives, for $P \in \mathrm{Fun}(S, V)$ and $s' \in S$,

$$T(P)(s') \;\simeq\; \int^{a\in S} \underline{\mathrm{Hom}}_{\tilde{S}}(a, s') \otimes P(a) \;=\; \int^{a} \mathrm{D}\left(\underline{\mathrm{Hom}}_S(a, s')\right) \otimes P(a).$$

On the other hand, Day convolution gives

$$(P \otimes \mathrm{D})(s') = \int^{a,b} \mathrm{Hom}_S(a \otimes b,\, s') \otimes P(a) \otimes \mathrm{D}(b).$$

Integrating over b first (Fubini for coends): for each fixed a, the closed-structure equivalence $\mathrm{Hom}_S(a \otimes b, s') \simeq \mathrm{Hom}_S(b, \underline{\mathrm{Hom}}_S(a, s'))$ followed by the co-Yoneda lemma $\int^b \mathrm{Hom}_S(b, Y) \otimes \mathrm{D}(b) \simeq \mathrm{D}(Y)$ at $Y = \underline{\mathrm{Hom}}_S(a, s')$ collapses the inner coend to $\mathrm{D}(\underline{\mathrm{Hom}}_S(a, s'))$, whence

$$(P \otimes \mathrm{D})(s') \;\simeq\; \int^{a} P(a) \otimes \mathrm{D}\left(\underline{\mathrm{Hom}}_S(a, s')\right).$$

The two displays agree, naturally in s', so $T \simeq (-) \otimes \mathrm{D}$ as endofunctors of $\mathrm{Fun}(S, V)$ — obtained from the closed structure of S alone, with no rigidity or separate projection formula. (Equivalently, by V-linearity it would suffice to compare the two on representables; the coend does it on all of $\mathrm{Fun}(S, V)$ at once.)

The monad structure, and conclusion. We promote this to an equivalence of monads. As f^* is lax monoidal with $f^*\mathbb{1} \simeq \mathrm{D}$, the monoidal adjunction supplies a canonical comparison: for $P \in \mathrm{Fun}(S, V)$,

$$P \otimes \mathrm{D} = P \otimes f^*\mathbb{1} \xrightarrow{\eta_P \otimes \mathrm{id}} f^* f_! P \otimes f^*\mathbb{1} \xrightarrow{\mathrm{lax}} f^*\left(f_! P \otimes \mathbb{1}\right) \simeq f^* f_! P = T(P),$$

where η is the unit of $f_! \dashv f^*$. This is a morphism of monads $(-) \otimes \mathrm{D} \to T$: the unit $\mathbb{1} \to \mathrm{D}$ of the algebra furnishes its unit, and the multiplication $\mathrm{D} \otimes \mathrm{D} \to \mathrm{D}$ — the lax constraint of $\mathrm{D} = f^* \mathbb{1}$ assembled over the Day coend — its multiplication. Its underlying natural transformation is an equivalence: by V-linearity it suffices to check this on the representable presheaves, where $f_!$ is the enriched left Kan extension computed by the Yoneda lemma and the comparison map is manifestly invertible (the verification Aoki performs). A monad morphism that is an underlying equivalence is an equivalence of monads. Therefore

$$\mathrm{PSh}^V(\tilde{S}^{\mathrm{op}}) \simeq \mathrm{Mod}_T(\mathrm{Fun}(S,V)) \simeq \mathrm{Mod}_{(-)\otimes \mathrm{D}}(\mathrm{Fun}(S,V)) \simeq \mathrm{Mod}_{\mathrm{D}}(\mathrm{Fun}(S,V)),$$

the last identification being $\mathrm{Mod}_{(-)\otimes A} \simeq \mathrm{Mod}_A$ for the free-module monad of an algebra A. Every step is canonical, so the equivalence is. □

Now, taking $V = 1\mathrm{Pr}$ and $S = \mathrm{Span}(C)$, we obtain the desired result.

Proposition 2.3 ([Sch26, Prop. 9.3])**.** *The natural functor*

$$\mathrm{Mod}_{\mathrm{D}(*)}(1\mathrm{Pr}) \longrightarrow \mathbb{K}_{\mathrm{D}}$$

is fully faithful, with essential image the "restricted[3]" *objects: $M \in \mathbb{K}_{\mathrm{D}}$ lies in it iff $\mathrm{D}(X) \otimes_{\mathrm{D}(*)} M(*) \to M(X)$ is an equivalence for all X. In particular,*

$$\mathbb{K}_{\mathrm{D}} = \mathrm{Mod}_{\mathrm{D}(*)}(1\mathrm{Pr}) \quad \Longleftrightarrow \quad \mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(X \times Y) \ \textit{for all } X, Y.$$

Proof. We work with the description $\mathbb{K}_{\mathrm{D}} = \mathrm{Mod}_{\mathrm{D}}(\mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr}))$ of Theorem 2.1 and the formula $\mathrm{Hom}_{\mathbb{K}_{\mathrm{D}}}([X]_{\mathbb{K}_{\mathrm{D}}}, M) = M(X)$ established there.

Evaluation is atomic. For each X, that formula says the corepresentable of $[X]_{\mathbb{K}_{\mathrm{D}}}$ is the evaluation functor

$$\mathrm{ev}_X = \mathrm{Hom}_{\mathbb{K}_{\mathrm{D}}}([X]_{\mathbb{K}_{\mathrm{D}}}, -) \colon \mathbb{K}_{\mathrm{D}} \longrightarrow 1\mathrm{Pr}, \qquad M \longmapsto M(X).$$

Colimits in $\mathbb{K}_{\mathrm{D}}$ are created by the forgetful functor to $\mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr})$, where they are computed pointwise; hence ev_X preserves all colimits, and it is 1Pr-linear. Thus every $[X]_{\mathbb{K}_{\mathrm{D}}}$ is 1Pr-atomic. In particular the unit $\mathbb{1} = [*]_{\mathbb{K}_{\mathrm{D}}}$ is 1Pr-atomic, with $\mathrm{End}_{\mathbb{K}_{\mathrm{D}}}(\mathbb{1}) = \mathrm{ev}_*(\mathbb{1}) = \mathrm{D}(*)$.

The natural functor and its right adjoint. Since $\mathrm{ev}_*(M) = M(*)$ is naturally a module over $\mathrm{End}_{\mathbb{K}_{\mathrm{D}}}(\mathbb{1}) = \mathrm{D}(*)$, the functor ev_* refines to a colimit-preserving functor

$$G \colon \mathbb{K}_{\mathrm{D}} \longrightarrow \mathrm{Mod}_{\mathrm{D}(*)}(1\mathrm{Pr}), \qquad M \longmapsto M(*).$$

The functor of the statement is its left adjoint — induction along $\mathrm{D}(*) = \mathrm{End}_{\mathbb{K}_{\mathrm{D}}}(\mathbb{1}) \to \mathbb{K}_{\mathrm{D}}$,

$$L \colon \mathrm{Mod}_{\mathrm{D}(*)}(1\mathrm{Pr}) \longrightarrow \mathbb{K}_{\mathrm{D}}, \qquad N \longmapsto N \otimes_{\mathrm{D}(*)} \mathbb{1},$$

[3] in the sense of [Gai25]

the unique 1Pr-linear colimit-preserving functor with $L(\mathrm{D}(*)) = \mathbb{1}$; one has $L \dashv G$.

Full faithfulness. L is fully faithful iff the unit $N \to GL(N)$ of the adjunction is an equivalence for every N. Here

$$GL(N) = \big(N \otimes_{\mathrm{D}(*)} \mathbb{1}\big)(*) \;\simeq\; N \otimes_{\mathrm{D}(*)} \mathbb{1}(*) \;=\; N \otimes_{\mathrm{D}(*)} \mathrm{D}(*) \;=\; N.$$

The middle equivalence is where atomicity enters: ev_* is colimit-preserving and 1Pr-linear, so it commutes with the relative tensor product $N \otimes_{\mathrm{D}(*)} (-)$ (a geometric realization of the bar construction); and $\mathbb{1}(*) = \mathrm{D}(*)$. Hence $GL \simeq \mathrm{id}$ and L is fully faithful.

The essential image. As L is fully faithful with right adjoint G, an object $M \in \mathbb{K}_\mathrm{D}$ lies in the essential image of L iff the counit $\varepsilon_M : LG(M) \to M$ is an equivalence. Evaluating $LG(M) = M(*) \otimes_{\mathrm{D}(*)} \mathbb{1}$ at X — again using that ev_X commutes with $M(*) \otimes_{\mathrm{D}(*)} (-)$, together with $\mathbb{1}(X) = \mathrm{D}(X)$ —

$$\big(LG(M)\big)(X) \;\simeq\; M(*) \otimes_{\mathrm{D}(*)} \mathbb{1}(X) \;=\; \mathrm{D}(X) \otimes_{\mathrm{D}(*)} M(*),$$

and ε_M at X is precisely the canonical map $\mathrm{D}(X) \otimes_{\mathrm{D}(*)} M(*) \to M(X)$. So M is in the image — i.e. *restricted* — iff this map is an equivalence for all X, as claimed.

The Künneth criterion. The restricted objects are closed under colimits and under 1Pr-tensoring: LG and id are both colimit-preserving and 1Pr-linear, so the locus where the natural transformation ε is invertible is closed under colimits and 1Pr-tensoring. Since $\mathbb{K}_\mathrm{D}$ is 1Pr-atomically generated by the representables $[Y]_{\mathbb{K}_\mathrm{D}}$ — i.e. generated under colimits and 1Pr-tensoring — the functor L is essentially surjective — equivalently an equivalence, i.e. $\mathbb{K}_\mathrm{D} = \mathrm{Mod}_{\mathrm{D}(*)}(1\mathrm{Pr})$ — iff every $[Y]_{\mathbb{K}_\mathrm{D}}$ is restricted. Finally, by the $\mathrm{Hom}_{\mathbb{K}_\mathrm{D}}$-formula,

$$[Y]_{\mathbb{K}_\mathrm{D}}(*) = \mathrm{Hom}_{\mathbb{K}_\mathrm{D}}(\mathbb{1}, [Y]_{\mathbb{K}_\mathrm{D}}) = \mathrm{D}(Y), \qquad [Y]_{\mathbb{K}_\mathrm{D}}(X) = \mathrm{Hom}_{\mathbb{K}_\mathrm{D}}([X]_{\mathbb{K}_\mathrm{D}}, [Y]_{\mathbb{K}_\mathrm{D}}) = \mathrm{D}(X \times Y),$$

so $[Y]_{\mathbb{K}_\mathrm{D}}$ is restricted iff $\mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y) \to \mathrm{D}(X \times Y)$ is an equivalence for all X. Ranging over Y as well, this is exactly the categorical Künneth formula. □

3 Transmutation

We now construct the transmutation functor $C^{\mathrm{op}} \to \mathrm{Gest}^{\mathrm{op}} = \mathrm{StRing}$. Note that for any $X \in C$, we can repeat Theorem 2.1 with the slice $C_{/X}$, defining

$$\mathbb{K}_{\mathrm{D},X} \coloneqq \mathrm{Mod}_\mathrm{D}(\mathrm{Fun}(\mathrm{Span}(C_{/X}), 1\mathrm{Pr})) \in \mathrm{CAlg}(2\mathrm{Pr}).$$

The pullback functor $\mathrm{Span}(C) \to \mathrm{Span}(C_{/X})$ induces, by passing to free presentable 2-categories, a symmetric monoidal functor

$$\mathrm{Fun}(\mathrm{Span}(C), 1\mathrm{Pr}) \to \mathrm{Fun}(\mathrm{Span}(C_{/X}), 1\mathrm{Pr}), \quad F \mapsto F\,|_{C_{/X}}\,.$$

Because D is compatible with this restriction, the two routes to D-modules — restrict then take modules, or take modules then restrict — agree; there is only one sense of "passing to modules over D".

For any map $f\colon Y \to X$ in C, the pullback functor $f^*\colon C_{/X} \to C_{/Y}$ induces a symmetric monoidal functor

$$f_1^*\colon \mathbb{K}_{\mathrm{D},X} \to \mathbb{K}_{\mathrm{D},Y},$$

so this assembles into a functor

$$C^{\mathrm{op}} \to \mathrm{CAlg}(2\mathrm{Pr}).$$

Proposition 3.1 ([Sch26, Prop. 9.4]). *For every $f\colon Y \to X$ the functor $f_1^*\colon \mathbb{K}_{\mathrm{D},X} \to \mathbb{K}_{\mathrm{D},Y}$ admits* isomorphic *left and right adjoints $f_{1!} \cong f_{1*}$, whose formation commutes with base change.*

Proof. Existence of the adjoints. Forgetting the $\mathrm{Mod}_{\mathrm{D}}$-structure to $\mathrm{Fun}(\mathrm{Span}(C_{/X}), 1\mathrm{Pr})$, the functor f_1^* is restriction (precomposition) along the span functor $\mathrm{Span}(u)\colon \mathrm{Span}(C_{/Y}) \to \mathrm{Span}(C_{/X})$, where $u\colon C_{/Y} \to C_{/X}$ is the forgetful functor left adjoint to base change ($u \dashv f^*$); both u and f^* preserve pullbacks, so $\mathrm{Span}(u)$ and $\mathrm{Span}(f^*)$ exist. Restriction along $\mathrm{Span}(u)$ has as left and right adjoints the Kan extensions $f_{1!} = \mathrm{Span}(u)_!$ and $f_{1*} = \mathrm{Span}(u)_*$, which exist because the target 1Pr is presentable. These adjoints are D-linear (by the projection formula) and survive $\mathrm{Mod}_{\mathrm{D}}(-)$ — passing to modules over D is compatible with base change — giving $f_{1!} \dashv f_1^* \dashv f_{1*}$ between $\mathbb{K}_{\mathrm{D},X}$ and $\mathbb{K}_{\mathrm{D},Y}$.

The ambient 3-category. We prove $f_{1!} \cong f_{1*}$ by ambidexterity, and the correct ambient for this is the 3-category $2\mathrm{Pr} = \mathrm{Mod}_{1\mathrm{Pr}}(1\mathrm{Pr})$: its objects are the presentable 2-categories such as the $\mathbb{K}_{\mathrm{D},X}$, its 1-morphisms the 1Pr-linear functors such as f_1^*, its 2-morphisms the 1Pr-linear transformations, its 3-morphisms the modifications. Crucially we do *not* work inside a single $\mathbb{K}_{\mathrm{D},X}$, which is only a 2-category, where the unit and counit of $f_1^* \dashv f_{1*}$ would have no room to acquire adjoints; in 2Pr they are 2-morphisms and can.

The criterion, and which unit/counit it consumes. Recall the general ambidexterity criterion ([Sch26, Prop. 4.4], after Lurie [Lur09, Rem. 3.4.22]): a 1-morphism F of a 3-category with a chosen right adjoint F^R admits a left adjoint F^L with $F^L \simeq F^R$ as soon as the unit $\eta\colon \mathrm{id} \to F^R F$ and counit $\varepsilon\colon FF^R \to \mathrm{id}$ *of the adjunction $F \dashv F^R$* themselves admit right adjoints. We apply it with $F = f_1^*$ and $F^R = f_{1*}$ in 2Pr; the relevant 2-morphisms are thus the unit and counit of $f_1^* \dashv f_{1*}$ — *not* those of $f_{1!} \dashv f_1^*$.

Discharging the hypotheses. That η and ε admit right adjoints is checked by descent on the categorical level. By evaluation at the monoidal unit, the unit and counit of the level-2 adjunction $f_1^* \dashv f_{1*}$ are identified with those of the analogous restriction/Kan-extension adjunction one categorical level down, inside $\mathrm{Fun}(\mathrm{Span}(C_{/-}), 1\mathrm{Pr})$, where their right-adjointability is the adjointability of the relevant span correspondences ($\mathrm{Span}(u)$ and the legs of the spans). At the bottom level this holds by the self-duality of span categories: every object of $\mathrm{Span}(C_{/Y})$ is self-dual, every correspondence admits both adjoints (its transpose), and the transposition equivalence fixes $\mathrm{Span}(u)$, so the data is compatible with $\mathrm{Span}(u)$ and propagates back up through $\mathrm{Mod}_{\mathrm{D}}(-)$. Hence η, ε admit right adjoints in 2Pr.

Ambidexterity. The criterion therefore applies and yields a left adjoint F^L of f_1^* with $F^L \simeq f_{1*}$; a left adjoint being unique, $F^L \simeq f_{1!}$, whence the canonical equivalence $f_{1!} \simeq f_{1*}$.

Base change. For a pullback square in C the induced square of slice forgetful functors is adjointable and the mate of the base-change square is an equivalence; passing through $\mathrm{Span}(-)$ and $\mathrm{Mod}_{\mathrm{D}}(-)$ (both base-change compatible) transports this to the Kan-extension adjoints, so the formation of $f_{1!}$ and of f_{1*} commutes with base change. Since $f_{1!} \simeq f_{1*}$ is the canonical equivalence extracted from the same adjunction data, it is natural in f and commutes with base change as well. □

This already yields a functor $C \to \mathrm{Gest}$, $X \mapsto \mathrm{Gest}(\mathbb{K}_{\mathrm{D},X})$, valued in 2-affine Gestalten. Being 2-affine, every map is automatically 2-proper; by Theorem 3.1 it is moreover 1-prim, the right adjoint f_{1*} supplying the primness datum. It is not, however, evidently 1-suave: that would need left adjoints for the pullback $\mathrm{Mod}_{\mathbb{K}_{\mathrm{D},X}}(2\mathrm{Pr}) \to \mathrm{Mod}_{\mathbb{K}_{\mathrm{D},Y}}(2\mathrm{Pr})$ *one level up*, whose existence is not clear. And promoting 1-prim/1-suave to 1-proper/1-étale would mean applying Theorem 3.1 to the iterated diagonals of f, which requires the very Künneth formula that may fail (Theorem 2.3). Either obstruction blocks us at this level — which is what forces us to iterate the construction.

Remark 3.2 (Why $\mathrm{Gest}(\mathbb{K}_{\mathrm{D},X})$ is 2-affine)**.** By [Sch26, Def. 1.18] a Stefanich ring is a sequence $A = (A_1, A_2, \dots)$ with $A_n \in \mathrm{CAlg}(n\mathrm{Pr})$ and $A_n \simeq \mathrm{End}_{A_{n+1}}(\mathbb{1})$ — where $0\mathrm{Pr} = \mathrm{An}$, $1\mathrm{Pr} = 1\mathrm{Pr}$ and $(n+1)\mathrm{Pr} = \mathrm{Mod}_{n\mathrm{Pr}}(1\mathrm{Pr})$ — and $O(\mathrm{Gest}(A))_n = A_n$. On the *unit* Stefanich ring its terms are the units $\mathbb{1}_{n\mathrm{Pr}} = (n-1)\mathrm{Pr}$, so the relevant term is $A_3 = 2\mathrm{Pr}$.

By [Sch26, Def. 6.5], B is *n-affine* if it lies in the essential image of $\mathrm{CAlg}(A_{n+1}) \to \mathrm{StRing}_{A/}$; equivalently, B is generated by its single component $B_n \in \mathrm{CAlg}(A_{n+1})$ through $B_{m+1} \simeq \mathrm{Mod}_{B_m}(A_{m+1})$ for $m \geq n$, the lower terms being recovered by looping. In words: an n-affine Gestalt carries no new categorical data above level n.

Now $\mathbb{K}_{\mathrm{D},X} = \mathrm{Mod}_{\mathrm{D}}(\mathrm{Fun}(\mathrm{Span}(C_{/X}), 1\mathrm{Pr}))$ is a commutative algebra in 2Pr, hence $\mathbb{K}_{\mathrm{D},X} \in \mathrm{CAlg}(2\mathrm{Pr}) = \mathrm{CAlg}(A_3)$ — exactly such a generator, at level 2. So $\mathrm{Gest}(\mathbb{K}_{\mathrm{D},X})$ is Gest applied to

$$\big(\mathrm{D}(X),\ \mathbb{K}_{\mathrm{D},X},\ \mathrm{Mod}_{\mathbb{K}_{\mathrm{D},X}}(2\mathrm{Pr}),\ \dots\big), \qquad \mathrm{D}(X) = \mathrm{End}_{\mathbb{K}_{\mathrm{D},X}}(\mathbb{1}),$$

whose only genuine datum is $\mathbb{K}_{\mathrm{D},X}$: the higher terms are iterated module categories and $\mathrm{D}(X)$ is the looping. This is 2-affineness — the higher analogue of an affine scheme $\mathrm{Spec}\, R$, whose tower $\big(\mathrm{D}(R), \mathrm{Mod}_{\mathrm{D}(R)}(1\mathrm{Pr}), \dots\big)$ is likewise iterated modules over one ring.

The level is sharp, and reconnects to Theorem 2.3. $\mathrm{Gest}(\mathbb{K}_{\mathrm{D},X})$ would already be 1-affine — that is, $\mathbb{K}_{\mathrm{D},X} \simeq \mathrm{Mod}_{\mathrm{D}(X)}(1\mathrm{Pr})$ — precisely when the categorical Künneth formula holds over $C_{/X}$. When it fails, $\mathbb{K}_{\mathrm{D},X} \supsetneq \mathrm{Mod}_{\mathrm{D}(X)}(1\mathrm{Pr})$ and $\mathrm{Gest}(\mathbb{K}_{\mathrm{D},X})$ is genuinely 2-affine, a 2-stack morally at the level of $B^2\mathbb{G}_m$. Whether the *full* transmutation $[X]_{\mathrm{D}} = \mathrm{Gest}(A_{\mathrm{D},X})$ is n-affine for some finite n depends on D:

this happens exactly when the kernel iteration stabilizes, each higher level contributing genuine data only where Künneth keeps failing. For SH it *does* stabilize, already at level 2: by [Sch26, Prop. 9.9] the gestalt $[\operatorname{Spec}\mathbb{Z}]_{\mathrm{SH}}$ is 2-affine and every projection $[X]_{\mathrm{SH}} \to [\operatorname{Spec}\mathbb{Z}]_{\mathrm{SH}}$ is 1-affine, so $[X]_{\mathrm{SH}}$ is 2-affine for all X and the tower terminates. By contrast there do exist Gestalten that are n-affine for no finite n — e.g. $B^\infty\mathbb{G}_m$ or the infinite-dimensional affine space [Sch26, Rem. 5.6] — but these are not transmutations of finite-type schemes.

It is worth pausing on *why* iterating helps, since climbing a level might seem only to defer the problem. The kernel construction does not *repair* Künneth — it *relocates* it. By design $\operatorname{Hom}_{\mathbb{K}_{\mathrm{D}}}([X]_{\mathbb{K}_{\mathrm{D}}}, [Y]_{\mathbb{K}_{\mathrm{D}}}) = \mathrm{D}(X \times Y)$: the level-1 discrepancy $\mathrm{D}(X \times Y) \neq \mathrm{D}(X) \otimes_{\mathrm{D}(*)} \mathrm{D}(Y)$ is no longer an obstruction but is *absorbed* into the level-2 object $\mathbb{K}_{\mathrm{D}}$, which now knows $\mathrm{D}(X \times Y)$ tautologically. What we have bought is a genuinely new six-functor formalism $X \mapsto \mathbb{K}_{\mathrm{D},X}$, valued one categorical degree higher, and it carries its *own* Künneth question $\mathbb{K}_{\mathrm{D},X\times Y} \overset{?}{\simeq} \mathbb{K}_{\mathrm{D},X} \otimes_{\mathbb{K}_{\mathrm{D}}} \mathbb{K}_{\mathrm{D},Y}$, which may fail afresh. Each pass thus converts the *failure* of Künneth at one level into honest *data* at the next, and asks the question again one rung up; iterating is relocating the obstruction again and again, and the Stefanich ring $A_{\mathrm{D},X}$ is the fixed point that records the answer at every level at once. For some D the relocation halts — for SH already at level 2 (Theorem 3.2) — but the construction is uniform whether or not it does.

Geometrically this is the passage from a ring to a space. The bottom datum $\mathrm{D}(X)$ — the level-1 sheaf theory, with functions $\mathcal{O}(X) = \operatorname{End}_{\mathrm{D}(X)}(\mathbb{1})$ — does not determine the gestalt $[X]_{\mathrm{D}}$, precisely because Künneth fails; it would, were $[X]_{\mathrm{D}}$ already 1-affine (Theorem 2.3). $\operatorname{Gest}(\mathbb{K}_{\mathrm{D},X})$ is the 2-affine approximation, one layer deeper but still blind above level 2, and the full transmutation $[X]_{\mathrm{D}} = \operatorname{Gest}(A_{\mathrm{D},X})$ is the honest geometric object, assembled from these approximations the way an ∞-stack is glued from affines: the genuine *space* in Gest attached to X, not the bare algebra $\mathrm{D}(X)$ — the level-∞ row of the ladder, with $[-]_{\mathrm{D}}$ recording every rung.

The failure diagnosed before the remark — that $\operatorname{Gest}(\mathbb{K}_{\mathrm{D},X})$ is only 1-prim, not yet 1-suave or 1-proper — is repaired by climbing one categorical level and repeating the construction. The assignment

$$C^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(2\mathrm{Pr}), \qquad X \longmapsto \mathbb{K}_{\mathrm{D},X},$$

itself admits, by Theorem 3.1, base-change-compatible left adjoints $f_{1!}$ ($\cong f_{1*}$) for every f; it therefore upgrades to a once-categorified six-functor formalism

$$\mathrm{Span}(C) \longrightarrow 2\mathrm{Pr} = \mathrm{Mod}_{1\mathrm{Pr}}(1\mathrm{Pr}),$$

valued one categorical degree above D. Applying the kernel construction to this sheaf theory produces the level-3 data, and iterating level by level yields — following Stefanich [Ste25] — a *Stefanich ring*

$$A_{\mathrm{D},X} = \big(\mathrm{D}(X),\ \mathbb{K}_{\mathrm{D},X},\ \dots\big) \in \mathrm{StRing}, \qquad \text{functorial in } X,\ \text{i.e. a functor } C^{\mathrm{op}} \to \mathrm{StRing}.$$

This is the level-∞ row of the ladder of categorical levels from the introduction: the single object that records every categorical degree of X at once.

Definition 3.3 (Transmutation)**.** With Gest := $\mathrm{StRing}^{\mathrm{op}}$, the *transmutation* of D is

$$[-]_{\mathrm{D}} \colon C \longrightarrow \mathrm{Gest}_{/\mathrm{Gest}(A_{\mathrm{D}})}, \qquad X \longmapsto [X]_{\mathrm{D}} = \mathrm{Gest}(A_{\mathrm{D},X}).$$

The target is the slice over $\mathrm{Gest}(A_{\mathrm{D}}) = [*]_{\mathrm{D}}$, the transmutation of the point; forgetting it reads $[-]_{\mathrm{D}}$ simply as the functor $C \to \mathrm{Gest}$ of the factorization diagram above.

Theorem 3.4 ([Sch26, Prop. 9.5])**.** *For every presentable six-functor formalism* D*, the transmutation* $[-]_{\mathrm{D}}$ *satisfies:*

(1) *it commutes with finite limits;*

(2) *for each $f \colon Y \to X$ in C, the map $[f]_{\mathrm{D}} \colon [Y]_{\mathrm{D}} \to [X]_{\mathrm{D}}$ is* 1*-étale and* 1*-proper ([Sch26, Def. 6.19 and 6.12]), with $f_{1!} \cong f_{1*}$ (the left and right adjoints to the kernel pullback $f_1^* \colon \mathbb{K}_{\mathrm{D},X} \to \mathbb{K}_{\mathrm{D},Y}$ agreeing);*

(3) *the composite $O(-)_1 \circ [-]_{\mathrm{D}}$ recovers the $(f^*, \otimes)$-functoriality of* D.

Moreover the construction is, in Scholze's words, "completely canonical".

Proof. For each $f \colon Y \to X$ the iteration furnishes, at every categorical level, the kernel pullback together with *both* of its adjoints, so Theorem 3.1 applies level by level: $[f]_{\mathrm{D}}$ is 1-prim and 1-suave, with $f_{1!} \cong f_{1*}$. The extra condition in n-suaveness — a right adjoint to the canonical map $(B/A)^!_m \to \mathbb{1}$ for $m \geq n+1$ [Sch26, Def. 6.16] — is automatic here, both adjoints being 1Pr-linear, so that the relevant objects are dualizable [Sch26, Lemma 6.22]. Promoting these to 1-properness and 1-étaleness only requires applying Theorem 3.1 to the iterated diagonals of f, legitimate once $[-]_{\mathrm{D}}$ preserves finite limits.

For that it suffices to treat binary products, since a fibre product reduces to one after passing to a slice. A product in Gest = $\mathrm{StRing}^{\mathrm{op}}$ is a relative tensor product of Stefanich rings (formed after spectrification, [Sch26, Cor. 5.5]), so the claim is a Künneth formula for the lower-star functors f_{n*}, one categorical level at a time. The base-change compatibility of the adjoints of Theorem 3.1 is exactly what makes each

$$C^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(n\mathrm{Pr}), \qquad X \longmapsto O(A_{\mathrm{D},X})_n,$$

a six-functor formalism (with respect to the right adjoints), and it is there that Künneth supplies the product.

Finally $O(A_{\mathrm{D},X})_1 = \mathrm{D}(X)$ is the bottom term of the Stefanich ring, with its $(f^*, \otimes)$-functoriality intact, so $O(-)_1 \circ [-]_{\mathrm{D}} = \mathrm{D}^{*,\otimes}$. □

Warning 3.5. $\mathcal{O}(-)_1 \circ [-]_{\mathrm{D}}$ recovers D only at the level of $(f^*, \otimes)$; the shriek functors are not seen by the target $\mathrm{CAlg}(1\mathrm{Pr})$. Whether the *full* formalism — the !-functors included — can be reconstructed from $[-]_{\mathrm{D}}$, by singling out a class of !-able maps of Gestalten and a six-functor formalism on them, is the subject of the appendix to *Scholze's* Lecture IX — not the present appendix — and is where Talk 11 takes over; the results there are, in Scholze's words, "not definitive".

4 Dictionary: six-functor notions vs. Gestalten

Having built $[-]_{\mathrm{D}}$, we can now read the geometric vocabulary of a map f directly off its image $[f]_{\mathrm{D}}$ in Gest: each six-functor property of f becomes a Gestalt-level property of $[f]_{\mathrm{D}}$.

By Theorem 3.4 every $[f]_{\mathrm{D}}$ is automatically 1-étale and 1-proper, with $f_{1!} \cong f_{1*}$; the categorical layers above the bottom are forced by ambidexterity and separate no maps. The whole dictionary therefore lives at layer 0, where the four Stefanich-ring conditions unwind directly to the six-functor notions of Section A. Concretely, the layer-0 datum of $[f]_{\mathrm{D}}$ is the unit map $\mathbb{1} \to (B/A)_1$ of the underlying map of Stefanich rings. This is a 1-morphism $[X] \to [Y]$ in $A_2 = \mathbb{K}_{\mathrm{D},X}$, carrying the Fourier–Mukai kernel of f. Asking it to admit a right adjoint is 0-primness [Sch26, Def. 6.9]; asking a left adjoint is 0-suaveness [Sch26, Def. 6.16]. By Theorem A.14 these say exactly that f is D-prim, resp. D-suave. Imposing the same on all iterated diagonals of f gives 0-properness and 0-étaleness [Sch26, Def. 6.12 and 6.19], which is the recursive Theorem A.18; that recursion terminates only for truncated f, whence the extra truncatedness clause for the cohomological notions. All four equivalences then follow directly from the definitions.

Remark 4.1 ([Sch26, Rem. 9.6])**.** In tabular form:

f in C	$[f]_{\mathrm{D}}$ in Gest
suave	0-suave
prim	0-prim
cohomologically étale	truncated $\wedge$ 0-étale
cohomologically proper	truncated $\wedge$ 0-proper

Remark 4.2 (Stack-theoretic perspective)**.** The suave and prim notions above have a clean stack-level form, due to Aoki [Aok26, Def. 4.5], stated in the absolute presentable kernel 2-category (with stacks the objects of its CAlg): a morphism $[X] \to [Y]$ is *weakly suave* if it admits an $[X]$-linear left adjoint, *prim* if it admits an $[X]$-linear right adjoint, and *suave* if it is weakly suave and $[Y]$ is dualizable over $[X]$. For the kernel 2-category of a six-functor formalism, checking that $Y \to X$ in C determines such a morphism reduces to this $[X]$-linear adjoint condition [Aok26, Example 4.8]. Aoki's *weakly suave* matches the object-level Theorem A.11 up to the standard reformulation "is a left adjoint" versus "admits an $[X]$-linear left adjoint" (via the self-duality $\mathbb{K}_{\mathrm{D},Y} \simeq \mathbb{K}_{\mathrm{D},Y}^{\mathrm{op}}$), while his *suave* adds Aoki's dualizability of $[Y]$ over $[X]$ — equivalently, by Theorem A.16, of the suave dual

— exactly the extra input promoting suaveness to cohomological smoothness. This is the stack-level reading behind the dictionary above: suaveness and primness of f become properties of $[f]_{\mathrm{D}}$ in Gest.

Outlook: towards the extension to stacks

One last question, and the most satisfying. The transmutation was built object by object — but does it see the *topology*? Tracking covers through $[-]_{\mathrm{D}}$ reveals where the D-topology really comes from.

Recall from [Sch25, Def. 5.13] the D-*topology* on C: a family $\{f_i : Y_i \to X\}$ (with each f_i in E and forming a cover for the canonical topology) is a D-cover iff it moreover satisfies *universal* !*-descent*, i.e. after any base change $X' \to X$ the upper-shriek functor

$$\mathrm{D}(X') \xrightarrow{\sim} \lim\Big(\prod_i \mathrm{D}(Y_i \times_X X') \rightrightarrows \prod_{i,j} \mathrm{D}(Y_i \times_X Y_j \times_X X') \cdots \Big)$$

is an equivalence.

Proposition ([Sch26, Prop. 9.7])**.** If $\{f_i : Y_i \to X\}$ satisfies universal !-descent, then $\coprod_i [Y_i]_{\mathrm{D}} \to [X]_{\mathrm{D}}$ is a cover (effective epimorphism) in Gest.

Corollary ([Sch26, Cor. 9.8])**.** Hence $[-]_{\mathrm{D}}$ extends to a functor

$$\mathrm{Shv}_{\mathrm{D}}(C)^{\aleph_1} \longrightarrow \mathrm{Gest}_{/[*]_{\mathrm{D}}}, \qquad X \longmapsto [X]_{\mathrm{D}},$$

from D-sheaves on C (with values in 1-étale Gestalten), commuting with countable colimits and finite limits.

The upshot is that $[-]_{\mathrm{D}}$ carries the D-topology on C into the canonical topology on Gest — every D-cover becomes a cover there — so in distilling a Grothendieck topology from a six-functor formalism one was already seeing, if only in secret, the canonical topology on Gestalten. Constructing the six-functor formalism *on* Gest itself — and pinning down the sense in which D can be recovered from it — is where Talk 11 takes over.

One-line summary

$\mathrm{D} \rightsquigarrow \mathbb{K}_{\mathrm{D}} \rightsquigarrow A_{\mathrm{D},X} \rightsquigarrow [X]_{\mathrm{D}} = \mathrm{Gest}(A_{\mathrm{D},X})$: the $(f^*, \otimes)$-part of the six-functor formalism *is* quasicoherent sheaves on its transmutation, canonically and for all D (the !-functors are another matter — see the Warning above) — so Gest is the universal coefficient category for presentable six-functor formalisms.

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A Six-functor formalisms, smoothness, and suaveness

A modern viewpoint, originating in Grothendieck's school, organizes "cohomology with coefficients" on a class of geometric objects C as a single coherent package: an assignment $X \mapsto \mathrm{D}(X)$ of a category of "coefficient systems" to each object, together with six functors relating these categories under morphisms.

An informal definition. A *six-functor formalism* on C assigns to each $X \in C$ a presentable symmetric monoidal stable category $\mathrm{D}(X)$, and to each morphism $f: Y \to X$ in C four functors

$$f^* \dashv f_*\colon\ \mathrm{D}(X) \rightleftarrows \mathrm{D}(Y), \qquad f_! \dashv f^!\colon\ \mathrm{D}(Y) \rightleftarrows \mathrm{D}(X),$$

together with the symmetric monoidal product $\otimes$ and internal hom $\underline{\mathrm{Hom}}$ on each $\mathrm{D}(X)$, such that:

- f^* is colimit-preserving and symmetric monoidal, and $f_!$ is colimit-preserving and satisfies the projection formula;
- base change: for every pullback square in C,

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

  the canonical maps $g^* f_! \to f'_! g'^*$ and $f^! g_* \to g'_* f'^!$ are equivalences;
- projection formula: $f_!(A \otimes f^* B) \simeq f_!(A) \otimes B$ for $A \in \mathrm{D}(Y)$, $B \in \mathrm{D}(X)$.

These data, with all their higher coherences, are most cleanly encoded as a 2-functor out of a span 2-category — the precise formulation we take up in Section A.1. First we record the geometric content of each functor.

The six functors carry distinct geometric content:

- f^* is *pullback*, f_* the *direct image*, $\otimes$ the *symmetric monoidal product* encoding cup products, and $\underline{\mathrm{Hom}}$ the right adjoint to $\otimes$.
- $f_!$ is the *exceptional / proper-supported pushforward*: it agrees with f_* when f is proper and with $j_\#$ (extension by zero) when f is an open immersion.
- $f^!$ is the *exceptional pullback*, encoding orientation and duality data; in many examples, for f smooth of relative dimension d, $f^! \simeq f^* \otimes \omega_f$ with ω_f the relative dualizing complex.

The three operations f_*, $f^!$, $\underline{\mathrm{Hom}}$ are *right adjoints* of f^*, $f_!$, $\otimes$ respectively, hence determined by the others. Modern formulations therefore typically specify only the three left functors $(f^*, f_!, \otimes)$ (a so-called *three-functor formalism*) and recover the rest by adjunction.

Three representative examples are:

- *Locally compact Hausdorff topological spaces* (Heyer–Mann [HM24]; see also [Sch25, Lecture VII]): here X ranges over (condensed) topological spaces and the coefficient category $\mathrm{D}(X)$ is the derived category of sheaves of Λ-modules on X, for a fixed coefficient ring Λ.
- *ℓ-adic étale cohomology of schemes* [Sch25, Appendix to Lecture VII].
- *The motivic formalism* SH [Sch25, Lecture XI] — the universal $\mathbb{A}^1$-invariant one, by Drew–Gallauer.

For pedagogical introductions complementing the present appendix, we recommend Scholze's lecture notes [Sch25], Gallauer's mini-course [Gal21] (the source of much of this section's exposition), and the textbook treatment in Cisinski–Déglise [CD19]. The encoding via span 2-categories adopted here is closest to Mann–Scholze [Man22; Sch25] and Cnossen–Lenz–Linskens [CLL25], to which we turn in the next subsection.

A.1 Span categories and suitable decompositions

The economical encoding of a six-functor formalism, initiated by Liu–Zheng [LZ24] and brought to its present form by Mann [Man22] and Scholze [Sch25], packages all the operations $(f^*, f_!, \otimes)$ and their coherences into a single lax symmetric monoidal functor on a span category. Given a wide subcategory $E \subset C$ closed under composition, base change, and diagonals (the last is needed for the kernel 2-category below — equivalently the inclusion $C_E \subseteq C$ preserves fibre products, [HM24, Def. 2.1.1 and Rem. 2.1.2]), the *span category* $\mathrm{Span}(C, E)$ has the same objects as C and morphisms $X \to Y$ given by spans

$$X \longleftarrow Z \xrightarrow{e} Y, \qquad e \in E,$$

composed by pullback; when C has finite products, $\mathrm{Span}(C, E)$ inherits a symmetric monoidal structure induced by the finite products of C, with unit the terminal object. (This monoidal structure is *not* Cartesian: the tensor $X \otimes Y \simeq X \times Y$ is induced by the *product* of C, and is in general neither the categorical product nor the coproduct of $\mathrm{Span}(C, E)$. Independently, $\mathrm{Span}(C, E)$ is semiadditive; but its biproduct $X \oplus Y \simeq X \sqcup Y$ comes from the *coproduct* of C, a different operation from the tensor.) A *three-functor formalism* on (C, E) is then a lax symmetric monoidal functor

$$\mathrm{D} \colon \mathrm{Span}(C, E) \longrightarrow \mathrm{Cat},$$

encoding f^* (backward legs), $f_!$ (forward legs), and $\otimes$ (monoidal structure) in a single coherent package; the right adjoints f_*, $f^!$, $\underline{\mathrm{Hom}}$ are recovered automatically and are therefore *properties* of D rather than additional data.

The challenge is that constructing a lax monoidal functor out of $\mathrm{Span}(C, E)$ directly is difficult. In practice, one starts from a much simpler piece of data, namely a functor

$$\mathrm{D}_0 \colon C^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Cat})$$

encoding only f^* and $\otimes$, and asks when it extends to $\mathrm{Span}(C, E)$. The key input is a *suitable decomposition* of E in the sense of Mann [Man22, Definition A.5.9]: a pair of wide subcategories $I, P \subset E$ (thought of as "étale-like" and "proper-like" morphisms respectively) such that every $e \in E$ factors as $e = p \circ i$ with $i \in I$ and $p \in P$, subject to mild cancellation and truncation axioms. The extension theorem then reads as follows. Suppose $i^* = \mathrm{D}_0(i)$ admits a left adjoint $i_\sharp$ and $p^* = \mathrm{D}_0(p)$ admits a right adjoint p_*, both satisfying the expected base change and projection formulae — in particular the mixed Beck–Chevalley equivalence $i_\sharp p'_* \simeq p_* i'_\sharp$. Then D_0 admits an essentially unique extension to a three-functor formalism D, in which

$$f_! = p_* \, i_\sharp \qquad \text{for any factorization } f = p \circ i.$$

Recently, Cnossen–Lenz–Linskens [CLL25] gave a conceptually streamlined account of this extension theorem. They enhance $\mathrm{Span}(C, E)$ to a 2-category whose 2-morphisms encode the adjunction data of $i_\sharp$ and p_* in a fully coherent way, and they prove that the inclusion $C^{\mathrm{op}} \hookrightarrow \mathrm{Span}_2(C, E)_{P,I}$ is a universal (I, P)-biadjointable functor. In this setup, Mann's extension theorem becomes a formal corollary; as a byproduct, the unit and counit witnessing $i_\sharp \dashv i^*$ are part of the structure of D, rather than non-canonical choices recoverable only after the fact.

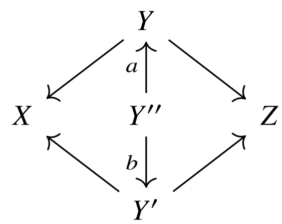
Construction A.1 (CLL 2-categorical span category)**.** The 2-category $\mathrm{Span}_2(C, E)_{P,I}$ has:

- *objects:* objects in C;
- 1-*morphisms* $X \to Z$: spans

$$X \xleftarrow{\;f\;} Y \xrightarrow{\;g\;} Z$$

  with $g \in E$;

- 2-*morphisms* between two such spans: diagrams



with $a \in P$ and $b \in I$.

Theorem A.2 ([CLL25, Theorem 5.17])**.** *Let C be a category with finite products and let $I, P \subseteq E \subseteq C$ be wide subcategories closed under base change such that*

(1) *I and P are left cancellable;*

(2) *every map in E factors as a map in I followed by a map in P;*

(3) *every map in $I \cap P$ is truncated.*

Let $\mathrm{D}_0 \colon C^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat})$ *be a functor satisfying:*

(1) *for every $i \colon a \to b$ in I, the functor $i^* \colon \mathrm{D}_0(b) \to \mathrm{D}_0(a)$ admits a left adjoint $i_\sharp$ satisfying the Beck–Chevalley condition with respect to pullbacks in C and the projection formula $i_\sharp(- \otimes i^*(-)) \simeq i_\sharp(-) \otimes (-)$;*

(2) *for every $p \colon a \to b$ in P, the functor $p^* \colon \mathrm{D}_0(b) \to \mathrm{D}_0(a)$ admits a right adjoint p_* satisfying the dual Beck–Chevalley condition and dual projection formula;*

(3) *for every pullback square in C with vertical maps in P and horizontal maps in I, the mixed Beck–Chevalley map $j_\sharp p_* \to q_* i_\sharp$ is an equivalence.*

Then the lax symmetric monoidal functor $(C^\times)^{\mathrm{op}} \to \mathrm{Cat}^\times$ corresponding to D_0 admits a unique extension to a lax symmetric monoidal 2-functor

$$\mathrm{D} \colon \mathrm{Span}_2(C, E)^{\otimes}_{P,I} \longrightarrow \mathrm{Cat}^\times.$$

Moreover, for any $e \in E$ with factorization $e = p \circ i$ ($i \in I$, $p \in P$), the exceptional pushforward is $e_! \simeq p_* i_\sharp$.

The above is the Mann-style extension formulation; equivalently, $C^{\mathrm{op}} \hookrightarrow \mathrm{Span}_2(C, E)_{P,I}$ is the universal (I, P)-biadjointable functor (left base change for I, right base change for P).

Thus, to construct a six-functor formalism it suffices to verify the three biadjointability conditions: left base change for I, right base change for P, and the mixed Beck–Chevalley equivalence. All coherences come for free from the universal property.

Throughout this subsection, (C, E) denotes a geometric setup and $\mathrm{D}\colon \mathrm{Span}(C, E) \to \mathrm{Cat}$ a three-functor formalism (or a six-functor formalism, when stated). Following [HM24] we write Cat for the 2-category of categories and Cat_2 for the 2-category of 2-categories.

Cohomological smoothness

With the formalism in hand, we turn to the geometric properties of a morphism that D can detect, beginning with smoothness. The motivating example is Poincaré duality.

Theorem A.3 (Poincaré duality, classical)**.** *Let $f\colon X \to Y$ be a submersion of topological manifolds of relative dimension d (i.e. locally of the form $Y \times \mathbb{R}^d \to Y$). Then $f_!$ admits a right adjoint $f^!$, and there is a sheaf $\omega_f := f^!(\mathbb{1}_Y) \in \mathrm{D}(X, \mathbb{Z})$, locally isomorphic to $\mathbb{Z}[d]$, with*

$$f^! \;\simeq\; f^* \otimes \omega_f.$$

The sheaf ω_f is the dualizing complex *of f. (For f proper one has instead $f_! \simeq f_*$; the two phenomena are independent.)*

A morphism in E which satisfies the analogue of Poincaré duality in D will be called *cohomologically smooth* (or D-*smooth*). Two requirements are immediate:

- There is an isomorphism $f^! \simeq f^* \otimes \omega_f$ for some sheaf ω_f;
- $\omega_f := f^!(\mathbb{1}_Y)$ is invertible.

But this is not yet enough. Consider a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

We expect f' to be D-smooth as well, and the dualizing complexes to be compatible with restriction:

$$g'^*\omega_f \;\simeq\; \omega_{f'}.$$

This compatibility is not automatic from pure category theory; it must be imposed.

Definition A.4 (Cohomologically smooth morphism)**.** A morphism $f\colon X \to Y$ in E is *cohomologically smooth* (or D-*smooth*) if it satisfies:

(1) The natural transformation

$$f^!(\mathbb{1}_Y) \otimes f^*(-) \longrightarrow f^!(-)$$

is an equivalence; we write $\omega_f \coloneqq f^!(\mathbb{1}_Y)$ and call it the *dualizing complex* of f.

(2) The dualizing complex ω_f is $\otimes$-invertible in $\mathrm{D}(X)$.

(3) For every pullback square as above with $g \in E$, the morphism f' also satisfies (1)–(2), and the canonical map $g'^* f^!(\mathbb{1}_Y) \to f'^!(\mathbb{1}_{Y'})$ is an equivalence.

Remark A.5. Cohomological smoothness in the sense of Theorem A.4 is weaker than the geometric notion of a smooth morphism. For instance, in the étale six-functor formalism $X \mapsto \mathrm{Shv}(X_{\text{ét}}, \mathrm{D}_{\mathrm{pf}}(\mathbb{Z}))$, every universal homeomorphism between schemes (a map of underlying topological spaces that remains a homeomorphism after pullback) is cohomologically smooth. The étale topology is insensitive to such maps, so they look like isomorphisms at the level of D — and isomorphisms are trivially D-smooth.

The kernel 2-category and Fourier–Mukai transforms

Verifying Theorem A.4 directly is awkward because of the base-change clause (3). The next definitions package D-smoothness into a single 2-categorical statement, after which the base-change clause is automatic.

Fix $Y \in C$, and let $C_E \subseteq C$ be the wide subcategory whose morphisms are those in E. The slice $(C_E)_{/Y}$, paired with the class of *all* morphisms, is itself a geometric setup, and D restricts to a functor

$$\mathrm{D}_Y \colon \mathrm{Span}((C_E)_{/Y}, \mathrm{all})^{\otimes} \to \mathrm{Span}(C, E)^{\otimes} \xrightarrow{\mathrm{D}} \mathrm{Cat}^{\times}.$$

Definition A.6 (Fourier–Mukai transform)**.** Given $X_1, X_2 \in (C_E)_{/Y}$ and $K \in \mathrm{D}(X_1 \times_Y X_2)$, the *Fourier–Mukai transform with kernel K* is the functor

$$\mathsf{F}_K \colon \mathrm{D}(X_1) \to \mathrm{D}(X_2), \qquad A \longmapsto (\pi_2)_!\big(\pi_1^* A \otimes K\big).$$

A span $X_1 \xleftarrow{h} Z \xrightarrow{k} X_2$ over Y induces a Fourier–Mukai transform whose kernel is $K = (h,k)_! \mathbb{1}_Z$.

Remark A.7. The terminology is borrowed from the classical analytic transform $T \colon C^\infty(X) \to C^\infty(Y)$, $f \mapsto \int_X K(x,y) f(x)\, dx$, determined by its kernel K. Theorem A.6 categorifies this construction: $\pi_1^* A$ pulls A back to a "function of two variables", $\otimes K$ multiplies pointwise by the kernel, and $(\pi_2)_!$ is integration along the first variable.

Specializing to $f \colon X \to Y$ in E:

- Setting $X_1 = X$, $X_2 = Y$, and $K_1 = \mathbb{1}_X \in \mathrm{D}(X \times_Y Y) = \mathrm{D}(X)$, the Fourier–Mukai transform is $A \mapsto f_! A$.

- Setting $X_1 = Y$, $X_2 = X$, and $K_2 = \omega_f = f^!(\mathbb{1}_Y) \in \mathrm{D}(Y \times_Y X) = \mathrm{D}(X)$, the transform is $B \mapsto f^*B \otimes \omega_f$.

Idea A.8. The desired identity $f^! \simeq f^* \otimes \omega_f$ should arise by exhibiting the kernels K_1 and K_2 as an adjoint pair in a 2-category whose objects are schemes over Y and whose 1-morphisms are kernels. Since 2-functors preserve adjunctions, the resulting adjunction is automatically compatible with arbitrary base change $g\colon Y' \to Y$, taking care of clause (3) of Theorem A.4 for free.

The invertibility of ω_f is a separate, relatively independent condition; this is what motivates the notion of suaveness below.

Definition A.9 (2-category of kernels)**.** Let $Y \in C$. The 2-*category of kernels* $\mathbb{K}_{\mathrm{D},Y}$ is the 2-category whose:

- *Objects* are objects of $(C_E)_{/Y}$, i.e., morphisms $X \to Y$ in E.
- 1-*morphism category* from X_1 to X_2 is $\mathrm{Fun}_Y(X_1, X_2) \coloneqq \mathrm{D}(X_1 \times_Y X_2)$.
- *Composition* of $M \in \mathrm{Fun}_Y(X_1, X_2)$ and $N \in \mathrm{Fun}_Y(X_2, X_3)$ is

  $$N \circ M \;\coloneqq\; (\pi_{13})_!\big(\pi_{12}^* M \otimes \pi_{23}^* N\big) \;\in\; \mathrm{D}(X_1 \times_Y X_3),$$

  where the π_{ij} are the projections from $X_1 \times_Y X_2 \times_Y X_3$.

The identity 1-morphism on X is $\Delta_{f!}(\mathbb{1}_X) \in \mathrm{D}(X \times_Y X)$, where $\Delta_f\colon X \to X \times_Y X$ is the relative diagonal.

Conceptually, $\mathbb{K}_{\mathrm{D},Y}$ is obtained from $\mathrm{Span}((C_E)_{/Y}, \mathrm{all})$, which is closed (being a span category over a category with finite limits) and hence self-enriched, by base-changing the self-enrichment along D_Y; see [Aok26, Definition 2.4].

There is a canonical 2-functor

$$\Psi_{\mathrm{D},Y} \;=\; \mathrm{Fun}_Y(Y, -) \;:\; \mathbb{K}_{\mathrm{D},Y} \longrightarrow \mathrm{Cat},$$

sending an object X to $\mathrm{D}(X)$ and a 1-morphism $M \in \mathrm{Fun}_Y(X_1, X_2)$ to the Fourier–Mukai transform $(\pi_2)_!\big(\pi_1^*(-) \otimes M\big)$.

For the reader's convenience, we record three facts about adjunctions in 2-categories that we use freely (proofs are standard, see [HM24, Appendix D]):

- Any 2-functor $F\colon \mathbb{C} \to \mathbb{D}$ preserves adjunctions; in fact, it preserves the entire mate correspondence [HM24, Lemma D.2.7].

- Adjunctions are unique up to a contractible space of choices: any two right adjoints to a given 1-morphism are canonically equivalent.
- A pair (f, g) with candidate unit η and counit ε such that the triangle composites are isomorphisms (not necessarily identities) can be corrected to a strict adjunction by modifying η.

Theorem A.10 (Smoothness via kernels, cf. [HM24, §4.5])**.** *A morphism $f : X \to Y$ in E is* D*-smooth if and only if*

(1) *The* 1*-morphism $\mathbb{1}_X \in \mathrm{Fun}_Y(X, Y) = \mathrm{D}(X)$ is a left adjoint in $\mathbb{K}_{\mathrm{D},Y}$. Denote its right adjoint by $\omega_f \in \mathrm{Fun}_Y(Y, X) = \mathrm{D}(X)$.*

(2) *ω_f is $\otimes$-invertible in $\mathrm{D}(X)$.*

When these hold, $\omega_f \simeq f^!(\mathbb{1}_Y)$.

Proof sketch. ($\Rightarrow$) Suppose f is D-smooth. By Theorem A.4(1), $f^!(-) \simeq f^*(-) \otimes \omega_f$. Combined with the adjunction $f_! \dashv f^!$ this becomes $f_! \dashv f^*(-) \otimes \omega_f$ at the level of functors. Translating to $\mathbb{K}_{\mathrm{D},Y}$, the kernel $\mathbb{1}_X$ corresponds to $f_!$ and ω_f to $f^*(-) \otimes \omega_f$, so $\mathbb{1}_X \dashv \omega_f$ in $\mathbb{K}_{\mathrm{D},Y}$. (This lift of the functor-level adjunction to one in $\mathbb{K}_{\mathrm{D},Y}$ rests on the explicit kernels above, not on any reflection property of $\Psi_{\mathrm{D},Y}$, which is not conservative in general; cf. the Künneth obstruction of Theorem 2.3.) Invertibility is clause (2).

($\Leftarrow$) Apply $\Psi_{\mathrm{D},Y}$ to the adjunction $\mathbb{1}_X \dashv \omega_f$. Since 2-functors preserve adjunctions, the resulting functors $\Psi_{\mathrm{D},Y}(\mathbb{1}_X) \simeq f_!(-)$ and $\Psi_{\mathrm{D},Y}(\omega_f) \simeq f^*(-) \otimes \omega_f$ form an adjunction in Cat_2. By uniqueness of right adjoints, $f^*(-) \otimes \omega_f \simeq f^!(-)$, which is clause (1) of Theorem A.4; setting $- = \mathbb{1}_Y$ gives $\omega_f \simeq f^!(\mathbb{1}_Y)$. The base-change clause (3) follows because $S \mapsto \mathbb{K}_{\mathrm{D},S}$ assembles into a 2-functor $\mathrm{Span}(C, E)^{\otimes} \to \mathrm{Cat}_2^{\times}$ [HM24, Theorem 4.2.4], so the adjunction $\mathbb{1}_X \dashv \omega_f$ is preserved by base change. (The cited functor is lax symmetric monoidal; only its underlying 2-functoriality is used here.) □

Suaveness and primness

Theorem A.10 decomposes cohomological smoothness as "left-adjointness in $\mathbb{K}_{\mathrm{D},Y}$" plus "invertibility of the dualizing complex". As foreshadowed, dropping the invertibility yields broader classes of objects.

Definition A.11 (Suave and prim objects)**.** Let $f : X \to Y$ be in E and $A \in \mathrm{D}(X)$.

(1) A is *f-suave* if, viewed as a 1-morphism $X \to Y$ in $\mathbb{K}_{\mathrm{D},Y}$, it is a left adjoint. Its right adjoint is the *f-suave dual* $\mathrm{SD}_f(A) \in \mathrm{D}(X)$. Write $\mathrm{Suave}_f(X) \subseteq \mathrm{D}(X)$ for the full subcategory of f-suave objects.

(2) A is *f-prim* if, viewed as a 1-morphism $Y \to X$ in $\mathbb{K}_{\mathrm{D},Y}$, it is a left adjoint. Its right adjoint is the *f-prim dual* $\mathrm{PD}_f(A) \in \mathrm{D}(X)$. Write $\mathrm{Prim}_f(X) \subseteq \mathrm{D}(X)$ for the full subcategory of f-prim objects.

(Equivalently, via the self-duality $\mathbb{K}_{\mathrm{D},Y} \simeq \mathbb{K}_{\mathrm{D},Y}^{\mathrm{op}}$, A is f-prim iff it is a *right* adjoint 1-morphism $X \to Y$; this is the form used by Heyer–Mann [HM24].)

Remark A.12 (Origin of the terminology)**.** The terms are due to Scholze (suave) and Hansen (prim); applications typically require D to be a six-functor formalism, although the definition makes sense for three-functor formalisms.

- *Suave* (*relaxed smoothness*): when $A = \mathbb{1}_X$ is f-suave, applying $\Psi_{\mathrm{D},Y}$ to the adjunction $\mathbb{1}_X \dashv \mathrm{SD}_f(\mathbb{1}_X)$ gives $f^*(-) \otimes \mathrm{SD}_f(\mathbb{1}_X) \simeq f^!(-)$. More generally, every f-suave A produces a relation between f^* and $f^!$.
- *Prim* (*primitive properness*): when $A = \mathbb{1}_X$ is f-prim, the dual statement gives $f_!(\mathrm{PD}_f(\mathbb{1}_X) \otimes -) \simeq f_*(-)$. Primness is the phenomenon that f_* is recovered from $f_!$ by twisting; this is strictly weaker than the proper-base-change identity $f_! \simeq f_*$, so it is "primitive" in this sense.

In a precise sense, A is f-suave (or f-prim) iff it is "finite-dimensional relative to f": it admits a reflexive duality.

Proposition A.13 (Reflexive duality, [HM24, §4.4])**.** *Let* D *be a six-functor formalism on* (C, E) *and* $f : X \to Y$ *in* E*.*

(1) *If* $A \in \mathrm{D}(X)$ *is* f*-suave, then* $\mathrm{SD}_f(A)$ *is also* f*-suave (with right adjoint* A *in* $\mathbb{K}_{\mathrm{D},Y}$*), and there is a natural isomorphism*

$$\mathrm{SD}_f(A) \otimes f^*(-) \;\simeq\; \underline{\mathrm{Hom}}\big(A, f^!(-)\big) :\ \mathrm{D}(Y) \to \mathrm{D}(X).$$

Setting $- = \mathbb{1}_Y$ *yields* $\mathrm{SD}_f(A) \simeq \underline{\mathrm{Hom}}(A, f^!(\mathbb{1}_Y))$*, exhibiting* $\mathrm{SD}_f(A)$ *as a relative Verdier dual of* A*. The bidual identity* $\mathrm{SD}_f(\mathrm{SD}_f(A)) \simeq A$ *then holds, and* SD_f *defines a contravariant equivalence*

$$\mathrm{SD}_f :\ \mathrm{Suave}_f(X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Suave}_f(X).$$

(2) *Dually, if* $A \in \mathrm{D}(X)$ *is* f*-prim, then* $\mathrm{PD}_f(A)$ *is also* f*-prim, with*

$$f_!\big(\mathrm{PD}_f(A) \otimes -\big) \;\simeq\; f_*\,\underline{\mathrm{Hom}}(A, -) :\ \mathrm{D}(X) \to \mathrm{D}(Y),$$

and the bidual $\mathrm{PD}_f(\mathrm{PD}_f(A)) \simeq A$ *holds. Consequently* PD_f *defines a contravariant equivalence*

$$\mathrm{PD}_f :\ \mathrm{Prim}_f(X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Prim}_f(X).$$

Suave morphisms and prim morphisms

Specializing Theorem A.11 to $A = \mathbb{1}_X$ yields classes of *morphisms*.

Definition A.14 (Suave and prim morphisms)**.** Let $f : X \to Y$ be in E.

(1) f is D-*suave* if $\mathbb{1}_X \in \mathrm{Suave}_f(X)$. As before we write $\omega_f := \mathrm{SD}_f(\mathbb{1}_X) \in \mathrm{D}(X)$ for its *dualizing complex*.

(2) f is D-*prim* if $\mathbb{1}_X \in \mathrm{Prim}_f(X)$. We then write $\delta_f := \mathrm{PD}_f(\mathbb{1}_X) \in \mathrm{D}(X)$ for the *co-dualizing complex* of f.

Remark A.15. Combining Theorem A.10 with Theorem A.14 gives

$$f \text{ is D-smooth} \iff f \text{ is D-suave and } \omega_f \text{ is invertible.}$$

By Theorem A.13, the dualizing complex is the relative Verdier dual $\omega_f \simeq f^!(\mathbb{1}_Y)$. The co-dualizing complex — the right adjoint in $\mathbb{K}_{\mathrm{D},Y}$ of the unit kernel (Theorems A.9 and A.11) — is computed from the identity kernel by [HM24, §4.5]:

$$\delta_f \simeq \pi_{2*}\Delta_{f!}\,\mathbb{1}_X,$$

where $\pi_2 : X \times_Y X \to X$ is the projection onto the second factor.

For D-suave f, the formula $\omega_f \otimes f^*(-) \xrightarrow{\sim} f^!(-)$ is equivalent (by adjunction) to $f^*(-) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\omega_f, f^!(-))$. Plugging in $\mathbb{1}_Y$ yields $\mathbb{1}_X \xrightarrow{\sim} \underline{\mathrm{Hom}}(\omega_f, \omega_f)$, and for dualizable ω_f this gives $\mathbb{1}_X \simeq \omega_f^\vee \otimes \omega_f$. We record this as an observation.

Observation A.16. For a D-suave morphism f, the dualizing complex ω_f is $\otimes$-invertible if and only if it is dualizable. Consequently the verification of D-smoothness reduces to checking suaveness and dualizability of ω_f.

We close with the base-change identities for suave and prim morphisms; they will guide the definitions of cohomologically étale and proper morphisms in the next subsubsection.

Lemma A.17 (Base change for suave and prim morphisms, [HM24, Lemma 4.5.13])**.** *Let* D *be a six-functor formalism on* (C, E) *and consider a pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in C_E.

(1) *If f is* D*-suave, the natural transformations*

$$g'_* f'^* \simeq f^* g_*, \quad g'_! f'^! \simeq f^! g_!, \quad f'^* g^! \simeq g'^! f^*, \quad g'^* f^! \simeq f'^! g^*$$

are all isomorphisms.

(2) *If f is* D*-prim, the natural transformations*

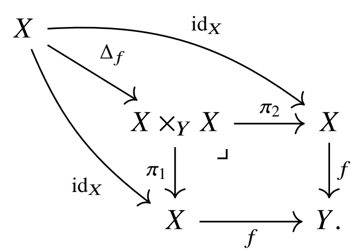
$$g^* f_* \simeq f'_* g'^*, \quad g^! f_! \simeq f'_! g'^!, \quad f_! g'_* \simeq g_* f'_!, \quad f_* g'_! \simeq g_! f'_*$$

are all isomorphisms.

Cohomological étaleness and properness

We now ask: under what conditions are there genuine isomorphisms $f^! \simeq f^*$ and $f_! \simeq f_*$, rather than mere twists by ω_f or δ_f?

A candidate map $f^! \to f^*$ is constructed using the diagonal. Consider



We have $f\pi_1 = f\pi_2$ and $\mathrm{id}_X = \pi_2\Delta_f$. Suppose inductively that Δ_f satisfies $\Delta_f^! \simeq \Delta_f^*$. Then

$$\begin{aligned} f^! &\simeq \mathrm{id}_X^* f^! \simeq \Delta_f^* \pi_2^* f^! \xrightarrow{\sim} \Delta_f^! \pi_2^* f^! \\ &\to \Delta_f^! \pi_1^! f^* \simeq (\pi_1 \Delta_f)^! f^* \simeq f^*, \end{aligned}$$

where $\pi_2^* f^! \to \pi_1^! f^*$ is induced (via right adjoints) from the base-change isomorphism $\pi_{1!}\pi_2^* \simeq f^* f_!$. By Theorem A.17, when f is D-suave the latter is an isomorphism, and so $f^! \simeq f^*$.

A dual construction gives $f_! \to f_*$, an isomorphism when Δ_f satisfies $\Delta_{f!} \simeq \Delta_{f*}$ and f is D-prim. To make the recursion well-founded we impose a truncatedness hypothesis: the recursion terminates at -2-truncated morphisms (i.e., isomorphisms), where the conclusion is trivial.

Definition A.18 (Cohomologically étale and proper morphisms)**.** Let D be a six-functor formalism on (C, E), and let $f : X \to Y$ be a truncated morphism in E (i.e., n-truncated for some $n \in \mathbb{N}$).

- f is D-*étale* (or *cohomologically étale*) if f is D-suave and Δ_f is either D-étale or an isomorphism.

- f is D-*proper* (or *cohomologically proper*) if f is D-prim and Δ_f is either D-proper or an isomorphism.

These definitions are well-founded because n-truncated morphisms have $(n-1)$-truncated diagonals.

Proposition A.19 (Characterization of étaleness and properness, cf. [HM24, §4.6])**.** *Let* D *be a six-functor formalism on* (C, E) *and* $f : X \to Y$ *in* E.

(1) *Suppose* Δ_f *is* D*-étale. There is a natural map* $f^! \to f^*$*, and the following are equivalent:*

  (a) f *is* D*-étale;*

  (b) $f^! \mathbb{1}_Y \simeq f^* \mathbb{1}_Y$ *in* $\mathrm{D}(X)$*;*

  (c) $f^! \xrightarrow{\sim} f^*$ *as functors.*

(2) *Suppose* Δ_f *is* D*-proper. There is a natural map* $f_! \to f_*$*, and the following are equivalent:*

  (a′) f *is* D*-proper;*

  (b′) $f_! \mathbb{1}_X \simeq f_* \mathbb{1}_X$ *in* $\mathrm{D}(Y)$*;*

  (c′) $f_! \xrightarrow{\sim} f_*$ *as functors.*

This completes the foundational vocabulary — cohomological smoothness, suaveness, primness, étaleness, properness — whose translation under the transmutation $[-]_{\mathrm{D}}$ is the subject of the dictionary in the main text.

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